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On the Mechanics of the Isolated Semicircular Canal under Arbitrary Accelerations

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1. Formulation of the Model

In the recent couple of decades the behaviour of the isolated semicircular canal has attracted broad attention. Mechanical modelling has been usually perfomed under the assumptions that (see Fig. 1):

(i) The semicircular canal is a rigid toroidal shell filled with a certain viscous Newtonian liquid, so-called "endolymph";

(ii) The canal ends in an ampulla (Fig. 1) which is hermetically barred by an elastic diaphragm called cupula;

(iii) The density of the diaphragm is equal to that of the liquid;

(iv) The viscosity is uniform and depends only on the temperature.

The described viscoelastic system responds only to angular accelerations occurring when it is forced to execute unsteady rotations. Then the liquid trails the canal walls and thus deflects the elastic diaphragm which in its turn deforms the receptive cells sensitive to deflexion. The latter results in a bioelectrical signal (impulse).

On the basis of the above scheme, Steinhausen [1] proposed a parametric equation for modelling the behaviour of the semicircular mechanics as follows:

$$I\frac{d^2\xi}{dt^2}+B\frac{d\xi}{dt}+K\xi=-I\alpha(t), \tag{1.1}$$

where ξ is the averaged relative angular displacement of the fluid, B the momentum of the viscous forces per unit mean angular fluid velocity, I the inertia momentum of the fluid and K is the momentum of the restoring elastic forces per unit angular displacement of the fluid. Respectively $\alpha(t)$ is the normal to the duct cross-section component of the applied angular acceleration if referred to certain resting coordinate system. It is important to note that this equation has a physical meaning for the semicircular canal only when $B \gg K$.

Further, the relationship between the applied acceleration and the pressure resulting on the diaphragm was sought in two different ways. It was Baum i n g e r [2] who obtained it, assuming that the velocity profile in the duct was parabolic. In other words, the pressure lost in the duct was represented by Poiseulle formula, but with a flux which was function of time. This simplification was valid only for slow acceleration. The approach of Bauminger could be called "hydrodynamic" since the flow in the duct played a decisive role. It was forwarded further in the paper by V a n B u s k i r k et al. [3], where a truly unsteady duct flow was employed. This significantly improved the hydrodynamic basis of the theory of semicircular canal in comparison with the quasi-steady assumptions. In that work only one type of loading was investigated, namely the case of a step input in angular velocity.

The other approach to the problem was described by O m a n and Y o u n g [4] who assumed that the motion of the fluid in the canal could be neglected. As a result, the total force acting on the diaphragm appeared to be represented only by the liquid inertia. Naturally, the pressure loss due to viscous drag in the duct was absent and this approach seemed to be applicable whithin very short time after loading when the viscous force could not develop.

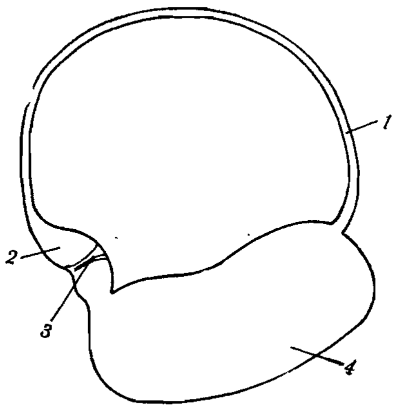

Fig. 1. A schematic diagram of a single semicircular canal
1 — duct, 2 — ampulla, 3 — cupula, 4 — utricle

The present paper can be considered to descend from the first (called above "hydrodynamic") approach. Here a derivation of a closed equation for the pressure is aimed when the acceleration is an arbitrary function of time

2. The Basic Equations

The major feature of the investigated system is that the radius of the cross-section of the duct is much smaller in comparison with the radius of the curvature (the radius of the torus, Fig. 2). It allows one to assume in first aproximation that the fluid flow in the torus is identical with the flow in a straight pipe. This, immediately, yields that the velocity components in the cross-section of the pipe are equal to zero and only the component in axial direction retains a nonzero value. Then the equations of Navier-Stokes reduce to (see S l e z k i n [5]):

$$\frac{\partial v_z}{\partial t} = -\frac{1}{\varrho}\frac{\partial p}{\partial z} + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right), \tag{2.1}$$

where the pressure gradient is a function only of time

$$\frac{\partial p}{\partial z} = \psi(t). \tag{2.2}$$

Before discussing the boundary conditions it is convenient to render (2.1) in terms of coordinate system connected with the duct. Then

$$v_z = u + R\omega(t), \tag{2.3}$$

where $\omega(t)$ is the angular velocity of the duct of that part of its motion which is in the direction of the torus axis. Obviously, the other components of the rotation, as well as the body forces do not matter, because of the closed character of the flow, i. e. the absence of free boundaries. Introducing now (2.2) and (2.3) into (2.1), one obtains

$$\frac{\partial u}{\partial t}+Ra(t)=-\frac{1}{\varrho}\psi(t)+\nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \tag{2.4}$$

where $a(t)=\dot{\omega}(t)$ is called angular acceleration.

