COMPUTER METHODS IN APPLIED MECHANICS AND ENGINEERING 22 (1980) 49–58

OSCILLATORY FULLY DEVELOPED VISCOUS FLOW IN A TOROIDAL TUBE

Chr. CHRISTOV and Z. ZAPRYANOV
Institute of Mechanics and Biomechanics, Bulgarian Academy of Sciences, Sofia, Bulgaria

Received 16 March 1979
Revised 3 October 1979

1. Introduction

The main purpose of earlier investigations of nonsteady motions of viscous fluid through pipes or channels was to analyse the propagation of sound waves and the stability problems of laminar motions.

Nonsteady viscous flows in straight and curved tubes appear in many modern industrial equipment and technology processes. That is why these unsteady viscous flows are nowadays of broad interest both theoretically and experimentally. Recently, many studies in this field have become the object of attention.

An exact solution of the pulsating laminar flow superposed on the steady motion in a straight pipe under the assumption of parallel flow to the pipe axis was presented by Uchida [1] in 1956. One of the simplest solutions of steady laminar flow in a circular tube is the approach proposed by Targ (see [2]), whose special assumption is that the velocity of convection in the inertia terms remains the same as it was at the inlet section—this eliminates the nonlinearity from the equation of motion and leads to a solution that is in good agreement with experimental results. A similar assumption for the case of unsteady laminar flow was made by Atabek and Chang [3], who examined the oscillatory viscous flow near the inlet of a circular tube and the flow developed far downstream where the velocity profile becomes independent of the coordinate along the tube axis. Using Targ's assumption, Atabek [4] considered flow development in the inlet length of a straight circular tube starting from a state of rest.

The mode of the fluid flow in a curved tube is characterized by the secondary flow field which is superimposed upon the axial velocity flow field. The nature of the toroidal viscous fluid motion as compared with a simple straight-tube flow causes relatively high average heat- and mass-transfer rates per unit of axial pressure drop as well as a significant distribution of transport rates.

The first theoretical study of fully developed steady flow in a curved tube with a circular section was carried out in 1927 by Dean [5], whose analysis was restricted to the case of small values of $K = (2a/L)(aW_m/\nu)^2$, where W_m is the mean velocity along a pipe, ν the kinematic viscosity and a the radius of the pipe, which is bent into a circle with a radius L.

McConaloque and Srivastava [6] extended Dean's results and introduced the parameter $D = 4\text{Re}(2a/L)^{1/2}$. Physically, this parameter can be considered as the ratio of the centrifugal force induced by the circular motion of the fluid to the viscous force.

Trusdel and Adler [7] obtained results for $D \leqq 3578$, and Austin and Seader focused their attention on the range $0 \leqq D \leqq 5000$. Recently, Zapryanov and Christov [8] and Christov [9] presented results of the numerical solution of the same problem obtained by the method of fractional steps in the neighborhood of $D = 7000$; in [9] the results are extended to $D = 20{,}000$.

Riley [11], [12] systematically applied the method of higher order boundary layer theory to study numerous problems related to vibrating bodies in an unbounded fluid.

In 1970 Lyne [13] considered unsteady flow in a toroidal pipe with a circular cross-section under the assumption that the pressure gradient along the pipe varies sinusoidally with respect to time with frequency ω. For simplicity, the radius of the pipe curvature was assumed to be large in comparison with its own radius. An asymptotic theory for small values of the frequency parameter $\beta = [2\nu/(\omega a^2)]^{1/2}$ was developed. For sufficiently small values of β it was found that the secondary flow in the interior of the pipe is opposite in nature to that of a steady pressure gradient.

Zalosh and Nelson [14] also studied fully developed laminar flow in a curved tube of circular cross-section under the influence of a pressure gradient oscillating sinusoidally in time. The solution was obtained for arbitrary values of the frequency parameter $\alpha = a(\omega/\nu)^{1/2}$ with the help of the numerical evaluation of the finite Hankel transformation. The numerical solution of the ordinary differential equations produced a consistent result at low and moderate frequencies of oscillation but presented difficulties at high frequencies since it was not easy to evaluate the Hankel transformations for high frequencies.

The purpose of this paper is to present the numerical analysis of the fully developed oscillatory laminar viscous flow in a curved circular tube. Results are presented for different values of the frequency parameters β and $\epsilon^2 = \bar{W}^2/(La\omega^2)$, where $\bar{W}$ is a typical velocity along the pipe.

