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NUMERICAL STUDY OF THE VISCOUS FLOW IN OSCILLATORY SPHERICAL ANNULI

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The method of fractional steps is applied for investigating the viscous flow field in the oscillatory spherical annuli. The numerical solution obtained enlarges the intervals of the frequency parameter M and amplitude of the torsional oscillations ε in comparison with known solutions. Typical flow fields are shown graphically.

0. Introduction

In recent years much attention has been given to the fluid dynamics of oscillatory flows (oscillations, pulsations, sonic and ultrasonic waves). This has been prompted by the observation that such flows are beneficial in a wide variety of processes e.g. extraction, absorption, drying and combustion of liquid fuels.

In 1932 Schlichting [1] found a periodic solution for the problem of an oscillating circular cylinder in an unbounded viscous fluid by using the method of two-dimensional nonsteady boundary layer equations. The same problem was also investigated by Andreas and Ingard [2]. In 1955 Lane [3] studied the problem of a translatorily oscillating sphere in an unbounded viscous fluid.

It has been established that when a solid boundary performs small-amplitude periodic vibrations in a viscous fluid, in addition to the anticipated fluctuating component of the flow, there is also a time independent streaming, which is induced due to the action of Reynolds stresses. This streaming is characterized by a so-called Reynolds number of steady part $R_s = \Omega_0 a^2/\nu$, where Ω_0 is a representative amplitude of the angular velocity of the body, a is a measure of the body and ν is a coefficient of kinematic viscosity. When $R_s \gg 1$ the induced streaming exhibits a boundary layer within which the Stokes layer is embedded. For a symetric cylinder Stuart [4] conjectured that the boundary layers which form for $R_s \gg 1$ ultimately collide and emerge as a jet-like flow along the axis of oscillations.

The torsional oscillations of cylinders and spheres have also been discussed both theoretically and experimentally. DiPrima and Liron [5] have pointed out that the torsional oscillations of a sphere in an unbounded viscous fluid induced a secondary flow in the planes containing the axis of rotation and calculated the effect of the flow on the torque acting on the sphere.

The beneficial effect of pulsating flow on mass transfer in liquid–liquid extraction apparatus has been studied by Krasuk and Smith [6]. A critical review on the mass transfer between solid spheres and oscillating flows has been given recently in [7].

The steady secondary streaming around a sphere has received little attention because of the difficulties associated with the three-dimensional analysis [8, 9]. An examination of the flow field induced by a viscous fluid drop immersed in another translatorily oscillating unbounded fluid was undertaken by Zapryanov and Stoyanova [10, 11]. Tabakova and Zapryanov [12] investigated the unsteady motion of a fluid between two concentric spheres when the inner sphere executes torsional oscillations while the outer one remains at rest. In that work the problem of high frequency oscillations was solved by the method of matched asymptotic expansions. Recently Duck and Smith [13] have studied the flow field induced by oscillating cylinders. Munson and Douglass [14] have presented theoretical and experimental results describing viscous incompressible flow in spherical annuli. However, their theoretical results are valid only for low frequency oscillations.

The purpose of the present paper is to develop a numerical method for intermediate frequency oscillations of the viscous liquid in spherical annuli. The full Navier–Stokes equations with appropriate boundary conditions are solved by the method of fractional steps, similar to the one used in [15], where numerical analysis for the fully developed oscillatory flow in a curved tube have been presented.

1. Equations of motion

Consider a viscous incompressible fluid between two concentric spheres of radii $r' = a$ and $r' = b$, $a > b$ (see Fig. 1). It is convenient to use spherical polar coordinates r', θ, ϕ with origins at the center of the spheres. Here r' is the radial coordinate, θ is the latitudinal angle, and ϕ is the longitudinal angle. Only rotationally symmetric motions, independent of ϕ are considered.

Each sphere may execute torsional oscillations with frequency ω and angular amplitude Ω_0^i ($i = 1$ corresponds to the inner sphere and $i = 2$ to the outer one), i.e. the angular velocity of each sphere is $\Omega_0^i \cos(\omega t + c_i)$, where c_i is the oscillation phase.

