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DETERMINATION OF THE INTERACTION BETWEEN THE PARTICULATE AND CONTINUOUS PHASE IN AN EMULSION VIA STOCHASTIC FUNCTIONAL EXPANSION

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Христо Иванов Христов. **Определение взаимодействия между частичковой и непрерывной фазой в эмульсии на основе стохастического функционального разложения.** В работе применяется функциональное разложение Вольтера-Винера случайных процессов с базисной функцией, являющейся маркированным точечным случайным полем. Принимая в качестве метки случайный радиус сферы, рассмотрено течение разреженных эмульсий из полидисперсных сферических капель. Получены уравнения для ядер функционального разложения. В качестве примера решена задача о взаимодействии частичковой и несущей фазы с первым порядком точности по концентрации. Получена замкнутая формула для силы взаимодействия и показано, что она зависит от вероятностного распределения частиц по размерам.

Christo Ivanov Christov. **Determination of the interaction between the particulate and continuous phase in an emulsion via stochastic functional expansion.** The Volterra-Wiener functional expansion with a marked point random function as a basis function is employed to treat the flow of polydisperse dilute emulsions when the mark can be interpreted as the radius of a spherical droplet out of the disperse phase. Governing equations for the first couple of kernels are obtained and solved within the first order of approximation with respect to the volume fraction of particles. In order to display the performance of the method proposed a formula for the drag force exerted by the continuous phase on the particulate one is derived. It is shown that this force depends on the probability distribution of radii of suspended spheres.

One of the most important fields of the modern hydrodynamics is the multiphase flow to whose modelling a vast amount of scientific publications is devoted. Roughly, these works can be broken into two major groups: phenomenological approach and statistical approach. In the former the motion of a suspension/emulsion is considered as a flow of several material continuums occupying the same geometrical space. Each of these continuums possesses its own rheological properties postulated in a way to obey the general laws of continuum mechanics and is coupled with the others through specifying the forces

acting upon a continuum as a result of the presence of the rest of continuums [1]. The statistical approach (see [2, 3]) as a rule plays a subsidiary role to the phenomenological one helping to identify the rheological and interaction coefficients.

Statistical approach, however, is much more consistent if the suspension/emulsion is treated as a single-phase liquid with random density and transport coefficients. Then the problem becomes purely mathematical and consits in devising an effective method for solving stohastic Navier-Stokes equations. For the case of low concentration of slowly drifting equi-sized spheres this approach is outlined in [4, 5] where on the basis of the method of stochasic functional expansions [6] an approximate solution of order $o(c)$ to the stochastic Navier—Stokes equations is obtained. The stress tensor of the averaged flow is calculated and shown to be comprised of an orthotropic pressure tensor whose axis of orthotropy is represented by the vector of relative (drift) velocity of particles and a viscous term which coincides with the famous Einstein result [7].

Further, the method of [6] has been forwarded in [8], [9] where are taken into account the possible exclusion effects when the spheres are restricted from intersecting each other and the effective moduli of heat conductivity [8] and elasticity [9] of a suspension are approximately calculated with order of accuracy $o(c^2)$. Recently the method of functional expansion has been extended to the case of basis function which includes still more information about the system of suspended particles [10], developing the necessary technique of stochastic functional expansions when the basis function is a general marked random point function.

The present paper is an application of the technique of [10] to the flow of dilute polydisperse emulsions of spherical droplets. In the last case the mark associated with a random point can be considered to be the random radius of a particle.

1. RANDOM DENSITY FUNCTION

Suspension is one of the most instructive examples of random field created by a system of random points, the latter being the set of centers of inclusions. Such stochastic functions are called random point functions [11], [12] and their statistical properties can be obtained from the respective properties of the generating set of random points [11]. The simplest case is when the points are identical and independent statistically: we then arrive to the so-called Poisson random point function which is an appropriate model for a suspension if only the inclusions are infinitely small [4, 5]. It is interesting to note here that the finite size affects the results only in the terms of order $O(c^2)$ (see [8, 9]). The more accurate modelling of a suspension requires to acknowledge not only the finite size of an inclusion but also the distribution of the size and the relative velocity. This means that the points cease to be identical and with each of them is associated a random variable (generally a vector) called mark [12]. In this instance the mark can be the radius of the sphere appearing at the random point considered, the vector of drift velocity of this sphere, the set of parameters governing the shape when the particle is not spherical, etc. Here we shall confine ourselves only to the case when the mark is just the radius of a suspended droplet in order to display the method proposed.

The basic random point function is the so-called random density function [11], defined as

(1) $$\varphi(\boldsymbol{x})=\sum_j \delta(\boldsymbol{x}-\boldsymbol{x}_j),$$

where $\delta(\,.\,)$ is Dirac delta-function and $\boldsymbol{x}_j$ is the set of random points. It is easily shown that the more complicated random point functions are derived from (1) through linear transformation [11, 12].

For the case when a mark is associated with each random point x_j, the so-called "generalized random density function" is introduced in [10]:

(2) $$\omega(\boldsymbol{x};\ \boldsymbol{u})\equiv\omega(\boldsymbol{z})=\sum_j \delta(\boldsymbol{z}-\boldsymbol{z}_j),$$

where $\boldsymbol{z}=(x_1,\ x_2,\ x_3,\ u_1,\ \ldots,\ u_m)$ is the outer product of the three-dimensional geometrical space R^3 and the m-dimensional mark space U. There are no obstacles to include the time and it is not done only for the sake of brevity of notations. Moreover, in the present theory of slow-drift suspensions time plays the role of a parameter. Respectively, $\delta(.)$ are Dirac deltas of $(m+3)$-th order. As it has been mentioned above the more complex random point functions are expressed via ω, nameley every filtered random point function is given by:

(3) $$y(\boldsymbol{x};\ \boldsymbol{u})=\int_R\int_U K(\boldsymbol{x}-\xi,\ \boldsymbol{u})\omega(\xi;\boldsymbol{u})\ d^3\xi d^m\boldsymbol{u}\equiv\sum_j K(\boldsymbol{x}-\boldsymbol{x}_j;\ \boldsymbol{u}_j).$$

Let us now, following [11], introduce the probability density functions $f_n(x_1,\ldots,\boldsymbol{x}_n;\ u_1,\ \ldots,\ u_n)$ for the sytem random points $\{z_j\}$ in the $(m+3)$-dimensional space, i. e.

(4) $$dP=f_n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n;\ \boldsymbol{u}_1,\ldots,\boldsymbol{u}_n)d^3\boldsymbol{x}_1\ldots d^3\boldsymbol{x}_n\,d^m\boldsymbol{u}_1\ldots d^m\boldsymbol{u}_n$$

is the probability of simultaneous occurrence of n points $\boldsymbol{x}$ each of them with its mark u in the intervals:

$$\boldsymbol{x}_i\leqq\boldsymbol{x}\leqq\boldsymbol{x}_i+d\boldsymbol{x}_i,\quad \boldsymbol{u}_i\leqq\boldsymbol{u}\leqq\boldsymbol{u}_i+d\boldsymbol{u}_i,\quad i=1,\ 2,\ldots,n.$$

In the case when the points are mutually independent in stochastic sense we arrive to the Poisson random fuction for which functions f_n are decomposed into the following way

(5) $$f_n=f_1(\boldsymbol{x}_1;\ \boldsymbol{u}_1)\ldots f_1(\boldsymbol{x}_n;\ \boldsymbol{u}_n)=\prod_i\prod[\gamma P(\boldsymbol{u}_i)],$$

where γ is the intensity of the associated counting function and $P(\boldsymbol{u}_i)$ is the probability density of the mark (see [10]). Here and henceforth it is assumed that random processes investigated are statistically homogeneous, so the number of points falling per unit volume (intensity of the counting function) is constant.