2. Formulation of the problem

We shall consider the unsteady, hydrodynamically fully developed, laminar flow of an incompressible viscous fluid in a curved circular tube. Fig. 1 shows the system of toroidal

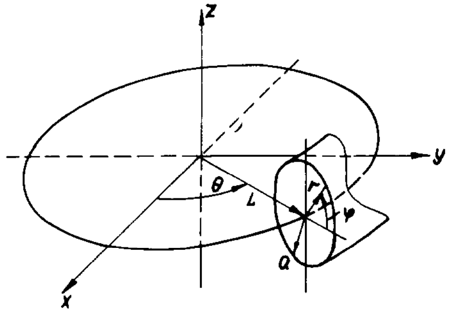

Fig. 1. The coordinate system.

coordinates r, φ, θ used in the description of the motion of the fluid through the pipe. The distance downstream the tube is measured by $L\theta$, where θ is the angular coordinate normal to the tube cross-section, and L is the radius of curvature of the coiled tube. A point P in the tube cross-section is the origin of the polar coordinates r, φ. Let u, v, w denote the velocity components in the r, φ, θ directions, respectively, and assume that the radius of the tube axis curvature is large in comparison with the radius of the tube. Let us also introduce the following transformations:

$$r = ar', \qquad u = \frac{\bar{W}^2}{L\omega}u', \qquad v = \frac{\bar{W}^2}{L\omega}v', \qquad w = \bar{W}w',$$

$$t' = \omega t, \qquad p' = \frac{p + \rho L\omega \bar{W}\cos \omega t}{\rho\delta\bar{W}^2}, \qquad \delta = \frac{a}{L}, \tag{1}$$

$$-\frac{\partial}{\partial\theta}\left(\frac{P}{\rho}\right) = L\bar{W}\omega \cos \omega t,$$

where p and ρ are the pressure and density of the fluid.

After the assumptions stated above are applied, the governing equations for the present problem in dimensionless form are

$$\frac{\partial w'}{\partial t'} - \frac{\epsilon^2}{r'}\frac{\partial(\psi, w')}{\partial(r', \varphi)} = \frac{1}{2}\beta^2\nabla^2 w' + \cos t', \tag{2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r'^2} + \frac{1}{r'}\frac{\partial}{\partial r'} + \frac{1}{r'^2}\frac{\partial^2}{\partial\varphi^2}, \qquad \frac{\partial(a, b)}{\partial(r', \varphi)} \equiv \frac{\partial a}{\partial r'}\frac{\partial b}{\partial\varphi} - \frac{\partial b}{\partial r'}\frac{\partial a}{\partial\varphi};$$

and the equation for the secondary flow is

$$\frac{\partial}{\partial t'}(\nabla^2\psi) - \frac{\epsilon^2}{r'}\frac{\partial(\psi, \nabla^2\psi)}{\partial(r', \varphi)} - \frac{2}{r'}\left(r'w'\frac{\partial w'}{\partial r'}\sin\varphi + w'\frac{\partial w'}{\partial\varphi}\cos\varphi\right) = \frac{1}{2}\beta^2\nabla^4\psi, \tag{3}$$

where

$$u' = \frac{1}{r'}\frac{\partial\psi}{\partial\varphi}, \qquad v' = -\frac{\partial\psi}{\partial r}.$$

The boundary conditions for these equations are

$$\psi = \frac{\partial\psi}{\partial r'} = 0 \quad \text{and} \quad w' = 0 \quad \text{at} \quad r' = 1, \tag{4}$$

and we also require the solution not to be singular within the pipe.

Because of symmetry we have

$$\psi(r', -\varphi) = -\psi(r', \varphi), \qquad w(r', -\varphi) = w(r, \varphi); \tag{5}$$

thus it is only necessary to consider the upper half of the circular region.