The velocity components v_r' and v_θ' are related to the stream function ψ' by

$$v_r' = \frac{1}{r'^2 \sin\theta} \frac{\partial \psi'}{\partial \theta}, \qquad v_\theta' = -\frac{1}{r' \sin\theta} \frac{\partial \psi'}{\partial r'}. \tag{1}$$

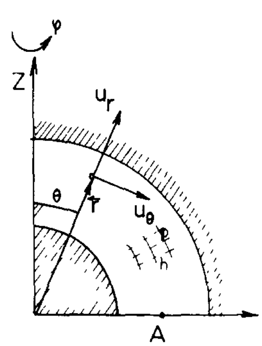

Fig. 1. The region between two spheres.

Variables can be rendered nondimensional as follows:

$$r' = ar\,, \qquad t' = \omega^{-1}t\,, \qquad \psi' = \varepsilon a^3\omega\psi_1\,, \qquad v_\phi = \varepsilon\omega a w_1\,. \tag{2}$$

Here $\varepsilon = \Omega_0/\omega$, where $\Omega_0 = (\Omega_0^{1^2} + \Omega_0^{2^2})^{1/2}$ is the representative angular velocity. Let also $a_i = \Omega_0^i/\Omega_0$. Then the governing equations of unsteady motion of an incompressible Newtonian fluid take the following form:

$$\frac{\partial D^2\psi_1}{\partial t} + \varepsilon\left[\frac{2\Omega_1}{r^2\sin\theta}\left(\frac{\partial\Omega_1}{\partial r}\cos\theta - \frac{\partial\Omega_1}{r\partial\theta}\sin\theta\right) - \frac{1}{r^2\sin\theta}\left(\frac{\partial\psi_1}{\partial r}\frac{\partial D^2\psi_1}{\partial\theta} - \frac{\partial\psi_1}{\partial\theta}\frac{\partial D^2\psi_1}{\partial r}\right)\right.$$
$$\left. + \frac{2D^2\psi_1}{r^2\sin\theta}\left(\frac{\partial\psi_1}{\partial r}\cos\theta - \frac{\partial\psi_1}{r\partial\theta}\sin\theta\right)\right] = \frac{1}{M^2}D^2\Omega_1\,, \tag{3}$$

$$\frac{\partial\Omega_1}{\partial t} - \frac{\varepsilon}{r^2\sin\theta}\left(\frac{\partial\psi_1}{\partial r}\frac{\partial\Omega_1}{\partial\theta} - \frac{\partial\psi_1}{\partial\theta}\frac{\partial\Omega}{\partial r}\right) = \frac{1}{M^2}D^2\Omega_1\,. \tag{4}$$

Here

$$D^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \quad \text{and} \quad \Omega_1 = r\sin\theta w_1\,.$$

The quantity $M^2 = a^2\omega/\nu$ is called a frequency parameter. The boundary conditions are

$$\psi = \partial\psi/\partial r = 0 \quad \text{at } r = 1 \text{ and } r = b/a = \lambda\,, \tag{5}$$

$$\Omega_1 = \alpha_1\sin^2\theta\cos(t + c_1) \quad \text{at } r = 1\,, \tag{6a}$$

$$\Omega_2 = \alpha_2\frac{b^2}{a^2}\sin^2\theta\cos(t + c_2) \quad \text{at } r = 2\,, \tag{6b}$$

and the flow symmetry requires

$$\Omega_1 = \psi_1 = D^2\psi_1 = 0 \quad \text{at } \theta = 0\,, \qquad \frac{\partial\Omega_1}{\partial\theta} = \psi_1 = D^2\psi_1 = 0 \quad \text{at } \theta = \tfrac{1}{2}\pi\,. \tag{7}$$

Thus it is only necessary to consider the first quarter of the region between the spheres (see Fig. 1).