When there exists a correlation between the position x and mark u, it can be acknowledged by means of functions f_n. For the simple case under investigation with mark being just a positive real number — the radius of the falling sphere — we can take into account the requirement of non-overlapping of spheres in the following way

(6) $$f_2(\boldsymbol{x}_1,\ \boldsymbol{x}_2;\ a_1,\ a_2)=Q_{12}(\boldsymbol{x}_1-\boldsymbol{x}_2)\ f_1(\boldsymbol{x}_1;\ a_1)f_1(\boldsymbol{x}_2;\ a_2),$$

where

$$Q_{ij}=\begin{cases}0 \text{ for } |\boldsymbol{x}_i-\boldsymbol{x}_j|\leqq a_1+a_2,\\ 1 \quad \text{otherwise,}\end{cases} \quad \text{and } R_{ij}\equiv 1-Q_{ij}.$$

Here is implicitly presumed that if spheres are situated far enough in order not to overlap they are statistically independent. This random field is obviously of the type of perfectly disordered ones. It is introduced in [10] and named PDSRR-field (Perfect Disorder of Spheres of Random Radii) and it is a straightforward generalization of the eatlier introduced in [8] PDS-field (Perfect Disorder of Spheres). The higher-order probability densities of PDSRR-field can be specified as

$$\begin{aligned}(7)\quad & f_n(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n;\ a_1,\ldots,a_n)\\ &=f_1(\boldsymbol{x}_1;\ a_1)\ldots f_1(\boldsymbol{x}_n;\ a_n)Q_{12}\ldots Q_{1n}Q_{23}\ldots Q_{2n}\ldots Q_{n-1n}.\end{aligned}$$

Having functions f_n one can obtain the statistics (moments, cumulants, etc.) of the random density function ω. This is possible due to results of S t r a t o n o v i c h [11] which apply to our case:

$$\begin{aligned}&\langle\omega(\boldsymbol{x};\ a)\rangle=f_1(\boldsymbol{x},\ a),\\ &\langle\omega(\boldsymbol{x}_1;\ a_1)\,\omega(\boldsymbol{x}_2,\ a_2)\rangle=f_1(\boldsymbol{x}_1;\ a_1)\delta[1,\ 2]+f_2(\boldsymbol{x}_1\ \boldsymbol{x}_2;\ a_1,\ a_2),\\ (8)\quad &\langle\omega(\boldsymbol{x}_1;\ a_1)\,\omega(\boldsymbol{x}_2,\ a_2)\,\omega(\boldsymbol{x}_3,\ a)\rangle=f_1(\boldsymbol{x}_1;\ a_1)\delta[1,\ 2]\delta[1,\ 3]\\ &+3\{\delta[1,\ 2]f_2(\boldsymbol{x}_1,\ \boldsymbol{x}_3;\ a_1,\ a_2)\}_s+f_3(\boldsymbol{x}_1,\ \boldsymbol{x}_2,\ \boldsymbol{x}_3;\ a_1,\ a_2,\ a_3),\end{aligned}$$

where it is denoted for brevity $\delta[i,\ j]=\delta(\boldsymbol{z}_i-\boldsymbol{z}_j)$ and $\{\ .\ \}_s$ is the symmetrization operation.

For the random density fuction under consideration (8) is reduced to

$$\begin{aligned}&\langle\omega(\boldsymbol{x};\ a)\rangle=\gamma P(a),\\ &\langle\omega(\boldsymbol{x}_1;\ a_1)\,\omega(\boldsymbol{x}_2,\ a_2)\rangle=\gamma P(a_1)\delta[1,\ 2]+\gamma^2P(a_1)P(a_2),\\ (9)\quad &\langle\omega(\boldsymbol{x}_1;\ a_1)\,\omega(\boldsymbol{x}_2;\ a_2)\,\omega(\boldsymbol{x}_3;\ a_3)\rangle=\gamma P(a_1)\delta[1,\ 2]\delta[1,\ 3]\\ &+3\gamma^2\{\delta[1,\ 2]Q_{13}P(a_1)P(a_3)\}_s+\gamma^3P(a_1)P(a_2)P(a_3)Q_{12}Q_{13}Q_{23},\\ &\ldots\end{aligned}$$

2. EMULSION AS A LIQUID WITH RANDOM DENSITY AND VISCOSITY

Consider a homogeneous suspension of spheres of radius a, the latter distributed with density $P(a)$, density ρ_1 and viscosity μ_1 (for rigid spheres $\mu_1\to\infty$). The spheres are randomly dispersed throughout a viscous liquid of viscosity μ_0 and density ρ_0. Let us also assume that velocity of a particle is a vector of non-random direction varying slowly with spatial coordinates but with a random amplitude which is a function of the random radius a, i. e. the amplitude is not an independent stochastic variable. This situation is easily attainable when the drift Reynolds number is small and therefore the relaxation time is short. Then the particles move virtually steady and has connection between the velocity amplitude and radius such as for the terminal velocity. A typical pattern of the flow under consideration is shown in Fig. 1. Let us note in passing that these requirements are man-

datory for all the phenomenological approaches if the mixture is to be considered as a continuous medium. In fact they are not specific for the stochastic approach but allow one to consider locally homogeneous random fields, simplifying deci-

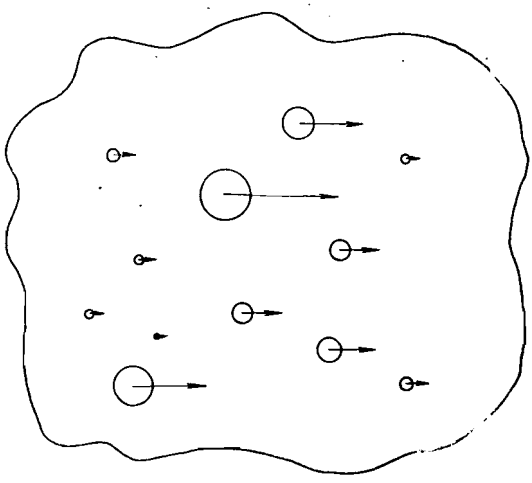

Fig. 1

sively the calculations. So we resort to the listed limitations in order to display the method in closed form.

Under the above assumptions the emulsion can be thought of as a single-phase liquid with random viscosity coefficient μ and random density ρ expressed as

$$\rho=\rho_0+(\rho_1-\rho_0)\sum_j h(\boldsymbol{x}-\boldsymbol{x}_j;\ a_j),$$

(10)

$$\mu=\mu_0+(\mu_1-\mu_0)\sum_j h(\boldsymbol{x}-\boldsymbol{x}_j;\ a_j),$$

where summation with respect to index j runs over all the random points x_j at which the centers of spherical inclusions are placed. In this formulae $h(x)$ is the so-called characteristic function:

$$h(\vec{\boldsymbol{x}})=\begin{cases}1 \text{ for } |\boldsymbol{x}|\leqq a,\\ 0 \text{ for } |\boldsymbol{x}|>a.\end{cases} \tag{11}$$

Obviously, the volume occupied by a single sphere is given by:

$$V_a=\int_{\mathbb{R}^3} h(\boldsymbol{x};\ a)d\boldsymbol{x}=\frac{4}{3}\pi a^3, \tag{12}$$

where subscript $\mathbb{R}^3$ of the intergral stands for an integration with respect to the entire three-dimensional space. Henceforth it will be omitted for the sake of brevity without fear of confusion.

Making use of the random density function of the previous section one can recast (10) as follows:

$$\rho=\rho_0+(\rho_1-\rho_0)\int\int h(\boldsymbol{x}-\vec{\xi};a)\,\omega(\vec{\xi};\ a)d^3\vec{\xi}da,$$

(13)

$$\mu=\mu_0+(\mu_1-\mu_0)\int\int h(\boldsymbol{x}-\vec{\xi};\ a)\,\omega(\vec{\xi};\ a)d^3\vec{\xi}da,$$

where integration with respect to a is over the interval $[0, \infty)$. In the same manner the body force is expressed

$$\boldsymbol{f}=\boldsymbol{f}_0+\int\int(\boldsymbol{f}_1-\boldsymbol{f}_0)\,h(\boldsymbol{x}-\vec{\xi};\ a)\,\omega(\vec{\xi};\ a)d^3\vec{\xi}da, \tag{14}$$

where $\boldsymbol{f}_1$ can be a random function of $(\boldsymbol{x}_j,\ a_j)$. The last formula implies that the stochasticity of the body force is in concert with that of particles which is a completely natural assumption since the external body forces are connected with the material of the medium.

The volume fraction (concentration) of the particulate phase is

$$c=\left\langle\int\int h(\boldsymbol{x}-\vec{\xi};\ a)\,\omega(\vec{\xi};\ a)d^3\vec{\xi}da\right\rangle=\gamma\int V_a P(a)da=\gamma V, \tag{15}$$

where V is the averaged volume of a particle.

In the same manner one gets

$$\langle\rho\rangle=\rho_0+(\rho_1-\rho_0)c,\quad \langle\mu\rangle=(\mu_1-\mu_0)c+\mu_0. \tag{16}$$

Here $<\mu>$ is not the effective viscosity of emulsion and even $<\mu>\to\infty$ when the particles become rigid.

Having now the coefficients ρ, μ and the external body force f one can state that the flow of emulsion is governed by the stochastic Navier — Stokes equations (see [4, 5])

$$\rho\left[\frac{\partial v}{\partial t}+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right]=-\nabla p+\nabla(\mu\nabla\boldsymbol{v})+\boldsymbol{f} \tag{17}$$

and the stochastic equation of continuity

$$\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\boldsymbol{v})=0, \tag{18}$$

where $\boldsymbol{v}$ and p are the random fields of velocity and pressure, respectively.