If we introduce the function

$$\zeta = -\nabla^2 \psi, \tag{6}$$

eq. (3) is reduced to

$$\frac{\partial \zeta}{\partial t'} - \frac{\epsilon^2}{r'} \frac{\partial(\psi, \zeta)}{\partial(r', \varphi)} + \frac{2}{r'}\left(r'w' \frac{\partial w'}{\partial r'} \sin\varphi + w' \frac{\partial w'}{\partial \varphi} \cos\varphi\right) = \frac{1}{2}\beta^2 \nabla^2 \zeta, \tag{7}$$

and because of the flow symmetry we set the boundary conditions

$$\psi = \zeta = \frac{\partial w}{\partial \varphi} = 0 \qquad \text{at} \quad \varphi = 0 \qquad \text{or} \qquad \varphi = \pi \tag{8}$$

and

$$\frac{\partial w}{\partial r} = 0 \qquad \text{at} \quad \varphi = \frac{\pi}{2} \qquad \text{and} \qquad r = 0. \tag{9}$$

3. Finite difference approximation and computational procedure

We suppose the semicircular region to be divided into a grid formed by a set of radial lines which are cut by a set of semicircular concentric arcs with the boundary $r = 1$ (fig. 2). The grid points are uniformly spaced with spacing $h = 1/(M - 1)$ in the direction r and $l = \pi/(N - 1)$ in the angle φ, where M and N are the numbers of the points in the r and φ directions, respectively.

In order to solve the problem defined by (2), (3), (6) and (7)–(9), we add the time derivative

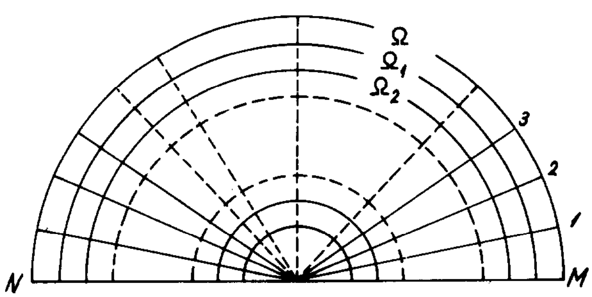

Fig. 2. Numerical grid in the tube cross-section.

of the function ψ in eq. (6):

$$\frac{\partial \psi}{\partial t_1} = \nabla^2 \psi + \zeta, \tag{6'}$$

where t_1 is the fictitious time parameter.

Some difficulties arise because there is no boundary condition for the vorticity ζ at the tube wall. That is why we have solved the equation inside the circle Ω_1 (fig. 2). At Ω_1 we put the condition

$$\zeta^{n+1} = -\Delta \psi^n, \tag{10}$$

where n is the number of the time stage. From the condition

$$\psi = \frac{\partial \psi}{\partial r} = 0 \qquad \text{at} \quad \Omega$$

we derive

$$\psi|_{\Omega_2} = 4\psi|_{\Omega_1}, \qquad \psi|_{\Omega} = 0.$$

Let

$$\Delta_\varphi A_{i,j} = \frac{A_{i,j+1} - A_{i,j-1}}{2r_i l} = \frac{\partial A}{\partial \varphi} + O(l^2),$$

$$\Delta_r A_{i,j} = \frac{A_{i+1,j} - A_{i-1,j}}{2h} = \frac{\partial A}{\partial r} + O(h^2),$$

$$\Lambda_\varphi A_{i,j} = \frac{A_{i,j+1} - 2A_{i,j} + A_{i,j-1}}{r_i^2 l^2} = \frac{\partial^2 A}{\partial \varphi^2} + O(l^2),$$

$$\Lambda_r A_{i,j} = \frac{(1 + 1/2h/r_i)A_{i+1,j} - 2A_{i,j} + (1 - 1/2h/r_i)A_{i,1,j}}{h^2} = \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial A}{\partial r} + O(h^2).$$

The calculation procedure is based on the numerical solution of the following finite difference approximation of the Navier–Stokes equations (2), (3) and (6). We have

$$\frac{\tilde{\psi} - \psi^{n,p}}{1/2\tau_1} = \Lambda_r \psi^{n,p} + \Lambda_\varphi \tilde{\psi} + \zeta^{n,p}, \tag{11}$$

$$\frac{\psi^{n,p+1} - \tilde{\psi}}{1/2\tau_1} = \Lambda_r \psi^{n,p+1} + \Lambda_\varphi \tilde{\psi} + \zeta^{n,p} \tag{12}$$

for $j = 2, 3, \ldots N-1$; $i = 2, 3, \ldots M-2$; $p = 1, 2, \ldots \mathsf{P}$.

From the boundary conditions we obtain

$$\psi_{1,j} = 0, \qquad \psi_{i,1} = 0, \qquad \psi_{1,N} = 0, \tag{13}$$

$$\psi_{M-2,j} = 4\psi_{M-1,j}, \qquad \psi_{M,j} = 0. \tag{14}$$

The finite difference schemes for the longitudinal velocity component w and the vorticity ζ have proved to be more complex.