Reference to (3) and (4) shows that in addition to the primary flow with velocity Ω_1, in torsional (azimuthal) direction there will also be a secondary flow with stream function ψ_1 in the planes containing the axis of the oscillations.

It is convenient to introduce the vorticity function

$$\zeta_1 = -D^2\psi_1 \tag{8}$$

reducing (3) to

$$\frac{\partial \zeta_1}{\partial t} - \frac{\varepsilon}{r^2 \sin\theta}\left[2\Omega_1\left(\frac{\partial \Omega_1}{\partial r}\cos\theta - \frac{\partial \Omega_1}{r\partial\theta}\sin\theta\right) + \left(\frac{\partial \psi_1}{\partial r}\frac{\partial \zeta_1}{\partial \theta} - \frac{\partial \psi_1}{\partial \theta}\frac{\partial \zeta_1}{\partial r}\right)\right.$$
$$\left. - 2\zeta_1\left(\frac{\partial \psi_1}{\partial r}\cos\theta - \frac{\partial \psi_1}{r\partial\theta}\sin\theta\right)\right] = \frac{1}{M^2} D^2 \zeta_1 . \tag{9}$$

In this way (4), (8) and (9) form a closed system of equations for the functions Ω_1, ζ_1 and ψ_1. It should be noted that there is no boundary condition for the vorticity ζ_1, but there are two boundary conditions for the stream function ψ_1 on both rigid walls.

The first attempt to numerically solve the equations (4), (8) and (9) was made by Pearson [16] for the steady-state case. In this work a computational scheme, which is an extension of the scheme that had been developed before in [15] is suggested especially for the unsteady case. The method of alternating-direction procedure is employed (for details see Yanenko [17]).

2. Computational procedure

The region of Fig. 1 is covered by a mesh with mesh spacing h and p in the r and θ directions respectively. τ denotes the increment in time. Let Φ_{ij}^n denote the value of any quantity at time stage $t^n = (n-1)\tau$ and at position $r_i = R_I + (i-1)h$, $\theta_j = (j-1)p$ where R_I is the inner radius. The computational procedure makes use of previously determined values of Ω and ζ at time t^n, and values of ψ at two time stages t^n and t^{n-1} together with numerical approximations to the equations, to determine the values of these quantities at time t^{n+1}. It is important to note that Pearson [16] used three time stages in computations, namely t^n, t^{n-1}, t^{n-2}.

It is useful to introduce the following new dependent variables by transformations:

$$\Omega_1 = \Omega \sin^2\theta , \qquad \zeta_1 = \zeta \sin^2\theta , \qquad \psi_1 = \psi \sin^2\theta . \tag{10}$$

Then (4), (8) and (9) transform to:

$$\frac{\partial \zeta}{\partial t} + \varepsilon\left[\frac{2\Omega}{r^3}\left(\frac{\partial \Omega}{\partial r} r\cos\theta - \frac{1}{\sin\theta}\frac{\partial \Omega \sin^2\theta}{\partial\theta}\right) - \frac{1}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial r}\frac{\partial \zeta\sin^2\theta}{\partial\theta} - \frac{\partial\zeta}{\partial r}\frac{\partial\psi\sin^2\theta}{\partial\theta}\right)\right.$$
$$\left. + \frac{2\zeta}{r^3}\left(\frac{\partial\psi}{\partial r} r\cos\theta - \frac{1}{\sin\theta}\frac{\partial\psi\sin^2\theta}{\partial\theta}\right)\right] = \frac{1}{M^2}\bar{D}^2\zeta , \tag{11}$$

$$\frac{\partial\Omega}{\partial t} - \varepsilon\frac{1}{r^2\sin\theta}\left(\frac{\partial\psi}{\partial r}\frac{\partial\Omega\sin^2\theta}{\partial\theta} - \frac{\partial\Omega}{\partial r}\frac{\partial\psi\sin^2\theta}{\partial\theta}\right) = \frac{1}{M^2}\bar{D}^2\Omega , \tag{12}$$

$$\bar{D}^2\psi = -\zeta \tag{13}$$

where

$$\bar{D}^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\right)\right] . \tag{14}$$

An important property of transformed equations is that it is no longer necessary to use the boundary conditions at $\theta = 0$. In addition the order of the operators at $\theta = 0$ is O(1), which plays a decisive role in constructing the approximation when a method of fractional steps is employed.