3. METHOD OF FUNCTIONAL EXPANSION

In the model proposed the only source of stochasticity is the random density function ω. Naturally the most effective way to treat the stochastic differential equations which define a nonlinear operator transforming the random fields of coefficients into the random fields of solution is to develop the random solution into series with respect to ω. The general way of doing that is outlined in [10], but it is not always possible to render the functional series orthogonal. Fortunately, in

the present case its still possible if Charlier polynomials are introduced (see for definition [13]) in the same manner as it is done in [8, 9] for the PDS-field. Following these works we write:

$$
\begin{aligned}
(19)\quad C^n_\omega(y_{i_1}, \ldots, y_{i_n};\ a_1, \ldots, a_n) &= \omega(y_{i_1};\ a_{i_1})\,[\omega(y_{i_2};\ a_{i_2}) - \delta[i_1,\ i_2]] \\
&\quad \ldots\ [\omega(y_{i_n};\ a_{i_n}) - \delta[i_1,\ i_n] - \ldots - \delta[i_{n-1},\ i_n]] \\
&\quad - \gamma\Big\{\sum_{(n,\,n-1)} \omega(y_{j_1};\ a_{j_1})\,[\omega(y_{j_2},\ a_{j_2}]] - \delta\,[j_1, j_2]] \ldots [\omega(y_{j_{n-1}};\ a_{n-1}) \\
&\quad - \delta[j_1,\ j_{n-1}] - \ldots - \delta[j_{n-2},\ j_{n-1}]]\,P(a_{j_n})\Big\} \\
&\quad + \ldots + (-\gamma)^{n-2} \sum_{(n,\,n-2)} \omega(y_{j_1};\ a_{j_1})\,[\omega(y_{j_2},\ a_{j_2}) - \delta[j_1,\ j_2]]\,P(a_{j_3}) \ldots P(a_{j_n}) \\
&\quad + (-\gamma)^{n-1} \sum_{\nu=1}^{n} \omega(y_\nu;\ a_\nu)\,P(y_1) \ldots P(y_{\nu-1}) \ldots P(y_n) + (-\gamma)^n P(y_1) \ldots P(y_n).
\end{aligned}
$$

Being reminded of the average characteristic of the random density function ω one obtains:

$$
\begin{aligned}
&\langle C^{(0)}_\omega \rangle = 1,\ \langle C^{(1)}_\omega(y;\ a)\rangle = 0, \\
&\langle C^{(2)}_\omega(y_1,\ y_2;\ a_1,\ a_2)\rangle = -\gamma^2 P(a_1)P(a_2)R_{12}, \\
(20)\quad &\langle C^{(3)}_\omega(y_1,\ y_2,\ y_3;\ a_1,\ a_2,\ a_3)\rangle = \gamma^3 P(a_1)\,P(a_2)P(a_3), \\
&\quad \times \{R_{12}R_{23} + R_{23}R_{31} + R_{31}R_{12} - R_{12}R_{23}R_{31}\}, \\
&\quad \ldots \\
&\langle C^{(n)}_\omega(y_1, \ldots, y_n;\ a_1, \ldots, a_n)\rangle = \gamma^n P(a_1) \ldots P(a_n), \\
&\times \Big\{\frac{1}{2}\ n(n-1)\ -\ \text{linear form of}\ \frac{1}{2}\ n(n-1)\ \text{functions}\ R_{ij},
\end{aligned}
$$

$i, j = 1, \ldots, n$. The least order is $(n-1)$}.

The fact that $C^{(n)}$is at least$(n\text{-}1)$-linear form of functions R_{ij}means that $< C_{(\omega)^{(n)}} > = 0$ only if all of the arguments y_i are well separated, i. e. $|y_i - y_j| \geqq a_i + a_j$. Respectively $< C_\omega{}^{(n)} > \neq 0$ is to be expected if at least$(n-1)$ pairs y_i, y_j of arguments are closer that the corresponding value $a_i + a_j$. For instance $< C_\omega{}^{(3)} > \neq 0$ when $|y_1 - y_2| \leqq a_1 + a_2$ and $|y_2 - y_3| \leqq a_2 + a_3$ regardless of the relationship between y_1 and y_3.

It is seen that the stochastic polynomials are not centered and orthogonal for arbitrary values of their arguments. Nevertheless the functionals

$$
\begin{aligned}
(21)\quad G^{(n)}_\omega[F] &= \iint \ldots \iint K^{(n)}_F(x - y_1, \ldots, x - y_r;\ a_1, \ldots, a_n) \\
&\quad \times C^{(n)}_\omega(y_1, \ldots, y_n;\ a_1, \ldots, a_n)\,d^3y_1 da_1 \ldots d^3y_n da_n
\end{aligned}
$$

can be rendered centered stochastic variables assuming that (see [8])

(22) $K^{(n)}_F(y_1, \ldots, y_n;\ a_1, \ldots, a_n) = 0$ if even for a single pair of indices i, j holds $|y_i - y_j| \leqq a_i + a_j$.

It is clearly seen that the asymmetry between the spatial coordinates and marks from (3) is retained in (21) which is the main point in constructing functional series with respect to the marked random point processes (see [10]).

The requirement (22) is completely natural for the case under consideration and leads to no loss of generality since in a system of spheres one never has a pair of points y_i, y_j situated closer than a_i+a_j. Then due to (22) one has:

$$\langle G_\omega^{(n)}[F]\rangle=0,\quad n=1,\ 2,\ \ldots \tag{23}$$

Moreover, Eq. (22) yields that functionals (21) are orthogonal in stochastic sense. The proof is similar to that for PDS-field [8] and we just state here the result

$$\langle G_\omega^{(n)}[F]\, G_\omega^{(m)}[H]\rangle=0,\quad n\neq m,$$

$$\langle G_\omega^{(n)}[F]G_\omega^{(n)}[H]\rangle=\iiiint\ldots\iiiint K_F^{(n)}(y-y_1;\ \ldots y-y_n;\ b_1,\ \ldots,\ b_n) \tag{24}$$

$$\times K_H^{(n)}(x-x_1,\ \ldots,\ x-x_n;\ a_1,\ \ldots,\ a_n)\sum_1^{n!} M_\omega^{(2)}(y_{i_1},\ x_{k_1};\ b_{i_1},\ a_{k_1})\ldots M_\omega^{(2)}(y_{i_n},$$

$$x_{k_n};\ b_{i_n},\ a_{k_n})\, d^3y_1 d^3x_1 da_1 db_1\ldots d^3y_n d^3x_n db_n da_n,$$

where

$$M_\omega^{(2)}(y,\ x;\ b,\ a)=\gamma P(b)\delta(y-x)\delta(a-b)-\gamma^2 P(a)P(b)R(y-x) \tag{25}$$

$$\equiv\langle C_\omega^{(1)}(y;\ b)\ C_\omega^{(1)}(x;a)\rangle.$$

The formula for the third moment can be obtained through obvious generalization of the respective formulae of [6] and [8], but it is too cumbersome so we shall cite here only the expression for the third moment of the first-order Charlier polynomials:

$$M_\omega^{(3)}\ (x,\ y,\ z;\ a,\ b,\ c)\equiv\langle C_\omega^{(1)}\ (x;\ a)\ C_\omega^{(1)}\ (y;\ b)\ C_\omega^{(1)}(z,\ c)\rangle=\gamma P(a)\delta \tag{26}$$

$$(x-y)\delta(x-z)\delta(a-b)\delta(a-c)-3\gamma^2\{\delta(x-y)\delta(a-b)R(x-z)P(a)P(c)\}$$

$$+\gamma^3[3\{R(x-y)R(z-z)\}_s-R(x-y)R(y-z)R(z-x)]P(a)P(b)P(c).$$

Here it should be mentioned that the kernels $K^{(n)}$ are symmetric functions of the spatial coordinates which is due to the symmetric properties of the Charlier polynomials $C_\omega^{(n)}$.

Now one can expand a random point function into functional series with respect to the PDSRR-field. In particular for the velocity and pressure of the stochastic liquid considered in the present work one has:

$$v=u(x,\ t)+\iint K^{(1)}(x-\vec{\xi};\ a;\ t)C_\omega^{(1)}(x;\ a)d^3\vec{\xi}da+\ldots \tag{27}$$

$$+\iint\ldots\iint K^{(n)}\ (x-\vec{\xi}_1,\ \ldots,\ \vec{x}-\vec{\xi}_n;\ a_1,\ \ldots,\ a_n)\ C_\omega^{(1)}\ (\vec{\xi}_1,\ \ldots,\ \vec{\xi}_n;\ a_1,\ \ldots,\ a_n)$$

$$\times d^3\vec{\xi}_1 da_1,\ \ldots\ d^3\vec{\xi}_n\, da_n+\ldots$$

$$p=q(\vec{x},\ t)+\iint Q^{(1)}(x-\vec{\xi};\ a;\ t)C_\omega^{(1)}\ (\vec{\xi};\ a)\ d^3\vec{\xi}da\ldots \tag{28}$$

$$+\iint\ldots\iint Q^{(n)}(x-\vec{\xi}_1,\ \ldots x-\vec{\xi}_n;\ a_1,\ \ldots,\ a_n)\, C_\infty^{(1)}\ (\vec{\xi}_1,\ \ldots,\ \vec{\xi}_n;\ a_1,\ \ldots,\ a_n)$$

$$+d^3\vec{\xi}_1 da_1\ldots d^3\ \vec{\xi}_n\, da_n+\ldots$$

We name the above expansion PDSRR-Wiener expansion.