At first we predict the values of w and ζ utilizing the following explicit schemes:

(i) For w:

$$\frac{w^{n,1}-w^n}{1/2\tau}=-\frac{\epsilon^2}{r_i}(\Delta_\varphi\psi^n\Delta_r w^n-\Delta_r\Delta_\varphi w^n)+\frac{\beta}{2}(\Lambda_\varphi+\Lambda_r)w^n+\cos t_n, \tag{15}$$

where $j=1,2,\dots N$, $i=2,3,\dots M-1$, and

$$w^{n,1}_{i,0}=w^{n,1}_{i,2},\qquad w^{n,1}_{i,N}=w^{n,1}_{i,N-1},\qquad w^{n,1}_{M,j}=0. \tag{16}$$

(ii) For ζ:

$$\frac{\zeta^{n,1}-\zeta^n}{1/2\tau}=-\frac{\epsilon^2}{r_i}(\Delta_\varphi\psi\Delta_r\zeta-\Delta_r\psi\Delta_\varphi\zeta)+\frac{\beta^2}{2}(\Lambda_\varphi+\Lambda_r)\zeta$$
$$-w\left(\sin\varphi_j\Delta_r W+\frac{\cos\varphi_j}{r_i}\Delta_\varphi w\right), \tag{17}$$

where $i=2,3,\dots M-1$, $j=2,3,\dots N-1$, and

$$\zeta^{n,1}_{1,j}=0,\qquad \zeta^{n,1}_{i,N}=\zeta^{n,1}_{1,N}=0. \tag{18}$$

Then we find $\psi^{n,1}$ from eqs. (11) and (12).

The computational procedure involves also the correstor operations:

(i) For w:

$$\frac{\tilde{w}-w^n}{1/2\tau}=F_w+\frac{\beta^2}{2}\Lambda_r\tilde{w}+\frac{\beta^2}{2}\Lambda_\varphi w^n, \tag{19}$$

$$\frac{w^{n,p+1}-\tilde{w}}{1/2\tau}=F_w+\frac{\beta^2}{2}\Lambda_r\hat{w}+\frac{\beta^2}{2}\Lambda_\varphi w^{n,p+1}, \tag{20}$$

where $i=2,3,\dots M-1$, $j=2,3,\dots N-1$, and

$$w^{n,p+1}_{M,j}=0,\qquad w^{n,p+1}_{i,1}=w^{n,p+1}_{i,2},\qquad w^{n,p+1}_{i,N}=w^{n,p+1}_{i,N-1},\qquad \tilde{w}_{1,N+1/2}=\frac{4\tilde{w}_{2,(N+1)/2}-\tilde{w}_{1,(N+1)/2}}{3} \tag{21}$$

$$F_w=-\frac{\epsilon^2}{2r_i}[\Delta_\varphi\psi^{n,p}\Delta_r w^{n,p}-\Delta_r\psi^{n,p}\Delta_\varphi w^{n,p}$$
$$+(\Delta_\varphi\psi^n\Delta_r w^n-\Delta_r\psi^n\Delta_r w^n)]+\frac{1}{2}(\cos t_n+\cos t_{n+1}). \tag{22}$$

(ii) For ζ:

$$\frac{\tilde{\zeta}-\zeta^n}{1/2\tau}=F_\zeta+\frac{\beta^2}{2}\Lambda_r\zeta^n+\frac{\beta^2}{2}\Lambda_\varphi\tilde{\zeta}, \tag{23}$$

$$\frac{\zeta^{n,p+1}-\tilde{\zeta}}{1/2\tau}=F_\zeta+\frac{\beta^2}{2}\Lambda_r\zeta^{n,p+1}+\frac{\beta^2}{2}\Lambda_\varphi\tilde{\zeta}, \tag{24}$$

where $i=2,3,\dots M-2$, $j=2,3,\dots N-1$, and

$$\tilde{\zeta}_{i,1}=\tilde{\zeta}_{i,N}=0, \qquad \zeta_{1,j}^{n,p+1}=0, \qquad \zeta_{M-1,j}^{n,p+1}=-(\Lambda_r+\Lambda_\varphi)\psi_{M-1,j}^{n,p}, \tag{25}$$