Since the function $\psi(t)$ in (2.4) is yet unknown, to couple the model one needs a connection between the pressure and the velocity. This connection should be sought in the law of diaphragm motion. It is broadly accepted now that the most efficient way of doing that is to introduce the elastic properties of the diaphragm as follows:

$$\Delta p_c=k\Delta V_c, \tag{2.5}$$

where Δp_c is the pressure difference from both sides of the cupula, ΔV_c is the volume enclosed between the undeformed and the actual shapes of the diaphragm and the empirical coefficient k is called paesure-volume modulus. Since the diaphragm plugs the ampulla without any leaking, the displacement volume ΔV_c is nothing else but an integral with respect to time from the total fluid flux $Q(t)$, i. e.

$$\Delta V_c=\int_0^t Q(t)dt. \tag{2.6}$$

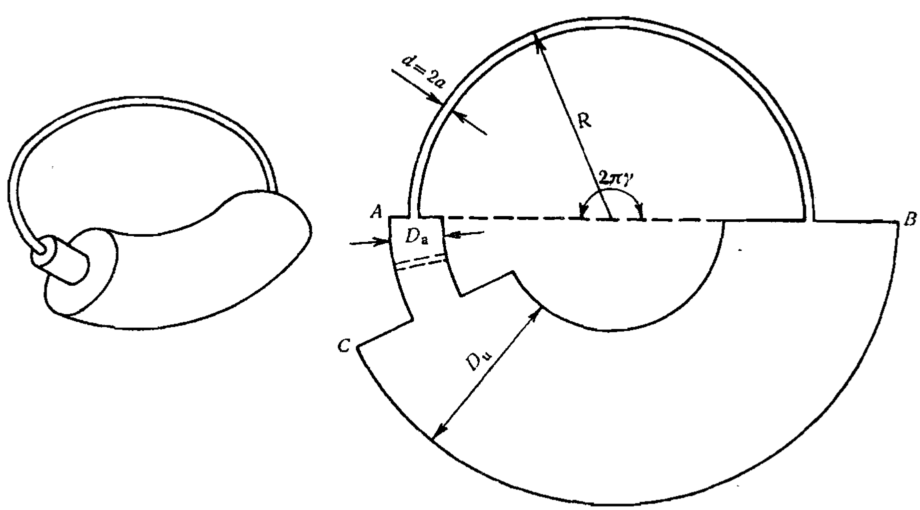

Fig. 2. An idealized model of the semicircular canal
$\widehat{AB}$ — a torodoidal segment representing the duct with circular cross-section (for human $d\approx 0.03$ cm, $R\approx 0.32$ cm); $\widehat{BC}$ — utricle ; $\widehat{AC}$ — ampulla ; $2\pi\gamma$ — central angle subtended by the narrow duct

At the time, the pressure drop in the duct is (see Van Buskirk et al. [3])

$$\Delta p_d=\int_0^s \frac{\partial p}{\partial z}dz=\varrho\psi(t)s, \tag{2.7}$$

where s is the length of the duct. The latter can be expressed as $2\pi\gamma R$, where γ is the share of the circle occupied by the duct (see Fig. 2). On the other hand, the utricle is wide enough to neglect the viscous drag and to assume that the unique reason for the pressure gradient there is the inertia of the liquid which yields

$$\Delta p_u = -2\pi(1-\gamma)\varrho\alpha(t)R^2. \tag{2.8}$$

In the last formula the length of the ampulla is automatically neglected.

Finally, the full pressure drop is

$$\Delta p_c = \Delta p_d + \Delta p_u = 2\pi\gamma\psi(t)R - (1-\gamma)2\pi\varrho\alpha(t)R^2. \tag{2.9}$$

Being reminded of the expression for the flux $Q(t)$ one can combine (2.9), (2.6) and (2.5) and obtain

$$2\pi\varrho k \int_0^t \int_0^a r\, u(r,\tau)\,dr\,d\tau = 2\pi\gamma\psi(t)R - (1-\gamma)2\pi\varrho\alpha(t)R^2. \tag{2.10}$$

It is evident now that (2.10) and (2.4) comprise a closed set for estimation of the two unknown functions $u(r, t)$ und $\psi(t)$.

