SIAM J. APPL. MATH.
Vol. 42, No. 6, December 1982

0036-1399/82/4206-0013 $01.00/0

A COMPLETE ORTHONORMAL SYSTEM OF FUNCTIONS IN $L^2(-\infty,\infty)$ SPACE*

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Abstract. A new complete orthonormal system of functions in the $L^2(-\infty,\infty)$ space is introduced. The system consists of two sequences composed of odd and even functions respectively. Unlike Hermite and Laguerre sets of functions which behave exponentially at infinity, the new system exhibits asymptotic behavior x^{-1} for the odd sequence and x^{-2} for the even one.

Formulae representing products, derivatives, etc. in series in the system are developed. A nonlinear differential equation with a requirement for summability of the square of its solutions, instead of boundary conditions, is solved. The example displays most of the features of the new method important for essentially nonlinear problems.

Introduction. In recent years a number of physical problems have frequently led to boundary value problems on infinite domains. These are the cases when there are no boundary conditions at certain points, but rather the solution is required to possess a summable square in that infinite domain. When the domain of the independent variable is the set of real numbers $\mathbb{R}\{x \in (-\infty,\infty)\}$, then the set of functions with summable squares in $\mathbb{R}$ is called $L^2(-\infty,\infty)$. A typical example of such an unusual boundary value problem lies in the theory of stochastical processes, namely the equations for the correlation and spectral functions. However, it is not a unique example, and numerous other mechanical and physical situations are concerned with infinite domains (or time intervals) in conjunction with a requirement for finite total energy of the system.

The numerical approaches, such as finite differences or elements, to problems in $L^2(-\infty,\infty)$ are subject to significant difficulties. It is enough to mention that the inevitable reducing of the infinite interval to a finite one introduces an artificial eigenvalue problem which is nonspecific for the original (infinite) domain. It can even happen that each of the finite-domain approximations has only a trivial solution, when the original problem possesses a nonzero one. Sometimes, the finite-domain problem has a solution only at some denumerable set of intervals with a special length corresponding to its eigenvalues. In addition, the related sequence of approximate solutions often converges slowly to the sought solution.

A method that is free of the above shortcomings is that of representing the solution in a series of some complete orthonormal (CON) sequence of functions, each of which belongs to the $L^2(-\infty,\infty)$ space. The most thoroughly developed are those based on Hermite and Laguerre polynomials. The former comprise the so-called set of Hermitian functions which is a CON sequence in $L^2(-\infty,\infty)$. The latter comprise the set of functions of the parabolic cylinder which is CON in $L^2(0,\infty)$ [1]. Both of them, however, are appropriate only for linear problems, because there exist no formulae to represent the product of two members of a sequence as a linear combination of functions from that sequence. In addition, the Hermite set behaves as $x^m \exp(-x^2/2)$ at infinity, and functions of the parabolic cylinder behave as $x^m \exp(-x/2)$ [1]. As will be shown below, the typical behavior required for solutions of nonlinear equations is that of $x^{-\alpha}$ ($\alpha \geqq 1$), as often as x^{-1}. This means that even if a formula for the products could be found, the representations would remain useless, since no infinite number of terms can secure the prescribed behavior at infinity.

* Received by the editors April 13, 1981.

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In the present work another CON sequence in $L^2(-\infty, \infty)$ which behaves as x^{-1} at infinity is introduced. Formulae for products and other properties are developed, and a nonlinear problem is solved as an example.

1. An example for the nonlinear boundary value problem in the $L^2(-\infty, \infty)$ space. Burgers [2] introduced the equation

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\partial^2 u}{\partial x^2} \tag{1.1}$$

as a substitute for the full system of Navier–Stokes equations in modelling the propagation of weak shock waves in a slightly compressible viscous fluid. Burgers' equation retains the major features of hydrodynamics, nonlinearity and viscosity, while being much simpler. Currently it is used as an instrument to investigate stochastic solutions. Though similar to a Navier–Stokes system, (1.1) is not unstable, and stochasticity of its solutions can be only prompted. The most convenient way to do that is by means of some random function of the spatial variable x taken as an initial condition [3]. Then the solution develops with time according to (1.1), but still appears to be a random function of x. One can calculate the cumulants of that random function and think of them as "experimental" data. Then they can be used for comparison with the theoretical predictions based on some kind of correlation equation(s) derived from (1.1) (cf. e.g. [4], [5]). Recently, the author [6] has introduced one more approach to the correlation problem for (1.1). For long times, when a kind of similitude of the random function can be assumed, the correlation equation from [6] reduces to