The numerical procedure is as follows:

Step 1. Extrapolate parabolically to obtain approximate value of ψ at all mesh points, at time $t^{n+1/2} = (n - 0.5)\tau$, i.e.

$$\psi_{ij}^{n+1/2} = \frac{3\psi_{ij}^{n} - \psi_{ij}^{n-1}}{2} + O(\tau^2)$$

Step 2. Using $\psi^{n+1/2}$ construct a second-order-approximation difference scheme for Ω:

$$\frac{\Omega_{ij}^{n+1/2} - \Omega_{ij}^{n}}{\frac{1}{2}\tau} = -\varepsilon \frac{\sin^2\theta_{j+1}\psi_{ij+1}^{n+1/2} - \psi_{ij-1}^{n+1/2}\sin^2\theta_{j-1}}{2pr^2\sin\theta_j} \frac{\Omega_{i+1j}^{n+1/2} - \Omega_{i-1j}^{n+1/2}}{2h} + \frac{\Omega_{i+1j}^{n+1/2} - 2\Omega_{ij}^{n+1/2} + \Omega_{i-1j}^{n+1/2}}{M^2h^2} + \cdots \text{(the terms in } \theta\text{-direction at } t^n \text{ stage)}. \tag{15}$$

$$\frac{\Omega_{ij}^{n+1} - \Omega_{ij}^{n+1/2}}{\frac{1}{2}\tau} = \varepsilon \frac{\psi_{i+1j}^{n+1/2} - \psi_{i-1j}^{n+1/2}}{2h} \frac{\Omega_{ij+1}^{n+1}\sin^2\theta_{j+1} - \Omega_{ij-1}^{n+1}\sin^2\theta_{j-1}}{2pr^2\sin\theta_j} + \frac{1}{p^2M^2}\left[\frac{\Omega_{ij+1}^{n+1}\sin^2\theta_{j+1}}{\sin\theta_j + \frac{1}{2}\sin\theta_j} - \left(\frac{\sin\theta_j}{\sin\theta_{j+1/2}} + \frac{\sin\theta_j}{\sin\theta_{j-1/2}}\right)\Omega_{ij}^{n+1} \frac{\sin^2\theta_{j-1}\Omega_{i-1}^{n+1}}{\sin\theta_{j-1/2}\sin\theta_j}\right] + \cdots$$

(the terms in r-direction at $t^{n+1/2}$ stage). (16)

It is easy to show that each of these systems of linear algebraic equations can be written in the form

$$a_j\Omega_{ij-1}^{n+1} - c_j\Omega_{ij}^{n+1} + b_j\Omega_{ij+1}^{n+1} = f_i, \quad j = 2, \ldots, N-1. \tag{17}$$

The condition $\partial\Omega/\partial\theta = 0$ at $\theta = \frac{1}{2}\pi$ is approximated by

$$\Omega_{iN}^{n+1} = \tfrac{1}{3}(4\Omega_{iN-1}^{n+1} - \Omega_{iN-2}^{n+1}).$$

The boundary condition at $\theta = 0$ is not required, because $a_2 = 0$ and (17) is a two-point difference equation at $j = 2$. One can find the similar form for (15)

$$a_i\Omega_{i-1j}^{n+1/2} - c_i\Omega_{ij}^{n+1/2} + b_i\Omega_{i+1j}^{n+1/2} = g_i, \quad i = 2, \ldots, L-1. \tag{18}$$

In addition, from the boundary conditions (6) one obtains

$$\Omega_{ij}^{n+1/2} = \alpha_1\cos(t^{n+1/2} + c_1) \quad \text{and} \quad \Omega_{ij}^{n+1/2} = \alpha_2\lambda^2\cos(t^{n+1/2} + c_2).$$

Then the algebraic systems (17) and (18) could be solved by the method of Gaussian elimination, as it is suggested, for example, by Samarsky and Nikolaev [18].