Obviously $\boldsymbol{u}(\boldsymbol{x}, t)$ is the ensemble averaged velocity of the emulsion and coincides with the mean-volume velocity defined in the multi-continuum theories. The mean-mass velocity is defined as

$$(29)\quad \boldsymbol{w}(\boldsymbol{x}, t)=\frac{\langle\rho\boldsymbol{v}\rangle}{\langle\rho\rangle}=\boldsymbol{u}+c\,\frac{\rho_1-\rho_0}{\langle\rho\rangle}\left\{\int\int\frac{1}{V}\,h(\boldsymbol{x};\ a)\,\boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)P(a)d^3\boldsymbol{x}da\right.$$

$$\left.-\int\int\int\int\frac{c}{V^2}\,h(\boldsymbol{x}_1;\ a)\,\boldsymbol{K}^{(1)}(\boldsymbol{x}_2,\ a_2)\,R_{12}P(a_1)P(a_2)d^3\boldsymbol{x}_1d^3\boldsymbol{x}_2da_1da_2\right\}.$$

It is convenient also to introduce the mean velocity of the particulate phase according to the formula

$$(30)\quad \boldsymbol{u}_1=\langle\boldsymbol{v}\int\int h(\boldsymbol{x};\ a)\,\omega(\boldsymbol{x},\ a)d^3\boldsymbol{x}\,da\rangle=c\boldsymbol{u}(\boldsymbol{x},\ t)$$

$$+c\left\{\int\int\frac{\vec{\boldsymbol{c}}}{V}\,h(\vec{\boldsymbol{x}};\ a)\,\boldsymbol{K}^{(1)}(\vec{\boldsymbol{x}};\ a)\,F(a)\,d^3\vec{\boldsymbol{x}}\,da\right.$$

$$\left.-\frac{c^2}{V^2}\int\int\int\int h(\boldsymbol{x}_1;\ a_1)\ \boldsymbol{K}^{(0)}(\boldsymbol{x}_2;\ a_2)\,R_{12}P(a_1)P(a_2)d^3\boldsymbol{x}_1d^3\boldsymbol{x}_2\,da_1da_2\right\}.$$

Respectively, for the velocity of continuous phase we have

$$(31)\quad \boldsymbol{u}_0=\boldsymbol{u}-\boldsymbol{u}_1$$

and then

$$(32)\quad \langle\rho\rangle\boldsymbol{w}=\rho_0\boldsymbol{u}_0+\rho_1\boldsymbol{u}_1=\rho_0(1-c)\,\boldsymbol{v}_0+\rho_1c\boldsymbol{v}_1,$$

where is denoted $\boldsymbol{u}_0=(1-c)\boldsymbol{v}_0$ and $\boldsymbol{u}_1=c_1\boldsymbol{v}_1$ in order to catch up with notations of phenomenological theories where $\boldsymbol{v}_0$ and $\boldsymbol{v}_1$ are called "weighted' velocities [1].

4. EQUATIONS FOR KERNELS

4.1. Some preliminaries

As has been argued above the stochastic fields are considered locally homogeneous so we neglect derivatives with respect to "global" coordinates in comparison with the "local' derivatives. In addition the time t is considered as a parameter. For this reason the global coordinates are already omitted in (27), (28).

For velocity gradient under these assumptions we have:

$$(33)\quad \nabla\boldsymbol{v}=\int\int\nabla_\xi\boldsymbol{K}^{(1)}(\boldsymbol{x}-\xi;\ a;\ t)\,C_\omega^{(1)}d^3\xi\,da$$

$$+\int\int\int\int(\nabla_{\xi_1}+\nabla_{\vec{\xi}_2})\boldsymbol{K}^{(2)}(\boldsymbol{x}-\vec{\xi}_1,\boldsymbol{x}-\vec{\xi}_2;\ a_1,\ a_2;\ t)C_\omega^{(2)}d^3\vec{\xi}_1\,da_1d^3\vec{\xi}_2\,da_2+\ldots$$

Here and henceforth the higher-order terms are discarded. All of the characteristic features of the method proposed are fully displayed by the first two kernels.

In the same manner the pressure gradient is manipulated.

Turning to second derivatives we do not face any additional difficulties in repeating the differentiation according to (33).

More complicated is the problem of calculating the time derivatives of v and p because time t enters the picture through the random density function due to the

fact that the random points generating the random density function are moving throughout the space. Let us denote the velocity of a particle by v_p. Then

$$\frac{dx_j(t)}{dt} = v_p(a_j). \tag{34}$$

It is seen that v_p is a random variable which depends solely on the radius of sphere a and, perhaps, on t or $\boldsymbol{x}$. The last dependence, however, is much weaker than on a so we disregard it. After neglecting the "global" derivatives one obtains:

$$\frac{\partial v}{\partial t} = \frac{\partial u}{\partial t} - \int\int v_p \cdot \nabla_\xi K^{(1)}(\boldsymbol{x} - \vec{\xi};\ a;\ t) d^3\vec{\xi} da\, C_\omega^{(1)}(\xi;\ a) \tag{35}$$

$$- \iiiint (x_p(a_1),\ \nabla_{\xi_1} + v_p(a_2) \cdot \nabla_{\xi_2}) K^{(2)}(\boldsymbol{x} - \vec{\xi}_1,\ \boldsymbol{x} - \vec{\xi}_2;\ a_1,\ a_2;\ t) C_\omega^{(2)} d^3\vec{\xi}_1 d^3\vec{\xi}_2 da_1 d_2 \ \ldots$$

The crucial point here is that $v_p(a) = v_p(a)v^\circ$, where v° is a constant vector representing the drift velocity and therefore the integrals (35) are of the same type of the original expansion (27) and only $K^{(1)}(a_j)$ is replaced by $v_p(a_j)K^{(1)}(a_j)$. The latter means that (35) is averaged in the same manner as (27).

Being consistent with the assumption of low drift velocity we shall in the following neglect the second order terms with respect to kernels K in comparison with first-order ones since K is of drift velocity order. The nonlinear effects are examined in [5], where it is shown that they lead to an orthotropic pressure tensor for the averaged flow. Further

$$\rho \frac{\partial v}{\partial t} = \langle \rho \rangle \frac{\partial u}{\partial t} - \langle \rho \rangle \int\int v_p \cdot \nabla_\xi K^{(1)}(\boldsymbol{x} - \vec{\xi};\ a)\, C_\omega^{(1)} d^3\vec{\xi}\, da \tag{36}$$

$$- \langle \rho \rangle \iiiint [v_p(a_1) \cdot \nabla_{\xi_1} + v_p(a_2) \cdot \Delta_{\xi_2}] K^{(2)}(\boldsymbol{x} - \vec{\xi}_1,\ \boldsymbol{x} - \vec{\xi}_2;\ a_1, a_2)\, C_\omega^{(2)} d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, da_1 da_2$$

$$- (\rho_1 - \rho_0) \iiiint h(\boldsymbol{x} - \vec{\xi}_2;\ a_2) \vec{v}_p(a_1) \cdot \nabla_{\xi_1} K^{(1)}(\boldsymbol{x} - \vec{\xi}_1)\, C_\omega^{(I)} C_\omega^{(1)} d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, da_1 da_2$$

$$- (\rho_1 - \rho_0) \iiiiii h(\boldsymbol{x} - \vec{\xi}_3;\ a_3)(v_p(a_1) \cdot \nabla_{\xi_1} + v_p(a_2) \cdot \nabla_{\xi_2}) K^{(2)}(\boldsymbol{x} - \vec{\xi}_1,\ \boldsymbol{x} - \vec{\xi}_2;\ a_1,\ a_2)$$

$$\times C_\omega^{(2)}(\vec{\xi}_1,\ \vec{\xi}_2;\ a_1,\ a_2)\, C_\omega^{(2)}(\vec{\xi}_3;\ a_3)\, d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, d^3\vec{\xi}_3\, da_1 da_2 da_3 + \ \ldots,$$

$$\rho(v \cdot \nabla) v = \langle \rho \rangle u \cdot \nabla u + \langle \rho \rangle \iint K^{(1)}(\boldsymbol{x} - \vec{\xi};\ a) \cdot \nabla u C_\omega^{(I)} d^3\vec{\xi}\, da \tag{37}$$

$$+ \langle \rho \rangle u \cdot \iint \nabla_\xi K^{(1)}(\boldsymbol{x} - \vec{\xi};\ a)\, C_\omega^{(I)} d^3\xi\, da$$