$$F_\zeta=-\frac{\epsilon^2}{2r_i}[(\Delta_\varphi\psi^{n,p}\Delta_r\zeta^{n,p}-\Delta_r\psi^{n,p}\Delta_\varphi\zeta^{n,p})$$
$$+(\Delta_\varphi\psi^n\Delta_r\zeta^n-\Delta_r\psi^n\Delta_\varphi\zeta^n)]-\frac{1}{2}\Big[w^{n,p}\Big(\sin\varphi_j\Delta_r w^{n,p}$$
$$+\frac{\cos\varphi_j}{r_i}\Delta_\varphi w^{n,p}\Big)+w^n\Big(\sin\varphi_j\Delta_r w^n+\frac{\cos\varphi_j}{r_i}\Delta_\varphi w^n\Big)\Big]. \tag{26}$$

The corrector iteration is terminated when the following criterion is satisfied:

$$\max\left\{\frac{\|w^{n,P}-w^{m,P-1}\|}{\|w^{n,P}\|},\frac{\|\zeta^{n,P}-\zeta^{n,P-1}\|}{\|\zeta^{n,P}\|}\right\}\leqq\epsilon_1=0.0001. \tag{27}$$

Since the boundary conditions for ψ do not depend on the fictitious time t_1 (i.e. superscript p), the solutions of (11) and (12) will be convergent simultaneously.

After that we can obtain the values of the vorticity on the tube wall $r=1$:

$$\zeta_{M,j}^{n,P}=-(\Lambda_r+\Lambda_\varphi)\psi^{n,P}|_{r=1}=-\frac{2}{h^2}\psi_{M-1,j}^{n,P}. \tag{28}$$

It is interesting to note that this solution algorithm is of order $O(\tau^2+h^2+l^2)$ with the exception of the points along $\varphi=0$ and $\varphi=\pi$, where the order of the numerical scheme is $O(\tau^2+h^2+l)$.

We have solved the physical problem (2), (3), (6′) and (7)–(9) beginning with the initial conditions

$$w_{i,j}^0=\psi_{i,j}^0=\zeta_{i,j}^0=0.$$

The iterative procedure is terminated when the following relative error criterion is satisfied:

$$\Big(\sum_{n=1}^{s}|w_{11}^{ks+n}-w_{11}^{(k-1)s+n}|\Big)\Big/\Big(\sum_{n=1}^{s}|w_{11}^{ks+n}|\Big)\leqq 0.001,$$

where s is the number of the steps in one period, and k is the number of the last calculating period.

4. Numerical results

Calculations were carried out for the grid sizes $h=0.05$ ($M=21$), $l=0.314$ ($N=11$) and $s\sim 100$. The stability condition is $\tau\approx 0.03$ ($\tau\sim h$), which has proved to be a very good one since $h=0.05$ and $l=0.314$. The above discretization procedure was found to converge satisfactorily when $k\leqq 7$.

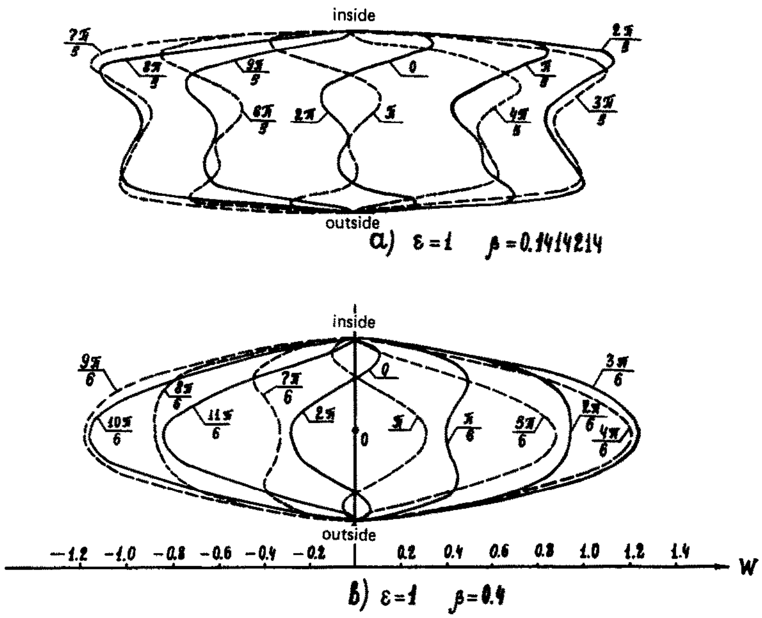

Fig. 3. Distribution of the axial velocity for different moments of time on the line of symetry $\varphi = 0$ and $\varphi = \pi$.