Step 3. Having Ω^n and Ω^{n+1}, approximate the non-linear term in the vorticity equation (12) as follows:

$$F = \frac{\varepsilon}{r_i^3}\left[\Omega_{ij}^n\left(\frac{\Omega_{i+1j}^n - \Omega_{i-1j}^n}{2h} r_i \cos\theta_j - \frac{\sin^2\theta_{j+1}\Omega_{ij+1}^n - \sin^2\theta_{j-1}\Omega_{ij-1}^n}{2p\sin\theta_j}\right)\right.$$

$$\left. + \Omega_{ij}^{n+1}\left(\frac{\Omega_{i+1j}^{n+1} - \Omega_{i-1j}^{n+1}}{2h} r_i \cos\theta_j - \frac{\Omega_{ij+1}^{n+1}\sin^2\theta_{j+1} - \Omega_{ij-1}^{n+1}\sin^2\theta_{j-1}}{2p\sin\theta_j}\right)\right]. \tag{19}$$

Therefore (12) becomes

$$\frac{\zeta_{ij}^{n+1/2} - \zeta_{ij}^n}{\frac{1}{2}\tau} = -\varepsilon\frac{\psi_{ij+1}^{n+1/2}\sin^2\theta_{j+1} - \psi_{ij-1}^{n+1/2}\sin^2\theta_{j-1}}{2pr_i\sin\theta_j}\left(\frac{\zeta_{i+1j}^{n+1/2} - \zeta_{i-1j}^{n+1/2}}{2h} + \frac{2}{r_i}\zeta_{ij}^{n+1/2}\right)$$

$$+ \frac{\zeta_{i+1j}^{n+1/2} - 2\zeta_{ij}^{n+1/2} + \zeta_{i-1j}^{n+1/2}}{M^2h^2} + F + \cdots$$

$$\text{(the terms in } \theta\text{-direction at } t^n \text{ stage)}. \tag{20}$$

In order to construct boundary conditions for vorticity one can use (13) with the condition $\partial\psi/\partial r = 0$ and finally obtain

$$\zeta_{1j}^{n+1/2} = -\frac{2}{h^2}\psi_{2j}^{n+1/2}, \qquad \zeta_{Mj}^{n+1/2} = -\frac{2}{h^2}\psi_{M-1j}^{n+1/2}.$$

For the second half-time step one has

$$\frac{\zeta_{ij}^{n+1} - \zeta_{ij}^{n+1/2}}{\frac{1}{2}\tau} = \varepsilon\frac{\psi_{i+1j}^{n+1/2} - \psi_{i-1j}^{n+1/2}}{2hr_i^2\sin\theta_j}\left(\frac{\zeta_{ij+1}^{n+1}\sin^2\theta_{j+1} - \zeta_{ij-1}^{n+1}\sin^2\theta_{j-1}}{2p} - 2\sin\theta_j\cos\theta_j\zeta_{ij}^{n+1}\right.$$

$$\left. + \frac{1}{p^2M^2}\left[\frac{\zeta_{ij+1}\sin^2\theta_{j+1}}{\sin\theta_{j+1/2}\sin\theta_j} + \left(\frac{\sin\theta_j}{\sin\theta_{j+1/2}} + \frac{\sin\theta_j}{\sin\theta_{j-1/2}}\right)\zeta_{ij}^{n+1} + \frac{\zeta_{ij}^{n+1}\sin^2\theta_{j-1}}{\sin\theta_{j-1/2}\sin\theta_j}\right]\right) + F + \cdots$$

$$\text{(the terms in } r\text{-direction at } t^{n+1/2} \text{ stage)}. \tag{21}$$