$$-\frac{1}{2}\left(\eta\frac{dB}{d\eta}+B\right)+B\frac{dB}{d\eta}=\frac{d^2B}{d\eta^2}, \tag{1.2}$$

where $B(\eta)$ is the kernel of the canonical representation introduced there. Respectively, the second cumulant is given by

$$Q(r)=\langle u(x,t)u(x+r,t)\rangle=\frac{\nu}{t}\int_{-\infty}^{\infty}B(\eta)B\left(\frac{r}{\sqrt{\nu t}}+\eta\right)d\eta,$$

and the correlation function is $K(r)=Q(r)/Q(0)$. Obviously, the above integral converges only if $B\in L^2(-\infty,\infty)$, i.e., if

$$\int_{-\infty}^{\infty}B^2(\eta)\,d\eta<+\infty. \tag{1.3}$$

The last condition is characteristic also for the theories in [4], [5], but the equations presented there are more complicated. For the purpose of the present work, the boundary value problem (1.2), (1.3) is enough to display the major features of such problems.

It is important here to mention the role of the nonlinear term in (1.2). Obviously it leads to the behavior $B\sim\eta^{-1}$ when $\eta\to\infty$. At that time, neglecting it yields $B\sim\operatorname{erfc}(\eta)$. Thus one arrives at the conclusion that the nonlinear problems in $L^2(-\infty,\infty)$ need a special CON sequence with asymptotic behavior η^{-1} at infinity.

2. Some preliminaries. As was mentioned above, the parabolic-cylinder functions are defined only on the semiaxis $(0,\infty)$. Changing the sign of the independent variable, one can easily define a similar set of functions for the negative interval $(-\infty, 0)$. If now the above functions are continued as zeros on the semiaxes where they are not defined (respectively negative or positive), then one obtains a system of functions defined on the entire real interval $\mathbb{R}$. Taking the Fourier transformation of these functions, one has the following set:

$$\rho_n = \frac{1}{\sqrt{\pi}} \frac{(ix-1)^n}{(ix+1)^{n+1}}, \qquad n = 0, 1, 2, \cdots, \tag{2.1}$$

which was first introduced by Wiener [7, p. 35] for positive values of n. Higgins [8, pp. 59–61] has briefly discussed the completeness and orthonormality of (2.1) as a straight corollary from the responsive properties of the parabolic-cylinder functions.

The most attractive feature in (2.1) is that the new system behaves as x^{-1} at infinity. Therefore it can form a basis for introducing a real-valued set of functions; however, it is important to derive more properties of this system. Obviously,

$$\frac{1}{\sqrt{\pi}} \frac{(ix-1)^{n-1}}{(ix+1)^{n+1}} = \frac{\rho_{n-1} - \rho_n}{2}. \tag{2.2}$$

Then

$$\frac{d\rho_n}{dx} = \frac{i}{2}[n\rho_{n-1} - (2n+1)\rho_n + (n+1)\rho_{n+1}]. \tag{2.3}$$

Correspondingly,

$$\frac{d^2\rho_n}{dx^2} = -\tfrac{1}{4}\{n(n-1)\rho_{n-2} - 4n^2\rho_{n-1} + [n^2 + (n+1)^2 + (2n+1)^2]\rho_{n+1} + (n+1)(n+2)\rho_{n+2}\}, \tag{2.4}$$

and so on.

The most important quality of (2.1) however, is that there exists an expression for the product of two members of the sequence

$$\rho_n\rho_k = \frac{1}{\pi} \frac{(ix-1)^{n+k}}{(ix+1)^{n+k+2}} = \frac{1}{2\sqrt{\pi}}(\rho_{n+k} - \rho_{n+k+1}), \tag{2.5}$$

which is of outstanding importance if one is bound to attack nonlinear problems. Furthermore,

$$x\frac{d\rho_n}{dx} = \tfrac{1}{2}[n\rho_{n-1} - \rho_n - (n+1)\rho_{n+1}]. \tag{2.6}$$

This last expression can prove to be very useful when equations of type (1.2) are considered.