As it has been mentioned above, this system was first introduced by Van Buskirk et al. [3]. It is hardly convenient to use the system of governing equations in its present form (2.4) and (2.10), because of the explicit dependence on the spatial coordinate r. Employing a new type of loading requires the solution of the full system again. It is much more desirable to have the hydrodynamic part analytically resolved and the number of the independent variables reduced to one (the time t). This is the subject of the next section of the present work.

The boundary conditions for $u(r, t)$ are those of nonsingularity in the centre of the polar coordinate system

$$r\frac{\partial u}{\partial r} < \infty, \quad r \to 0 \tag{2.11}$$

and nonslipping on the duct walls

$$u = 0, \quad r = a, \tag{2.12}$$

because $u(r, t)$ is the relative velocity.

3. Reduction of the Governing System

The equations (2.4) and (2.10) comprise a linear system which allows one to integrate (2.4) for arbitrary values of the pressure gradient $\psi(t)$ and then to substitute it in (2.10). The matter of integration is not novel itself. Following Slezkin [5], one breaks the problem into two steps. The first one is obtaining the solution of the problem of suddenly applied constant pressure gradient from the following boundary value problem:

$$\begin{aligned} &\frac{\partial u_0}{\partial t} = A + \nu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_0}{\partial r}\right), \\ &u_0(r, 0) = 0, \\ &u_0(a, t) = 0, \quad \lim_{r\to 0}\left(r\frac{\partial u_0}{\partial r}\right) < \infty, \end{aligned} \tag{3.1}$$

where A possesses a dimension of pressure gradient density. The solution of (3.1) was given by Gromeka in 1882 (see, for reference Slezkin, [5]) in the form

$$(3.2) \qquad u_0(r,t)=\frac{Aa^2}{4\nu}\left[1-\frac{r^2}{a^2}-8\sum_{k=1}^{\infty}\frac{J_0(\lambda_k r/a)}{\lambda_k^3 J_1(\lambda_k)}\exp\left(-\lambda_k^2\frac{\nu}{a^2}t\right)\right],$$

where λ_k are the roots of the following transcendental equation

$$J_0(\lambda_k)=0$$

and J_0 and J_1 are the first two Bessel functions. The flux of this special flow is

$$(3.3) \qquad Q_0(t)=\frac{\pi Aa^2}{8\nu}\left[1-32\sum_{k=1}^{\infty}\frac{1}{\lambda_k^4}\exp\left(-\frac{\lambda_k^2\nu}{a^2}t\right)\right].$$

Having the solution (3.2), the original problem (with arbitrary pressure) can be solved by means of Duhamell integral (see Slezkin [5])

$$(3.4) \qquad u(r,t)=\frac{1}{A}\left[g(0)u_0(r,t)+\int_0^t u_0(r,t-\tau)g(\tau)d\tau\right],$$

where the function $g(t)$ is suggested from (2.4)

$$g(t)=-\frac{1}{\varrho}\psi(t)-Ra(t).$$

After integration, (3.4) reduces to a formula for the flow flux

$$(3.5) \qquad Q(t)=\frac{\pi a^4}{8\nu}\left[g(0)F_0(t)+\int_0^t F_0(t-\tau)\dot{g}(\tau)d\tau\right]=-\frac{\pi a^4}{8\nu}\int_0^t \dot{F}_0(t-\tau)g(\tau)d\tau,$$

where

$$(3.6) \qquad F_0(t)=1-32\sum_{k=1}^{\infty}\frac{1}{\lambda_k^4}\exp\left(-\frac{\lambda_k^2}{a^2}\nu t\right)$$

and is acknowledged that $F_0(0)=0$.

In order to evaluate the integral $\int_0^t Q(t_1)dt_1$, which enters the left-hand side of (2.10), it it necessary to transform the related double integral

$$\int_0^t Q(t)dt=\frac{\pi a^4}{8\nu}\int_0^t\int_0^{t_1}\dot{F}_0(t_1-\tau)g(\tau)d\tau dt_1.$$

The region of integration is shown on Fig. 3. It is obvious that if the sequence of integration is to be interchanged, then region of integration is expressed by

$$0\leq\tau\leq t,\ \tau\leq t_1\leq t,$$

which yields

$$\int_0^t Q(t_1)dt_1 = \frac{\pi a^4}{8\nu}\int_0^t d\tau g(\tau) \int_\tau^t \dot{F}(t_1-\tau)dt_1$$

$$= \frac{\pi a^4}{8\nu}\int_0^t d\tau g(\tau)[F_0(t-\tau)-F_0(0)] = \frac{\pi a^4}{8\nu}\int_0^t F_0(t-\tau)g(\tau)d\tau.$$