Step 4. Using the calculated values of ζ_{ij}^{n+1} on all interior mesh points to solve (13) for ψ_{ij}^{n+1}. In order to employ again the method of fractional steps, a fictitious time is introduced:

$$\frac{\psi_{ij}^{k+1/2} - \psi_{ij}^k}{\frac{1}{2}\tau_1} = \frac{\psi_{ij+1}^{k+1/2}\sin^2\theta_{j+1}}{p^2\sin\theta_{j+1/2}\sin\theta_j} - \frac{1}{p^2}\left(\frac{\sin\theta_j}{\sin\theta_{j+1/2}} + \frac{\sin\theta_j}{\sin\theta_{j-1/2}}\right)\psi_{ij}^{k+1/2}$$

$$+ \frac{\psi_{ij-1}^{k+1/2}\sin^2\theta_{j-1}}{p^2\sin\theta_{j-1/2}\sin\theta_j} + \zeta_{ij}^{n+1} + \cdots$$

$$\text{(the terms in } r\text{-direction at } t_1^k \text{ stage)}. \tag{22}$$

$$\frac{\psi_{ij}^{k+1}-\psi_{ij}^{k+1/2}}{\frac{1}{2}\tau_1}=\frac{\psi_{i+1j}^{k+1}-2\psi_{ij}^{k+1}+\psi_{i-1j}^{k+1}}{h^2}+\zeta_{ij}^{n+1}+\cdots$$

(the terms in θ-direction at $t_1^{rk+1/2}$ stage). (23)

The difference boundary conditions for the above equations are

$$\psi_{2j}^{k+1}=\tfrac{1}{4}\psi_{3j}^{k+1}, \qquad \psi_{L-1j}^{k+1}=\tfrac{1}{4}\psi_{L-2j}^{k+1}, \tag{24}$$

$$\psi_{1j}^{k+1}=\psi_{Lj}^{k+1}=0. \tag{25}$$

In this way the two boundary conditions for ψ are satisfied but it is not necessary to solve (13) on the mesh lines $i=2$ and $i=L-1$.

The iteration is terminated when the following criterion is satisfied:

$$\max_{i,j}|(\psi_{ij}^{k+1}-\psi_{ij}^{k})/\tau_1\psi_{ij}^{k+1}|\leqq\Delta=0.0001. \tag{26}$$

If K is the first value of k that makes (26) true, one takes $\psi^{n+1}=\psi^K$.

Steps 1–4 represent the full step in physical time iteration, whose order of approximation is $O(h^2+p^2+\tau^2)$.

3. Results and discussion

The proposed finite-difference scheme can be used for solving both the steady and unsteady problems. To check the accuracy of the scheme an initial result has been obtained for steady motion. In this case the boundary conditions (6) do not include the function $\cos(\omega t+c_i)$ and the physical time becomes an iterative parameter. The iteration has been conducted until convergence:

$$\max\left\{\frac{\sum_{i,j}|\zeta_{ij}^{n+1}-\zeta_{ij}^{n}|}{\tau\sum_{i,j}|\zeta_{ij}^{n+1}|},\frac{\sum_{i,j}|\Omega_{ij}^{n+1}-\Omega_{ij}^{n}|}{\tau\sum_{i,j}|\Omega_{ij}^{n+1}|}\right\}\leqq 0.001. \tag{27}$$

Fig. 2 shows a comparison between the present result and Pearson's result for the stream lines when Re = 100. It is seen that the agreement is satisfactory.