Finally, one more property will be derived. It is easy to show that $\lim(x\rho_n) = -i/\sqrt{\pi}$ when $x \to \infty$. Let $\Phi_n(x) = x\rho_n - \lim_{x\to\infty}(x\rho_n)$. Then

$$\phi_n(x) = \frac{x(ix-1)^n}{(ix+1)^{n+1}} = \frac{x[(ix-1)^n - (ix+1)^n]}{\sqrt{\pi}(ix+1)^{n+1}} + \frac{i}{\sqrt{\pi}(ix+1)}.$$

Denoting for brevity $A = ix - 1$ and $B = ix + 1$, one can render $\rho_n(x)$ as follows:

$$\begin{aligned}\rho_n(x) &= \frac{1}{2i} \frac{(A+B)(A^n - B^n)}{\sqrt{\pi}B^{n+1}} + \frac{i}{\sqrt{\pi}(ix+1)} \\ &= \frac{A-B}{2i} \frac{(A+B)\sum_{k=0}^{n-1} A^k B^{n-1-k}}{\sqrt{\pi}B^{n+1}} + i\rho_0(x) \\ &= i\frac{\sum_{k=0}^{n-1} A^{k+1}B^{n-1-k} + \sum_{k=0}^{n-1} A^k B^{n-k}}{\sqrt{\pi}B^{n+1}} + i\rho_0.\end{aligned}$$

Finally,

$$\phi_n(x)=\begin{cases} i\left(\rho_n+2\sum_{k=0}^{n-1}\rho_k\right) & \text{for } n\geqq 1,\\ i\rho_0 & \text{for } n=0.\end{cases} \tag{2.7}$$

3. The real-valued system. It was shown in the previous section that the set of functions (2.1) possesses all desired properties needed for solving nonlinear problems in $L^2(-\infty,\infty)$ space. Unfortunately, it is a complex-valued set of functions. This presents a significant inconvenience when a nonlinear problem is to be solved. It is easy, however, to construct a real-valued set of functions on the basis of (2.1). As would be expected, the new system is comprised of two sequences, namely:

$$S_n=\frac{\rho_n(x)+\rho_{-n-1}(x)}{i\sqrt{2}},\qquad n=0,1,2,\cdots, \tag{3.1}$$

$$C_n=\frac{\rho_n(x)-\rho_{-n-1}(x)}{\sqrt{2}},\qquad n=0,1,2,\cdots. \tag{3.2}$$

Both sequences are orthonormal and each member of (3.1) is orthogonal to all members of (3.2); each member of (3.2) is also orthogonal to all members of (3.1). Here we need to mention that (3.1) and (3.2) can be defined for negative n through the relations

$$S_{-n}=S_{n-1}\quad\text{and}\quad C_{-n}=-C_{n-1}. \tag{3.3}$$

The functions S_n and C_n can be easily expressed in an explicit way:

$$S_0=\sqrt{\frac{2}{\pi}}\frac{-x}{x^2+1},\qquad C_0=\sqrt{\frac{2}{\pi}}\frac{1}{x^2+1},$$

$$S_1=\sqrt{\frac{2}{\pi}}\frac{-x^3+3x}{(x^2+1)^2},\qquad C_1=\sqrt{\frac{2}{\pi}}\frac{3x^2-1}{(x^2+1)^2},$$

$$S_2=\sqrt{\frac{2}{\pi}}\frac{-x^5+10x^3-5x}{(x^2+1)^3},\qquad C_2=\sqrt{\frac{2}{\pi}}\frac{5x^4-10x^2+1}{(x^2+1)^3},$$

$$\cdots\qquad\qquad\cdots$$

$$S_n=\sqrt{\frac{2}{\pi}}\frac{\sum_{k=1}^{n+1}x^{2k-1}(-1)^{n+k}\binom{2n+1}{2k-1}}{(x^2+1)^{n+1}},\qquad C_n=\sqrt{\frac{2}{\pi}}\frac{\sum_{k=1}^{n+1}x^{2k-2}(-1)^{n+k+1}\binom{2n+1}{2k-2}}{(x^2+1)^{n+1}}.$$

The first few S_n are shown in Fig. 1, and the first few C_n in Fig. 2. One can note that S_n are odd functions while C_n are even ones, and draw out a similarity between them and the ordinary sin and cos functions.