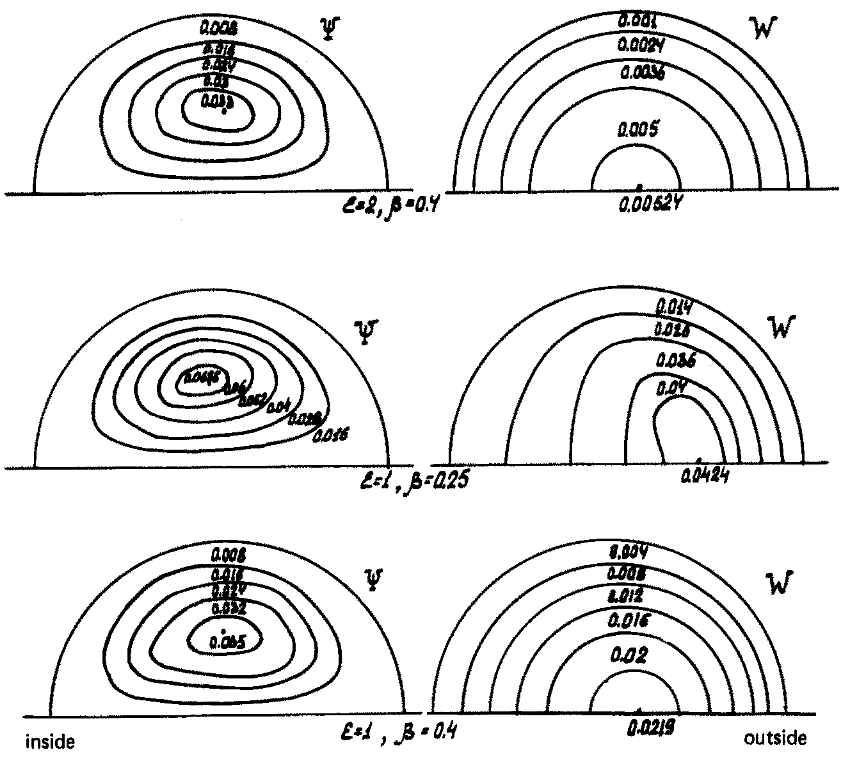

Fig. 4. Steady streaming and isolines for different values of ϵ and β.

Fig. 3 presents the calculated profiles of the longitudinal velocity w for different moments of time, $\epsilon = 1$, $\beta = 0.1414214$ and $\beta = 0.4$.

When the fluid passes through a curved tube, the pressure gradient directed towards the centre of curvature is set up across the tube to balance the centrifugal force arising due to the curvature. The fluid in the middle of the pipe moves outwards, and the fluid above and below the pipe moves inwards. The curves of constant streamfunction ψ and constants w for the steady component of the flow are presented in fig. 4 for different values of ϵ and β.

Fig. 5 shows the curves of constant dimensionless axial velocity w and constant streamfunction ψ for the typical boundary layer situation when $\epsilon = 1$ and $\beta = 0.05$.

It is instructive to note that in this case there is a zero streamfunction within the flow and a vortex in the boundary layer near the tube wall together with the basic vortex in the upper part of the tube.

In fig. 6 the calculated profiles of the axial velocity at $\varphi = 0$ and $\varphi = \pi$ for different values of ϵ and β are plotted. As opposed to the case of the fully developed steady flow in a curved tube, the axial velocity w distribution has a peak nearer to the inner wall in the case of the oscillatory flow.

One can conclude that the method of fractional steps combined with iteration in the form of the predictor and corrector procedure has been successfully utilized when solving the problem of the hydrodynamically fully developed oscillatory flow of a viscous incompressible fluid in a curved circular tube.