$$+ \langle \rho \rangle \iiiint [u \cdot (\nabla_{\xi_1} + \nabla_{\xi_2}) + (\nabla \vec{u})^T] \cdot K^{(2)}(\boldsymbol{x} - \vec{\xi}_1,\ \boldsymbol{x} - \vec{\xi}_2;\ a_1,\ a_2)$$

$$\times C_\omega^{(2)}(\vec{\xi}_1,\ \vec{\xi}_2;\ a_1, a_2)\, d^3\vec{\xi}_1 d^3\xi_2 da_1 da_2 + (\rho_1 - \rho_0) u \cdot \nabla u \iint h(\boldsymbol{x} - \vec{\xi};\ a)\, C_\omega^{(I)} d^3\vec{\xi}\, da$$

$$+ (\rho_1 - \rho_0) \iiiint h(\boldsymbol{x} - \vec{\xi}_2; a_2)[u \cdot \nabla_\xi + (\nabla \vec{u})^T] K^{(1)}(\boldsymbol{x} - \vec{\xi}_1;\ a_1)$$

$$\times C_\omega^{(I)} C_\omega^{(I)} d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, da_1 da_2$$

$$+ (\rho_1 - \rho_0) \iiiiii [u \cdot (\nabla_{\xi_1} + \nabla_{\xi_2}) + (\nabla u)^T] K^{(2)}(\boldsymbol{x} - \xi_1,\ \boldsymbol{x} - \xi_2;\ a_1,\ a_2)$$

$$\times h(-\boldsymbol{x}\xi_3;\ a_3)\, C_\omega^{(2)}(\vec{\xi}_1,\ \vec{\xi}_2;\ a_1,\ a_2)\, C_\omega^{(I)}(\vec{\xi}_3;\ a_3)\, d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, d^3\vec{\xi}_3\, da_1 da_2 da_3 + \ \ldots,$$

(38)
$$\rho v=\langle\rho\rangle u+\langle\rho\rangle \iint K^{(1)}(x-\vec{\xi};\ a)\, C_{\omega}^{(I)} d^3\vec{\xi}\, da$$
$$+\langle\rho\rangle \iiiint K^{(2)}(x-\vec{\xi}_1, x-\vec{\xi}_2;\ a_1,\ a_2)\, C_{\omega}^{(2)} d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, da_1 da_2$$
$$+(\rho_1-\rho_0)\, u \iint h(x-\vec{\xi};\ a)\, C_{\omega}^{(I)} d^3\vec{\xi}\, da$$
$$+(\rho_1-\rho_0) \iiiint h(x-\vec{\xi}_2;\ a_2)\, K^{(1)}(x-\vec{\xi}_1;\ a_1)\, C_{\omega}^{(1)} C_{\omega}^{(I)} d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, da_1 da_2$$
$$+(\rho_1-\rho_0) \iiiiiii h(x-\vec{\xi}_3;\ a_3)\, K^{(2)}(x-\vec{\xi}_1,\ x-\vec{\xi}_2;\ a_1,\ a_2)\, C_{\omega}^{(I)}(\vec{\xi}_3;\ a_3)$$
$$\times C_{\omega}^{(2)}(\vec{\xi}_1,\ \vec{\xi}_2;\ a_1,\ a_2)\, d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, d^3\vec{\xi}_3\, da_1 da_2 da_3+\ldots$$

In the end the viscous term is given by

(39)
$$\mu\nabla v=\langle\mu\rangle\nabla u+\langle\mu\rangle \iint \nabla_{\xi} K^{(1)}(x-\vec{\xi};\ a)\, C_{\omega}^{(I)} d^3\vec{\xi}\, da$$
$$+(\mu_1-\mu_0)\nabla u \iint h(x-\vec{\xi};\ a)\, C_{\omega}^{(I)} d^3\vec{\xi}\, da$$
$$+\langle\mu\rangle \iiiint (\nabla_{\xi_1}+\nabla_{\xi_2})\, K^{(2)}(x-\vec{\xi}_1;\ x-\vec{\xi}_2;\ a_1,\ a_2)\, C_{\omega}^{(2)} d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, da_1 da_2$$
$$+(\mu_1-\mu_0) \iiiint h(x-\vec{\xi}_2;\ a_2)\, \nabla_{\xi} K^{(1)}(x-\xi_1,\ a_1)\, C_{\omega}^{(I)} C_{\omega}^{(1)} d^3\xi_1\, d^3\vec{\xi}_2\, da_1 da_2$$
$$+(\mu_1-\mu_0) \iiiiiii h(x-\vec{\xi}_3;\ a_3)(\nabla_{\xi_1}+\nabla_{\xi_2})\, K^{(2)}(x-\vec{\xi}_1,\ x-\vec{\xi}_2;\ a_1,\ a_2)$$
$$\times C_{\omega}^{(I)}(\vec{\xi}_3;\ a_3)\, C_{\omega}^{(2)}(\vec{\xi}_1,\ \vec{\xi}_2;\ a_1,\ a_2)\, d^3\vec{\xi}_1\, d^3\vec{\xi}_2\, d^3\vec{\xi}_3\, da_1 da_2 da_3+\ldots$$

4.2. Equations for the average quantities (Zeroth-order kernels)

The mere averaging of Eqs (27), (28) with formulae from the previous subsection acknowledged gives a system for the average characteristics $u, q, <\rho>$. The averaged equation of continuity reads

(40)
$$\frac{\partial\langle\rho\rangle}{\partial t}+\nabla\cdot(\langle\rho\rangle w)=0,$$

or, which is the same,

$$\frac{\partial\langle\rho\rangle}{\partial t}+\nabla\cdot(\langle\rho\rangle u)=-\nabla\cdot\Big[c(\rho_1-\rho_0)\int\int\frac{1}{V}h(x;\ a)\, K^{(1)}(x;\ a)\, P(a)\, d^3x da$$
(41)
$$+\frac{c^2}{V^2}\int\int\int\int h(x_1;\ a_1)\, K^{(1)}(x_2;\ a_2)\, R_{12} P(a_1)\, P(a_2)\, d^3x_1 d^3x_2 da_1 da_2\Big].$$

The averaged equations of Navier—Stokes yield the following:

(42)
$$\langle\rho\rangle\Big[\frac{\partial w}{\partial t}+(w\cdot\nabla)\, w\Big]=-\nabla q+\langle f\rangle$$
$$+\nabla\cdot\Big\{\langle\mu\rangle\nabla u+(\mu_1-\mu_0)\, c\int\int\frac{1}{V}h(x;\ a)\, K^{(1)}(x;\ a)\, P(a)\, d^3x da\Big\}.$$

One should be reminded here that derivatives in (40) — (42) are with respect to the global coordinates and they are retained since there are no derivatives with respect to the local coordinates.

Eqs (40), (42) and (29) form the set for estimating the zeroth-order kernels. This set, however, is not closed unless kernels $K^{(1)}$, $K^{(2)}$, $Q^{(1)}$ and $Q^{(2)}$ are specified.

4.3. Equations for the first-order kernels

Here and henceforth we shall consider only terms of order γ^2 or c^2 and less since we display the method only within an accuracy of $0(c^2)$, This is possible due to the virial properties of the PDSRR-Wiener expansion which are proved in the same manner as for Poisson — Wiener expansion in [6] and PDS-Wiener expansion in [8]. Viriality means that each higher-order functional contributes to the averaged quantities a value proportional to the corresponding degree of concentration c or intensity γ.

Following precisely the idea of [6] we multiply (17) and (18) by $C_{\omega}^{(1)}(\boldsymbol{o}, a)$ and take the ensemble average. The second equation gives

$$
\begin{aligned}
(43)\qquad & -(\rho_1-\rho_0)\,\gamma\,P(a)\,\boldsymbol{v}_p(a)\cdot\nabla h(\boldsymbol{x};\ a)\\
& +\gamma\,P(a)\,\nabla\cdot[\langle\rho\rangle K^{(1)}(\boldsymbol{x};\ a)+(\rho_1-\rho_0)\,h(\boldsymbol{x};\ a)\,K^{(1)}(\boldsymbol{x};\ a)]\\
& +(\rho_1-\rho_0)\,\gamma^2\,P(a)\iint \boldsymbol{v}_p(b)\cdot\nabla h(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,d^3\vec{\xi}\,P(b)\,db\\
& -\gamma^2\,P(a)\,\nabla\cdot\{[\langle\rho\rangle+(\rho_1-\rho_0)\,h(\boldsymbol{x};\ a)]\iint K^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\\
& \times d^3\xi\,db+(\rho_1-\rho_0)\,K^{(1)}(\boldsymbol{x};\ a)\iint h(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db\\
& +(\rho_1-\rho_0)\iint h(\boldsymbol{x}-\vec{\xi};\ b)\,K^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db\}\\
& -\gamma^2\,P(a)(\rho_1-\rho_0)\,\nabla\cdot\{\boldsymbol{u}\iint h(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\xi\,db\\
& -2\iint h(\boldsymbol{x}-\vec{\xi};\ b)\,K^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\vec{\xi};\ a,\ b)\,P(b)\,d^3\vec{\xi}\,db\},
\end{aligned}
$$