Now, all of the properties of system (2.1) can be automatically translated for S_n and C_n, namely:

$$\frac{dS_n}{dx}=\frac{1}{2}[nC_{n-1}-(2n+1)C_n+(n+1)C_{n+1}], \tag{3.4S}$$

$$\frac{dC_n}{dx}=-\frac{1}{2}[nS_{n-1}-(2n+1)S_n+(n+1)S_{n+1}], \tag{3.4C}$$

$$x\frac{dS_n}{dx}=\frac{1}{2}[nS_{n-1}-S_n-(n+1)S_{n+1}], \tag{3.5S}$$

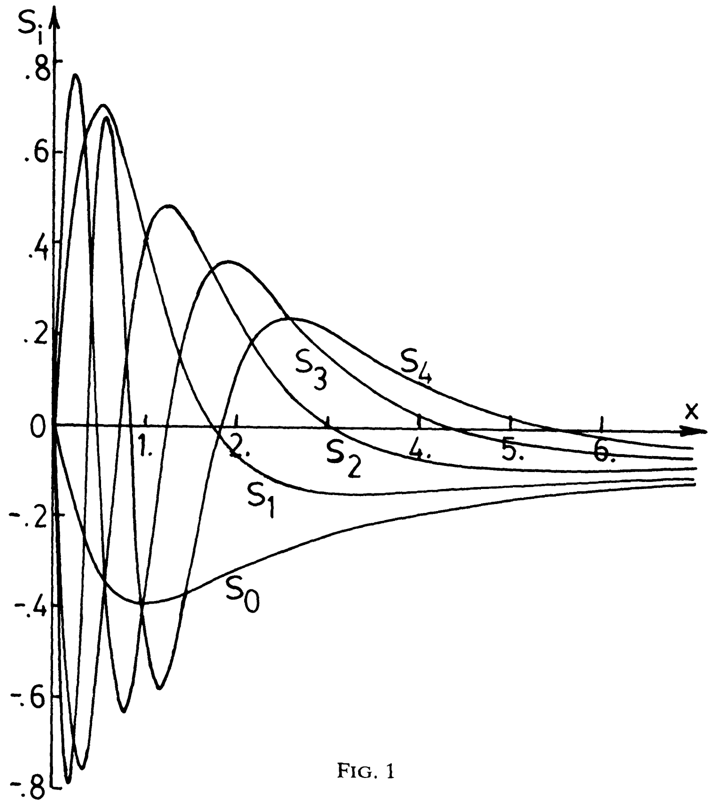
S_i
.8
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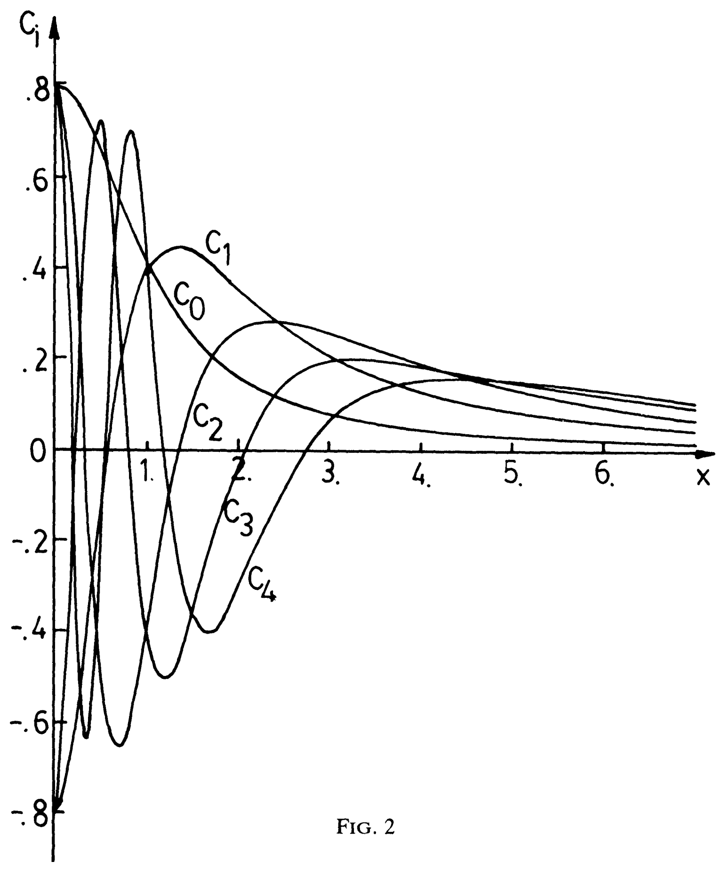
FIG. 1