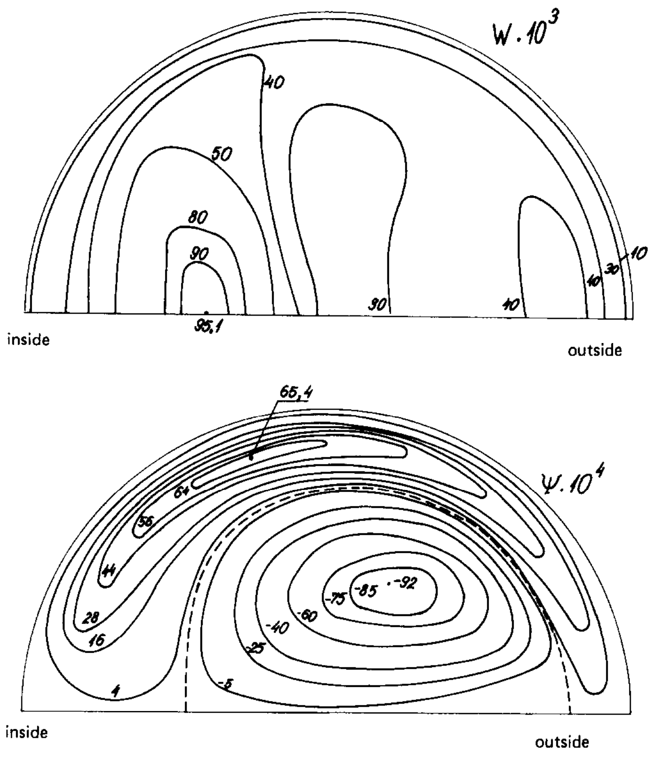

Fig. 5. Steady streaming and isolines for axial velocity at $\epsilon = 1$ and $\beta = 0.05$.

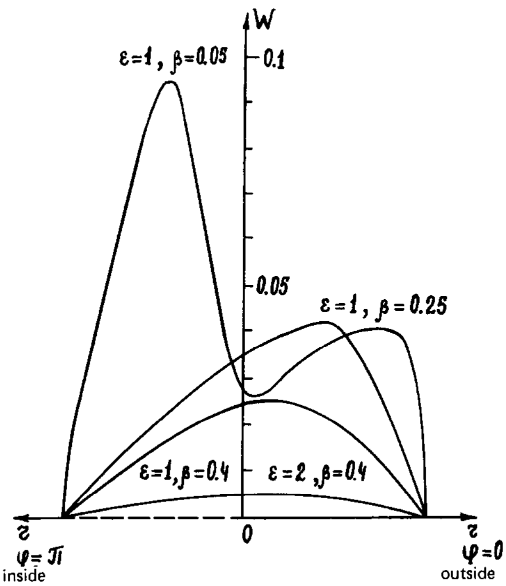

Fig. 6. Isolines for axial velocity at $\varphi = 0$ and $\varphi = \pi$ for different values of ϵ and β.

References

[1] S. Uchida, The pulsating viscous flow superposed on the steady laminar motion of incompressible fluid in a circular pipe, ZAMP 7 (1956) 403–422.

[2] N.A. Slezkin, Dynamics of viscous incompressible fluids [Russian] (Gostekhizdat, Moscow, 1955).

[3] H.B. Atabec and C.C. Chang, Oscillatory flow near the entry of a circular tube, ZAMP 12 (1961) 185–201.

[4] H.B. Atabek, Development of flow in the inlet length of a circular tube starting from rest, ZAMP 13 (1962) 417–430.

[5] W.R. Dean, Note on the motion of fluid in a curved pipe, Phil. Mag. 4 (1927) 208–223.

[6] D.J. McConalogue and R.S. Srivastava, Motion of fluid in a curved tube, Proc. Roy. Soc. 307-A (1968) 37–53.

[7] L.C. Truesdell, Jr. and B.J. Adler, Numerical treatment of fully developed laminar flow in helically coiled tubes, AIChE J. 16 (1970) 1010–1015.

[8] Z. Zapryanov and C. Christov, Fully developed viscous flow in toroidal pipe [Bulgarian], Theor. Appl. Mech. 8 (1977) 11–17.

[9] C.I. Christov, Fully developed viscous incompressible flow in toroidal tube for a wide range of Dean's number [Russian], Chisl. Meth. Mekh. Sploshnoi Sredy 9 (1978) 122–138.

[10] N.N. Yanenko, The method of fractional steps (McGraw-Hill, 1971).

[11] N. Riley, Oscillatory viscous flows—review and extension, Int. Math. Appls. 3 (1967) 419–434.

[12] N. Riley, Unsteady laminar boundary layers, SIAM Rev. 17 (1975) 274–297.

[13] W.H. Lyne, Unsteady viscous flow in a curved pipe, J. Fluid Mech. 45 (1971) 13–31.

[14] R.G. Zalosh and W.G. Nelson, Pulsating flow in a curved tube, J. Fluid Mech. 59 (1973) 693–705.