Substituting the last equality in (2.10), it can be obtained that

(3.7) $$k\frac{\pi a^4}{8\nu\varrho}\int_0^t F_0(t-\tau)\left[-\frac{1}{\varrho}\psi(\tau)-Ra(\tau)\right]d\tau = \frac{1}{\varrho}R\gamma\psi(t)-(1-\gamma)a(t)R^2,$$

where the expression for $g(t)$ is also acknowledged. Introducing nondimensional variables according to the scheme

$$t=\frac{a^2}{\nu}t',\ \tau=\frac{a^2}{\nu}\tau',\ a=\frac{\nu}{a^4}a',\ \psi=\frac{R\varrho\nu^2}{a^4}\psi',\ f'(t')=\psi'(t')-\frac{1-\gamma}{\gamma}a'(t'),$$

one easily obtains

(3.8) $$f'(t)=-\varepsilon\int_0^{t'} F_0(t'-\tau')\left[f'(\tau')+\frac{1}{\gamma}a'(\tau')\right]d\tau',$$

where

(3.9) $$\varepsilon=\frac{ka^6}{8\varrho\nu^2R}.$$

The equation (3.8) is a closed one for the only unknown function $f(t)$ which is related to the pressure. Here it should be mentioned that the right-hand side of this equation fully represents the hydrodynamics in first approximation with respect to the geometrical small parameter a/R. If one is bound

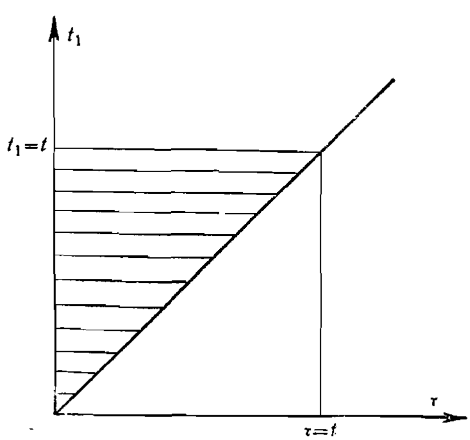

Fig. 3. The region of integration

to vary the type of the diaphragm rheology (even to nonlinear or nonelastic ones), he is free to do that whithout reconsidering the hydrodynamics any more. This is the main advantage of (3.8). In addition, unlike the system (2.4) and (2.10) which contains one more independent variable r, the new equation can be easily resolved numerically due to its simplest structure.

4. Some Asymptotic Considerations

It is interesting to outline some major traits of the general equation (3.8). Further, the primes denoting the nondimensional variables can be omitted without fear of confusion. The first limiting case which is to be considered is that of large ("long") times. The long times are those for which the kernel of the equation approaches unity, i. e. $F_0(t) \rightarrow 1$. The intrinsic characteristic time of function F_0 is defined as usual:

$$t^* = \left(\frac{dF_0}{dt} \bigg|_{t=0} \right)^{-1}.$$

After substituting here (3.6) and admitting the well-known expression for the sum of reciprocal squares of the quantities λ_k (see for details Slezkin [5]), is obtained that $t^*=1/8$. It is evident now that the time can be called "long" if $t \gg 1/8$. When this condition is satisfied, then (3.8) reduces to

$$f(t) = -\varepsilon \int_0^t \left[f(\tau) + \frac{1}{\gamma} \alpha(\tau) \right] d\tau$$

and after differentiation with respect to time one has

$$\frac{df}{dt} = -\varepsilon f(t) - \frac{1}{\gamma} \varepsilon \alpha(t). \tag{4.1}$$

The general solution of this equation is

$$f(t) = \exp(-\varepsilon t) \left[C - \frac{1}{\gamma} \varepsilon \int_0^t \exp(\varepsilon t_1) \alpha(t_1) dt_1 \right], \tag{4.2}$$

where the constant C is to be estimated after asymptotic matching with the solution for small times.