C_i
.8
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4.
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x
C_0
C_1
C_2
C_3
C_4

FIG. 2

(3.5C) $$x\frac{dC_n}{dx}=\frac{1}{2}[nC_{n-1}-C_n-(n+1)C_{n+1}],$$

(3.6SS) $$S_nS_k=\frac{1}{2\cdot 2}[C_{n+k+1}-C_{n+k}+C_{n-k}-C_{n-k-1}],$$

(3.6CC) $$C_nC_k=\frac{1}{2\cdot 2}[C_{n+k+1}-C_{n+k}-C_{n-k}+C_{n-k-1}],$$

(3.6SC) $$S_nC_k=\frac{1}{2\cdot 2}[-S_{n+k+1}+S_{n+k}+S_{n-k}-S_{n-k-1}],$$

(3.7S) $$xS_n-\lim_{x\to\infty}(xS_n)=\begin{cases} C_n+2\sum_{k=0}^{n-1}C_k & \text{for } n\geqq 1,\\ C_0 & \text{for } n=0;\end{cases}$$

(3.7C) $$xC_n-\lim_{x\to\infty}(xC_n)\equiv xC_n=\begin{cases} -S_n+2\sum_{k=0}^{n-1}S_k & \text{for } n\geqq 1,\\ -S_0 & \text{for } n=0.\end{cases}$$

4. Application to the model equation from § 1. The CON system S_n, C_n was developed in previous sections. Now it is very important to show how it performs in boundary value problems, such as (1.2), (1.3). As will be seen, the nonlinear problem outlined above offers a situation in which all major outstanding features of the new CON system can be displayed.

Equation (1.2) admits only odd functions as a solution, which enables one to make use only of the sequence S_n. On the other hand, it is obvious that if $B(\eta)\in L^2(-\infty,\infty)$, then $B(\beta\eta)\in L^2(-\infty,\infty)$ also. The latter allows one to consider a more general equation,

(4.1) $$-\beta\left(z\frac{dy}{dz}+y\right)+4\sqrt{2\pi}\,y\frac{dy}{dz}=\frac{d^2y}{dz^2},$$

where $\eta=\sqrt{2\beta}\,z$ and $B=4\sqrt{\pi/\beta}/y(z)$. The independent parameter β can be employed in various ways, in particular for reducing the number of nonzero coefficients in the representation of the solution. Below we will show that in the case of model equation (4.1), a successful choice of β can reduce the entire representation to just one term.

The equation (4.1) can be integrated with respect to the independent variable z, i.e.,

(4.2) $$-\beta[zy-\lim_{z\to\infty}(zy)]+2\sqrt{2\pi}\,y^2=\frac{dy}{dz}.$$

Here the importance of properties (3.7) becomes evident, because of $\lim_{z\to\infty}(zy)=0$, one cannot obtain a solution in $L^2(-\infty,\infty)$ for (4.2). Taking into account the fact that y is an odd function, we can seek the solution in the form,

(4.3) $$y=\sum_{n=0}^{\infty}a_nS_n(z).$$

Then, incorporating (3.6SS), one obtains

$$2\sqrt{2\pi}\,y^2=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}a_na_k2\sqrt{2\pi}\,S_nS_k=\sum_n\sum_k a_na_k(C_{n+k+1}-C_{n+k}+C_{n+k}-C_{n-k-1}).$$

Employing (3.3), the last relation, is rendered to

$$2\sqrt{2\pi}\, y^2 = \sum_{m=1}^{\infty}\left[-\sum_{n=0}^{m} a_n a_{m-n} + \sum_{n=0}^{m-1} a_n a_{m-1-n}\right] C_m + a_0^2 C_0. \tag{4.4}$$