The most representative characterization of oscillatory motion is, perhaps, the steady part of the flow in the cross-section plane. One can obtain this secondary steady streaming by averaging the solution with respect to time. The calculation of the solution can be done in the same way as for the steady-state case and only the condition of convergence is different for the oscillatory case. For the latter, a convergence of two consecutive periods in time is required, which ensures that the periodic solution is obtained. Following [15], this criterion is realized in the form:

$$\frac{\sum_{n=1}^{s}|\psi_{M-2,2}^{ks+n}-\psi_{M-2,2}^{(k-1)s+n}|}{\sum_{n=1}^{s}|\psi_{M-2,2}^{ks+n}|}\leqq 0.005$$

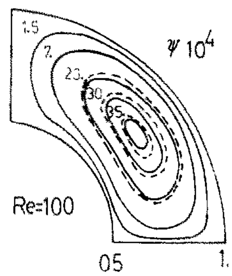

Fig. 2. The steady-state-case secondary flow at Re = 100. Inner sphere held fixed, outer rotates. ——, Pearson's [16] result; ----- present result (only where it differs from [16]).

where s is the number of the time steps in one period, and k is the number of the last completely calculated period. After checking under the above criterion (28) it is necessary to calculate one more period of time and estimate the time averaged values $\Omega^{(s)}$, $\zeta^{(s)}$, $\psi^{(s)}$.

It was mentioned above that a regular asymptotic solution to the problem of viscous flow in oscillatory spherical annuli when the frequency parameter is small has been given in [14]. Since the numerical solution is to extend the theoretical possibilities for intermediate values of the parameters, it was important to match it with asymptotic solutions in the region of parameters in which both methods of solution are valid. Such a region for [14] is where M approaches unity and ε is not large. The present numerical solution has proved to differ slightly at $M = 1$ and $\varepsilon \leqq 10$ if divided by ε. For comparison the modulus of velocity $|v(\theta = \frac{1}{2}\pi, r = \frac{1}{2}(a+b))|$ in a special point A (see Fig. 1) has been calculated when the outer sphere executes torsional oscillations. When $a = 0.5$, $b = 1$ and with mesh size $L = 21$, $N = 16$, this value is about 1.1×10^{-4}, while in [14] it is 1.3×10^{-4}.

Tabakova and Zapryanov [12] as well as Munson and Douglass [14] have investigated the case of high frequency ($M \gg 1$) oscillations theoretically and experimentally, respectively. Unfortunately the values of R_s are not given in [14] and comparison is not possible. Turning to the comparison with [12] it is important to note that the frequency parameter M of that work was based on the radius of the outer sphere, rather than on the inner sphere as in the present work. This is the reason why henceforth M will appear with two values, if referred to. The first value corresponds to the present work, while the value in brackets is the corresponding value from [12]. Once again the values of parameters for which both the numerical and the asymptotic solutions are accurate are needed. Such a value appears to be $M = 10$ (25). For comparison the maximum of the stream function is chosen. The present results have shown that $|\psi^{(s)}_{max}/\varepsilon| = 0.0167$ at $\varepsilon = 1$ while in [12] it is equal to 0.0142. This difference can be attributed to the rough mesh ($L = 31$, $N = 16$) and to the artificial scheme viscosity being of the same order of magnitude as the physical viscosity. Fig. 4 shows that result.

In Fig. 5 the steady streaming for one truly intermediate value of the frequency parameter $M = 4.4$ (10) can be seen. It is evident that for one and the same value $R_s = 100$ the order of $\psi^{(s)}$ is the same even for quite different values of the frequency parameter M, namely Fig. 3b and Fig. 5. This shows that for intermediate M the secondary steady streaming also depends chiefly on R_s. This fact is well known from [14] and [12] when $M \gg 1$ and $M \ll 1$, respectively.

Another interesting case, which could not be treated asymptotically is when R_s is not small. Conducting further the analogy with oscillatory motion in curved pipes, where in fact the value $R_s/762$ was important, one can expect, in the present case, that small values of R_s are those up