where the symmetry of kernel $\vec{K}^{(2)}$ with respect to the spatial coordinates is acknowledged. From the momentum equation it is obtained:

$$
\begin{aligned}
(44)\qquad & -\gamma\,P(a)\,[\langle\rho\rangle+(\rho_1-\rho_0)\,h(\boldsymbol{x};\ a)]\,\boldsymbol{v}_p(a)\cdot\nabla K^{(1)}(\boldsymbol{x};\ a)\\
& +\gamma^2 P(a)\iint[\langle\rho\rangle+(\rho_1-\rho_0)\,h(\boldsymbol{x}-\vec{\xi};\ b)]\,\boldsymbol{v}_p(b)\cdot\nabla_{\xi}K^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)R(\xi;\ a+b)P(b).d^3\xi\,db,\\
& -\gamma^2 P(a)\,\boldsymbol{v}_p(a)\cdot\nabla K^{(1)}(\boldsymbol{x};\ a)\iint h(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi},\ a+b)\,P(b)\,d^3\vec{\xi}\,db\\
& -\gamma^2\,P(a)\,h(\boldsymbol{x};\ a)\iint \boldsymbol{v}_p(b)\cdot\nabla_{\xi}K^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)\,R(\xi;\ a+b)\,P(b)\,d^3\vec{\xi}\,db\\
& -2(\rho_1-\rho_0)\,\gamma^2\,P(a)\iint h(\boldsymbol{x}-\xi;\ b)\,\boldsymbol{v}_p(b)\cdot(\nabla_{\xi_1}+\nabla_{\xi_2})\,K^{(2)}(\boldsymbol{x}-\vec{\xi};\ \boldsymbol{x};\ a,\ b)\\
& \times[1-R(\vec{\xi};\ a+b)]\,P(b)\,d^3\vec{\xi}\,db\\
& +(\rho_1-\rho_0)\,\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,\gamma\,P(a)[h(\boldsymbol{x};\ a)-\gamma\iint h(\boldsymbol{x}-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db]
\end{aligned}
$$

$$+\gamma P(a)[\langle\rho\rangle+(\rho_1-\rho_0)h(\boldsymbol{x};\ a)]\{\boldsymbol{u}\cdot\nabla \boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)+\boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)\cdot\nabla \boldsymbol{u}$$

$$-\gamma\iint[\boldsymbol{u}\cdot\nabla+(\nabla \boldsymbol{u})^T]\boldsymbol{K}^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db\}$$

$$-\gamma^2 P(a)(\rho_1-\rho_0)[\boldsymbol{u}\cdot\nabla+(\nabla \boldsymbol{u})^T\cdot]\boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)\iint h(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db$$

$$+2\gamma^2 P(a)(\rho_1-\rho_0)\iint h(\boldsymbol{x}-\vec{\xi};\ b)[1-R(\vec{\xi};\ a+b)][\boldsymbol{u}\cdot(\nabla_{\xi_1}+\nabla_{\xi_2})$$

$$+(\nabla \boldsymbol{u})^T\cdot]\boldsymbol{K}^{(2)}(\boldsymbol{x}-\vec{\xi},\ \boldsymbol{x};\ a,\ b)P(b)d^3\vec{\xi}\,db$$

$$-\gamma^2 P(a)(\rho_1-\rho_0)\iint h(\boldsymbol{x}-\vec{\xi};\ b)[\boldsymbol{u}.\nabla+(\nabla \boldsymbol{u})^T\cdot]\boldsymbol{K}^{(1)}(\boldsymbol{x}-\vec{\xi},\ b)R(\vec{\xi};a+b)P(b)d^3\vec{\xi}db$$

$$=-\gamma P(a)[\nabla Q^{(1)}(\boldsymbol{x};\ a)-\gamma\iint Q^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db$$

$$+\gamma P(a)[(\boldsymbol{f}_1-\boldsymbol{f}_0)h(\boldsymbol{x};\ a)-\gamma\iint(\boldsymbol{f}_1-\boldsymbol{f}_0)h(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\xi\,db$$

$$+\gamma P(a)\nabla\cdot\{[\langle\mu\rangle+(\mu_1-\mu_0)h(\boldsymbol{x};\ a)][\nabla \boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)$$

$$-\gamma\iint\nabla_\xi \boldsymbol{K}^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db]$$

$$+(\mu_1-\mu_0)\nabla \boldsymbol{u}[h(\boldsymbol{x};\ a)-\gamma\iint h(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db]$$

$$-(\mu_1-\mu_0)\gamma\nabla \boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)\iint h(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db$$

$$-(\mu_1-\mu_0)\gamma\iint h(\vec{\boldsymbol{x}}-\vec{\xi};\ b)\nabla_\xi \boldsymbol{K}^{(1)}(\boldsymbol{x}-\vec{\xi};\ b)R(\vec{\xi};\ a+b)P(b)d^3\vec{\xi}\,db$$

$$+(\mu_1-\mu_0)2\gamma\iint h(\boldsymbol{x}-\vec{\xi};\ b)(\nabla_{\xi_1}+\nabla_{\xi_2})\boldsymbol{K}^{(2)}(\boldsymbol{x}-\vec{\xi};\ \boldsymbol{x};\ a,\ b)P(b)d^3\vec{\xi}\,db\}.$$

4.4. Equations for the second-order kernels

Equations for the second-order kernels are obtained by multiplying (17) and (18) by $C_\omega^{(2)}(\boldsymbol{o}, \boldsymbol{z}; a, b)$ and taking the average:

(45)
$$\gamma^2 P(a)P(b)\nabla\cdot\{2\langle\rho\rangle\boldsymbol{K}^{(2)}(\boldsymbol{x};\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)$$

$$+(\rho_1-\rho_0)\cdot 2[h(\boldsymbol{x};\ a)+h(\boldsymbol{x}-\boldsymbol{z};\ b)]\boldsymbol{K}^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)$$

$$+(\rho_1-\rho_0)[1-R(\boldsymbol{z};\ a+b)][h(\boldsymbol{x};\ a)\boldsymbol{K}^{(1)}(\boldsymbol{x}-\vec{\boldsymbol{z}};\ b)+h(\boldsymbol{x}-\vec{\boldsymbol{z}};\ b)\boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)]\}=0,$$

(46)
$$-(\rho_1-\rho_0)2[h(\boldsymbol{x};\ a)+h(\boldsymbol{x}-\boldsymbol{z};\ b)][v_p(a)+v_p(b)]\cdot\nabla \boldsymbol{K}^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)$$

$$-(\rho_1-\rho_0)[h(\boldsymbol{x};\ a)v_p(b)\cdot\nabla\boldsymbol{K}^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)v_p(a)\cdot\nabla\boldsymbol{K}^{(1)}(\boldsymbol{x};a)][1-R(\boldsymbol{z};a+b)]$$

$$+2\langle\rho\rangle\boldsymbol{K}^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)\cdot\nabla \boldsymbol{u}$$

$$+(\rho_1-\rho_0)[h(\boldsymbol{x};\ a)\boldsymbol{K}^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)\boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)]\cdot\nabla \boldsymbol{u}[1-R(\boldsymbol{z};\ a+b)]$$

$$+(\rho_1-\rho_0)\boldsymbol{u}\cdot[b(\boldsymbol{x};\ a)\boldsymbol{K}^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)\boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)][1-R(\boldsymbol{z};\ a+b)]$$

$$+(\rho_1-\rho_0)2[h(\boldsymbol{x};\ a)+h(\boldsymbol{x}-\boldsymbol{z};\ b)][\boldsymbol{u}\cdot\nabla+(\nabla \boldsymbol{u})^T\cdot]\boldsymbol{K}^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)$$

$$=-2\nabla Q^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)+\nabla\cdot\{\nabla \boldsymbol{K}^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)$$

$$\times[\langle\mu\rangle+(\mu_1-\mu_0)[h(\boldsymbol{x},\ a)+h(\boldsymbol{x}-\boldsymbol{z},\ b)]]$$

$$+(\mu_1-\mu_0)[1-R(\boldsymbol{z};\ a+b)][h(\boldsymbol{x},\ a)\nabla \boldsymbol{K}^{(1)}(\boldsymbol{x}-\boldsymbol{z},\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)\nabla \boldsymbol{K}^{(1)}(\boldsymbol{x};\ a)]\}.$$