The relaxation properties of the semicircular canal can be derived from (4.2), assuming that the load is applied only for a finite time, i. e.

$$\alpha(t) = \begin{cases} \alpha(t) & \text{for } t \leq t_0, \\ 0 & \text{for } t > t_0. \end{cases}$$

In this case the entire complex in the brackets of the right-hand side of (4.2) turns to be a constant and the relaxation is governed by the exponential term outside the brackets. Obviously, the nondimensional relaxation time is $T'_1 = 1/\varepsilon$. Respectively,

$$T_1 = \frac{8\mu R}{k a^4} \tag{4.3}$$

is the dimensional time of relaxation. This result is fully compatible with the conclusions derived from the ordinary differential equation (1.1) when the second derivative and the nonhomogeneous term are neglected under the above assumptions on the type of loading.

The other limiting case, namely the case of small times, is more complicated. Here it is important to know the exact dependence of the angular acceleration α on the time t. Only when α possesses no characteristic time (for example, it is a delta-function) or when the characteristic time of α is much larger than the "short" time $t=1/8$ of the system, only then the characteristic time of the motion coincides with that of the function F_0 and the dimensional short time is just

(4.4) $$T_2 = a^2/8\nu.$$

It is interesting to asses the magnitude of the nondimensional parameter ε (see (3.9)). There exists a well established opinion that the radius of the duct $a=0.015$ cm, the radius of the curvature $R=0.32$ cm, the viscosity coefficient $\mu=0.00852$ g/(cm. s) and the density of endolymph $\varrho=1.0\,\mathrm{g/cm^3}$. Then $T_1=a^2/\nu\varepsilon \approx 0.0264/\varepsilon$. Unfortunately, the pressure-volume modulus k is an unknown value and it is necessary to seek for other ways to evaluate ε. One possible way is to employ the experimental data concerning the large-time constant T_1. In accordance with the experiment of Butchvarov [6], this quantity can be taken approximately $T_1=16$ s. Thus one has

(4.5) $$\varepsilon \approx 0.00165,$$

which is a small quantity. The last property allows one to seek for an asymptotic solution of eq. (3.8). The outer solution (see, for terminology Van Dyke [7]) is just (4.2). The inner solution, i. e. the solution for short times can be sought in the form of the following asymptotic expansion:

(4.6) $$f=f_0+\varepsilon f_1+\varepsilon^2 f_2+\cdots$$

Introducing (4.6) into (3.8) and equating the terms with equal powers of the small parameter ε, one obtains

(4.7) $$f_0=0,\qquad f_1=-\int_0^t F_0(t-\tau)\frac{1}{\gamma}\,\alpha(\tau)\,d\tau$$

. . . .

Neglecting the terms of order $O(\varepsilon^2)$ and higher, one finds the constant $C=-\varepsilon/\gamma$ through matching with the inner solution (4.7). After applying the standard procedure of summing the inner end outer solutions and substracting their commomn part, an uniformly valid asymptotic solution is obtained:

(4.8) $$f(t)=-\frac{\varepsilon}{\gamma}\int_0^t\left[\sum_{k=1}^{\infty}\frac{\exp(-\lambda_k^2(t-\tau))}{\lambda_k^4}-\exp(-\varepsilon(t-\tau))\right]\alpha(\tau)d\tau.$$

This formula gives a practical way to calculate the response of the semicircular canal to arbitrary angular accelerations.

5. Conclusions

In the present work a new equation for estimating the pressure acting on the diaphragm of a semicircular canal has been derived. The hydrodynamics equations has been integrated without any restrictions on the type of loading (applied angular acceleration) and the obtained pressure equation is valid for arbitrary accelerations, which appears to be novel in the literature. Another important feature of the new equation is the existence of only one independent variable, namely time t. It is a significant simplification in comparison with the known solutions to the problem, where there exists one more variable — the polar coordinate r in the cross-section of the duct.

The asymptotic considerations of the proposed equation has confirmed the conclusion of Van Buskirk et al. [3] that for practically important loads the ordinary-differential-equation approach is accurate enough. In addition, an asymptotic uniformly valid solution of first order of approximation with respect to the small parameter of the problem has been found for arbitrary accelerations.

The equation obtained can be directly employed in predicting the pressure behaviour of the semicircular canal if the loading function is known.

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О механике полуокружного канала при произвольных ускорениях

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(Резюме)

Путем интегрирования гидродинамической задачи выводится одномерное уравнение для определения давления на упругую диафрагму (купулу) полуокружного канала с учетом произвольного нестационарного движения эндолимфы. В качестве независимой переменной остается только время. Последнее дает возможность вводить разные реологические зависимости для поведения купулы, не возвращаясь заново к решению гидродинамической задачи. Исследовано асимптотическое поведение решения полученного уравнения при „малых“ и „больших“ временах. Построено равномерно пригодное асимптотическое решение.