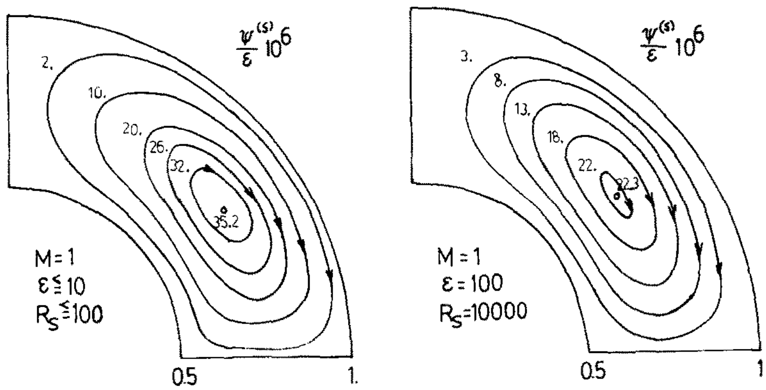

Fig. 3. Steady streaming at low frequency. Inner sphere held fixed, outer oscillates.

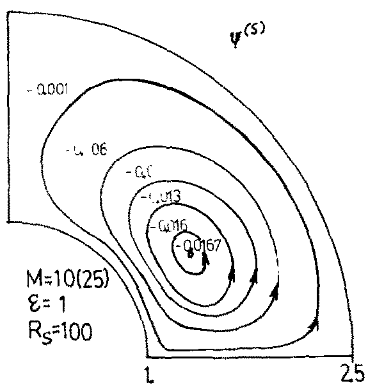

Fig. 4. Steady streaming at high frequency. Outer sphere held fixed, inner oscillates.

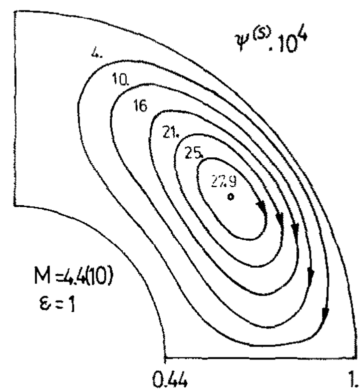

Fig. 5. Steady streaming at intermediate frequency. Inner sphere held fixed, outer oscillates.

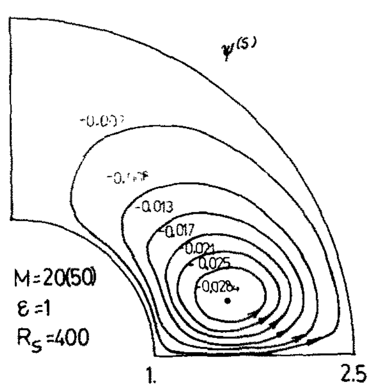

Fig. 6. Steady streaming at high Reynolds number of the steady part $R_s = 400$. Outer sphere held fixed, inner oscillates.

to 100. In Fig. 6 the steady part of the secondary motion when $R_s = 400$ employes a sufficiently high frequency parameter $M = 20$ (50) is shown. It is remarkable that the steady streaming exhibits a jet character near the equatorial plane.

4. Conclusions

In the present work a difference scheme of second order of approximation is developed for solving the problem of viscous incompressible flow in oscillatory spherical annuli. The accuracy of the proposed scheme is verified by the standard procedure of changing the mesh and time steps as well as by a comparison with a known solution for the steady-state viscous motion in spherical annuli.

For the case of oscillatory motion a criterion for 'periodic' convergence is established and results for various values of the governing parameters are obtained. A comparison with the known asymptotic solutions for the two extreme cases of the frequency parameter, namely $M \ll 1$ and $M \gg 1$ is performed. The agreement for low M proves satisfactory, while the other extreme shows differences up to 20%. This is attributed to the Stokes boundary layer of thickness $O(M^{-1})$ which develops with an increase of M and yields a necessity of more fine mesh than that used in the work.

The main attention is focused on the steady part of the secondary motion which takes place because of the curvature of the flow. The calculations show that the steady motion is governed exclusively from the 'Reynolds number of the stationary part' R_s which is observed in both asymptotical limits too. Thus the suggested finite-difference scheme permits one to bridge the gap between singular and regular asymptotic solutions to the problem, thereby securing a theoretical solution for the region of intermediate values of the parameters. However, it could be applied, if necessary, even to the high frequency case, but only with a more dense grid mesh.

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