5. DECOUPLING THE HIERARCHY OF EQUATIONS

The presence of the small parameter c (or which is the same — γ) in the governing set of equations for kernels enables one to seek for a solution in the form of asymptotic expansion with respect to the small parameter:

$$
\begin{aligned}
&K^{(1)}=K_0^{(1)}+\gamma K_1^{(1)}+\ldots, && K^{(2)}=K_0^{(2)}+\gamma K_1^{(2)}+\ldots,\\
&Q^{(1)}=Q_0^{(1)}+\gamma Q_1^{(1)}+\ldots, && Q^{(2)}=Q_0^{(2)}+\gamma Q_1^{(2)}+\ldots
\end{aligned}
\tag{47}
$$

Introducing these formulae into equations we get two closed systems for kernels $K_0^{(1)}$, $K_0^{(2)}$, $Q_0^{(1)}$, $Q_0^{(2)}$:

$$
\begin{aligned}
&-(\rho_1-\rho_0)\,v_p(a)\cdot\nabla h(\boldsymbol{x};\ a)\\
&+\nabla\cdot[\rho_0 K_0^{(1)}(\boldsymbol{x};\ a)+(\rho_1-\rho_0)h(\boldsymbol{x};\ a)\,K_0^{(2)}(\boldsymbol{x};\ a)]=0,
\end{aligned}
\tag{48}
$$

$$
\begin{aligned}
&-\rho_0 v_p(a)\cdot\nabla K_0^{(1)}(\boldsymbol{x};\ a)-(\rho_1-\rho_0)\,v_p(a)\cdot\nabla K_0^{(1)}(\boldsymbol{x};\ a)\,h(\boldsymbol{x};\ a)\\
&+(\rho_1-\rho_0)\,\boldsymbol{u}\cdot\nabla uh(\boldsymbol{x};\ a)+[\rho_0+(\rho_1-\rho_0)\,h(\boldsymbol{x};\ a)][\boldsymbol{u}\cdot\nabla+(\nabla\boldsymbol{u})^T\cdot]\,K_0^{(1)}(\boldsymbol{x};\ a)\\
&=-\nabla Q_0^{(1)}(\boldsymbol{x};\ a)+(\boldsymbol{f}_1-\boldsymbol{f}_0)\,h(\boldsymbol{x};\ a)\\
&+\nabla\cdot\{[\mu_0+(\mu_1-\mu_0)\,h(\boldsymbol{x};\ a)]\,\nabla K_0^{(1)}(\boldsymbol{x};\ a)+(\mu_1-\mu_0)\,\nabla uh(\boldsymbol{x};\ a)\},
\end{aligned}
\tag{49}
$$

$$
\begin{aligned}
&2\rho_0\nabla\cdot K_0^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a;\ b)+(\rho_1-\rho_0)\nabla\cdot\{2\,[h(\boldsymbol{x};\ a)+h(\boldsymbol{x}-\boldsymbol{z};\ b)]\,K_0^{(2)}\\
&+[1-R(\boldsymbol{z};\ a+b)][h(\boldsymbol{x};\ a)\,K_0^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)\,K_0^{(1)}(\boldsymbol{x};\ a)]\}=0,
\end{aligned}
\tag{50}
$$

$$
\begin{aligned}
&-(\rho_1-\rho_0)[1-R(\boldsymbol{z};\ a+b)][h(\boldsymbol{x};\ a)\,v_p(a)\cdot\nabla K_0^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)\\
&+h(\boldsymbol{x}-\boldsymbol{z};\ b)\,v_p(b)\cdot\nabla K_0^{(1)}(\boldsymbol{x};\ a)]+2\rho_0 K_0^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)\cdot\nabla\boldsymbol{u}\\
&-2(\rho_1-\rho_0)[h(\boldsymbol{x};\ a)+h(\boldsymbol{x}-\boldsymbol{z};\ b)][v_p(a)+v_p(b)]\cdot\nabla K_0^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)\\
&+(\rho_1-\rho_0)[1-R(\boldsymbol{z};\ a+b)][h(\boldsymbol{x},\ a)\,K_0^{(1)}(\vec{\boldsymbol{x}}-\vec{\boldsymbol{z}};\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)\,K_0^{(1)}(\boldsymbol{x};\ a)]\cdot\nabla\boldsymbol{u}\\
&+(\rho_1-\rho_0)[1-R(\boldsymbol{z};\ a+b)]\,\boldsymbol{u}\cdot[h(\boldsymbol{x};\ a)\,K_0^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)+h(\boldsymbol{x}-\boldsymbol{z};\ b)\,K_0^{(1)}(\boldsymbol{x};\ a)]\\
&+(\rho_1-\rho_0)\,2\,[h(\boldsymbol{x};\ a)+h(\boldsymbol{x}-\boldsymbol{z};\ b)][\boldsymbol{u}\cdot\nabla+(\nabla\boldsymbol{u})^T\cdot]\,K_0^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)\\
&=-2\nabla Q_0^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z},\ a,\ b)+\nabla\cdot\{\nabla K_0^{(2)}(\boldsymbol{x},\ \boldsymbol{x}-\boldsymbol{z};\ a,\ b)[\mu_0+(\mu_1-\mu_0)\,h(\boldsymbol{x};\ a)\\
&+(\mu_1-\mu_0)\,h(\boldsymbol{x}-\boldsymbol{z};\ b)]+(\mu_1-\mu_0)[1-R(\boldsymbol{z};\ a+b)][h(\boldsymbol{x};\ a)\,\nabla K_0^{(1)}(\boldsymbol{x}-\boldsymbol{z};\ b)\\
&+h(\boldsymbol{x}-\boldsymbol{z};\ b)\,\nabla K_0^{(1)}(\boldsymbol{x};\ a)]\}.
\end{aligned}
\tag{51}
$$

As will be seen further the interpretation of kernels $K_0^{(1)}$, $Q_0^{(1)}$ is very important. They can be viewed as the velocity and pressure created by the motion of a single particle in addition to the flow velocity $\boldsymbol{u}$.

For kernels $K_1^{(1)}$ and $Q_1^{(1)}$ we have

$$
\begin{aligned}
&\nabla\cdot\{(\rho_1-\rho_0)\,V_a K_0^{(1)}(\boldsymbol{x};\ a)+\rho_0 K_1^{(1)}(\boldsymbol{x};\ a)+(\rho_1-\rho_0)\,h(\boldsymbol{x};\ a)\,K_1^{(1)}(\boldsymbol{x};\ a)\}\\
&+(\rho_1-\rho_0)\iint v_p(a)\cdot\nabla h(\boldsymbol{x}-\vec{\boldsymbol{\xi}};\ b)\,R(\vec{\boldsymbol{\xi}};\ a+b)\,P(b)\,d^3\vec{\boldsymbol{\xi}}\,db\\
&-\nabla\cdot\iint\{[\rho_0+(\rho_1-\rho_0)\,h(\boldsymbol{x};\ a)]\,K_0^{(1)}(\boldsymbol{x}-\vec{\boldsymbol{\xi}};\ b)+(\rho_1-\rho_0)[K_0^{(1)}(\boldsymbol{x};\ a)\,h(\boldsymbol{x}-\vec{\boldsymbol{\xi}};\ b)
\end{aligned}
\tag{52}
$$

$$+h(x-\vec{\xi};\ b)\,K_0^{(1)}(x-\vec{\xi};\ b)]\}\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db$$

$$-(\rho_1-\rho_0)\,\nabla\cdot\iint\{uh(x;\ a)-2h(x-\vec{\xi};\ b)\,K_0^{(2)}(x,\ x-\vec{\xi};\ a,\ b)\}\,P(b)\,d^3\xi\,db=0, \tag{53}$$

$$-(\rho_1-\rho_0)\,V_a v_p(a)\cdot\nabla K_0^{(1)}(x;\ a)-[\rho_0+(\rho_1-\rho_0)h(x;\ a)]\,v_p(a)\cdot\nabla K_1^{(1)}(x;\ a)$$

$$+\rho_0\iint v_p(b)\cdot\nabla K_0^{(1)}(x-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db$$

$$+(\rho_1-\rho_0)\iint[h(x-\vec{\xi};\ b)\,v_p(b)\cdot\nabla K_0^{(1)}(x-\vec{\xi};\ b)$$

$$+h(x-\vec{\xi};\ b)\,v_p(a)\cdot\nabla K_0^{(1)}(x;\ a)$$

$$+h(x;\ a)\,v_p(b)\cdot\nabla K_0^{(1)}(x-\vec{\xi};\ b)]\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db$$

$$-2(\rho_1-\rho_0)\iint h(x-\vec{\xi};\ b)\,v_p(b)\cdot\nabla K^{(2)}(x-\vec{\xi},\ x;\ a,\ b)[1-R(\vec{\xi};\ a+b)]P(b)d^3\vec{\xi}\,db$$

$$-(\rho_1-\rho_0)\,u\cdot\nabla u\iint h(x-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\xi\,db$$