Turning to the term on the right-hand side of (4.2), it is easy to show by means of (3.4S) that

$$\frac{dy}{dz} = \sum_{m=0}^{\infty} \tfrac{1}{2}[(m+1)a_{m+1} - (2m+1)a_m + m a_{m-1}]C_m; \tag{4.5}$$

correspondingly,

$$zy(z) - \lim_{z\to\infty}(zy) = \sum_{m=0}^{\infty} [a_m + 2_{k=m+1} a_k]C_m. \tag{4.6}$$

Substituting (4.4), (4.5) and (4.6) into (4.2), one finally obtains the following algebraic system of equations for the unknown coefficients a_n:

$$-\beta\left[a_0 + 2\sum_{k=1}^{\infty} a_k\right] + a_0^2 = \tfrac{1}{2}(a_1 - a_0),$$

$$-\beta\left[a_m + 2\sum_{k=m+1}^{\infty} a_k\right] + \left[-\sum_{n=0}^{m} a_n a_{m-n} + \sum_{n=0}^{m-1} a_n a_{m-1-n}\right] \tag{4.7}$$

$$= \tfrac{1}{2}[(m+1)a_{m+1} - (2m+1)a_m + m a_{m-1}] \quad \text{for } m = 1, 2, \cdots.$$

This system is infinite and nonlinear. As usual, its solution can be approached by cutting off at some prescribed $m = M$ and retaining only the first $m+1$ unknowns $a_0, a_1, \cdots, a_M$. The convergence of such a procedure with increasing M is not obvious, and the proof needs special efforts in any particular case. This kind of situation is well known from the nonlinear periodic problems when solved by means of Fourier series. Though there are doubts that a theorem about convergence of approximation can be easily proved, the solution of the reduced system is not an enigma itself. For this reason, in the present work we do not show the string of approximate solutions (say with $M = 1, 2$, etc.) as is usually done for numerical "proof" of convergence. Rather, we will show below how the parameter β can be used to reduce the nonzero coefficients a_n.

In other words, it is interesting to seek a solution with $a_n = 0$ for $n \geqq 1$, and only $a_0 \neq 0$. Then (4.7) reduce to

$$a_0^2 = (\beta - \tfrac{1}{2})a_0 \quad \text{and} \quad a_0^2 = \tfrac{1}{2}a_0. \tag{4.8}$$

The solution of (4.8) exists only if $\beta = 1$ and is given by

$$y = \frac{1}{2}S_0(z) = -\frac{1}{2}\sqrt{\frac{2}{\pi}}\,\frac{z}{z^2+1}. \tag{4.9}$$

The substitution of (4.9) into (4.2) confirms that it really is a solution. The explanation of this success in reduction of the number of terms in the representation of the solution is that (4.2) is of first order, and therefore the coefficient a_0 enters only the first two equations (see (4.8)), and then a_0 and β become two unique unknowns with two equations defining them.

For more general equations (of higher order, for example) the above radical simplification of the system cannot be expected to apply. Nevertheless, the parameter β can be successfully used in accelerating the convergence of the reduced-system solutions to the one for the infinite system.

Returning to the original variables B and η, the solution (4.9) can be rewritten as

$$B(\eta) = -4\frac{\eta}{\eta^2+2}, \tag{4.10}$$

which is exactly the solution of (1.2) found in [6].

Concluding remarks. The CON system of real-valued functions $\{S_n, C_n\}$ introduced in the present work appears to be novel, though a simple consequence of a known complex-valued CON sequence in $L^2(-\infty, \infty)$. The subsequence S_n consists of odd functions and C_n of even ones, with asymptotic behavior x^{-1} and x^{-2} respectively.

Formulae representing the derivatives (as well as a couple of other specific linear transformations of S_n and C_n) in series in the system have been developed. The central point, however, is that a formula for a product of two members of the system, in series in the system, has been derived. It has to be mentioned, for comparison, that the other two known CON systems with summable squares on infinite interval, namely the sets of Hermite and Laguerre functions, do not possess such representations for the products.

Hence, the new system proves to be a unique tool for handling the various nonlinear equations, as well as certain linear ones with nonhomogeneous coefficients. To display the remarkable performance of the new system in infinite boundary-value problems, a typical example has been solved in closed form.

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