$$+(\rho_1-\rho_0)\,V_a[u\cdot\nabla K_0^{(1)}(x;\ a)+K_0^{(1)}(x;\ a)\cdot\nabla u]$$

$$+[\rho_0+(\rho_1-\rho_0)h(x;\ a)][u\cdot\nabla K_1^{(1)}(x;\ a)+K_1^{(1)}(x;\ a)\cdot\nabla u]$$

$$-[\rho_0+(\rho_1-\rho_0)h(x;\ a)]\iint[u\cdot\nabla+(\nabla u)^T\cdot]K_0^{(1)}(x-\vec{\xi};\ b)\,R(\vec{\xi};a+b)\,P(b)\,d^3\vec{\xi}\,db$$

$$-(\rho_1-\rho_0)[u\cdot\nabla+(\nabla u)^T\cdot]\,K_0^{(1)}(x;\ a)\iint h(x-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,R(b)\,d^3\vec{\xi}\,db$$

$$-(\rho_1-\rho_0)\iint h(x-\vec{\xi};\ b)[u\cdot\nabla+(\nabla u)^T\cdot]\,K_0^{(1)}(x-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db$$

$$+(\rho_1-\rho_0)\iint h(x-\vec{\xi};b)[1-R(\vec{\xi};\ a+b)][u.\nabla+(\nabla u)^T.]K_0^{(2)}(x,\ x-\vec{\xi};\ a,\ b)\,P(b)\,d^3\vec{\xi}\,db$$

$$=-\nabla Q_1^{(1)}(x;\ a)+\iint Q^{(1)}(x-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db$$

$$+\nabla\cdot[\mu_0 V_a\nabla K_0^{(1)}(x;\ a)]+\nabla\cdot\{[\mu_0+(\mu_1-\mu_0)h(x,\ a)]\nabla K_1^{(1)}(x;\ a)\}$$

$$-\nabla\cdot\iint\nabla K_0^{(1)}(x-\vec{\xi};\ b)R(\vec{\xi};\ a+b)\ [\mu_0+(\mu_1-\mu_0)h(x;\ a)]\,P(b)\,d^3\vec{\xi}\,db$$

$$-\nabla\cdot[(\mu_1-\mu_0)\nabla u\iint h(x-\vec{\xi};\ b)\,R(\vec{\xi},\ a+b)\,P(b)\,d^3\vec{\xi}\,db]$$

$$-(\mu_1-\mu_0)\nabla\cdot\iint[h(x-\vec{\xi};\ b)\nabla K_0^{(1)}(x;\ a)+h(x;\ a)\nabla K_0^{(1)}(x-\vec{\xi};\ b)]\,d^3\vec{\xi}\,P(b)\,db$$

$$+2(\mu_1-\mu_0)\nabla\cdot\iint h(x-\vec{\xi};\ b)\nabla K_0^{(2)}(x-\vec{\xi};\ \ x;\ a,\ b)\,P(b)\,d^3\vec{\xi}\,db$$

$$-(f_1-f_0)\iint h(x-\vec{\xi};\ b)\,R(\vec{\xi};\ a+b)\,P(b)\,d^3\vec{\xi}\,db.$$

6. GOVERNING EQUATIONS FOR THE PARTICULATE PHASE

The particulate phase occupies the random volume

$$V_r=\iint h(x-\vec{\xi};\ a)\omega(\vec{\xi};\ a)d^3\vec{\xi}da=\iint h(x-\vec{\xi};\ a)[c_\omega^{(1)}(\vec{\xi};\ a)+\gamma]d^3\vec{\xi}\,da \tag{54}$$

and the averaged properties connected with this phase are obtained through multiplying by V_r and taking the average, see in this instance Eq. (30). On this base governing equations for the particulate phase can be derived. Confining ourselves to the first order of approximation $o(c)$ we retain only first-order kernels in functional representation and then multiply the two sides of (17) by V_r from (54) and take the ensemble average. Different terms of (17) contribute the following:

$$\left\langle \rho \frac{\partial v}{\partial t} V_r \right\rangle = (\rho_1 - \rho_0) \frac{\partial u}{\partial t} \gamma V$$

$$-\gamma \rho_1 \iint v_p(a) \cdot \nabla K_0^{(1)}(\vec{\xi};\ a)\, h(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da$$

$$\langle \rho (v \cdot \nabla) v V_r \rangle = \gamma V (\rho_1 - \rho_0) u \cdot \nabla u$$

$$+\rho_1 \gamma \iint h(\vec{\xi};\ a)[u \cdot \nabla + (\nabla u)^T \cdot] K_0^{(1)}(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da$$

$$\langle \Delta p V_r \rangle = \gamma \iint Q_0^{(1)}(\vec{\xi};\ a)\, h(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da$$

$$\langle f V_r \rangle = \gamma V (f_1 - f_0)$$

$$\langle \nabla \cdot (\mu \nabla v) V_r \rangle = \gamma \iint \nabla \cdot [\mu_1 \nabla K_0^{(1)}(\vec{\xi};\ a)]\, h(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da$$

$$+\gamma (\mu_1 - \mu_0) \iint \nabla \cdot [\nabla u h(\vec{\xi};\ a)]\, h(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da.$$

Being consistent with the stated above assumptions we discard the gradient of the averaged flow ∇u in comparison with the gradient of the flow created by an inclusion. Finally, the governing equations for the particulate phase are

$$(55)\quad (\rho_1 - \rho_0) c \left[\frac{\partial u}{\partial t} + u \cdot \nabla u\right] - \gamma \rho_1 \iint h(\vec{\xi};\ a)\, v_p(a) \cdot \nabla K_0^{(1)}(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da$$

$$+\gamma \rho_1 \iint h(\vec{\xi};\ a)[(\nabla u)^T \cdot + u \cdot \nabla] K_0^{(1)}(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da$$

$$= -\gamma \iint \nabla Q_0^{(1)}(\vec{\xi};\ a)\, h(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da + c(f_1 - f_0)$$

$$+\gamma \iint \nabla \cdot [\mu_1 \nabla K_0^{(1)}(\vec{\xi};\ a)]\, h(\vec{\xi};\ a)\, P(a)\, d^3\vec{\xi}\, da.$$

The left-hand side of the last equation is the averaged acceleration of the particulate phase. The last term in the right-hand side is the external body force acting upon the particulate phase and the rest of the terms in the right-hand side are responsible for the interaction between the particulate and the continuous phases. Let us denote this force by F_{c-p}. Then the volume integrals are recast into surface integrals

$$(56)\qquad F_{c-p} = \gamma \int \!\!\!\!\!\!\bigcirc\!\!\!\!\!\!\bigcirc [-Q_0^{(1)}(\vec{\xi};\ a)\, n + \mu_1 n \cdot \nabla K_0^{(1)}(\vec{\xi};\ a)]\, ds P(a)\, da,$$

where n is the normal to the surface of a sphere vector and ds is an infinitesimal area element from that surface.

7. THE FIRST-ORDER SOLUTION

As has been shown above the complete description of the averaged motion of an emulsion can be obtained within the order of accuracy $o(c)$ provided the kernels $K_0^{(1)}$ and $Q_0^{(1)}$ are specified. These are governed by the Eqs.(48) and (49) which

contain discontinuous coefficients. So the solutions are to be understood in the sense of generalized functions. The latter means that we seek for kernels which are continuous functions along with its normal derivatives at the surface of a particle.

Within the adopted order of approximation we arrive to the following two sets of equations:

$$\left|\begin{array}{l} \nabla\cdot[\rho_1 K_0^{(1)}(x;\ a)]=0, \\ \rho_1[u-v_p(a)]\cdot\nabla K_0^{(1)}(x;\ a)=-\nabla Q_0^{(1)}(x;\ a) \\ +\nabla\cdot[\mu_1\nabla K_0^{(1)}(x;\ a)]+(f_1-f_0) \text{ for } |x|\leqq a \end{array}\right. \tag{57}$$

and

$$\left|\begin{array}{l} \nabla\cdot[\rho_0 K_0^{(1)}(x;\ a)]=0, \\ \rho_0[u-v_p(a)]\cdot\nabla K_0^{(1)}(x;\ a)=-\nabla Q_0^{(1)}(x;\ a)+\nabla\cdot[\mu_0\nabla K_0^{(1)}(x;\ a)] \\ \qquad\qquad \text{for } |\vec{x}|>a \end{array}\right. \tag{58}$$

with the following conditions:

$$K_0^{(1)}(x;\ a)\big|_{\substack{|x|\to a \\ |x|<a}} = K_0^{(1)}(x;\ a)\big|_{\substack{|x|\to a, \\ |x|>a}} \tag{59}$$

$$\left[-Q_0^{(1)}(x;\ a)n+\mu_1\frac{\partial}{\partial n}K_0^{(1)}(x;\ a)\right]_{\substack{|x|\to a \\ |x|<a}} = \left[-Q_0^{(1)}(x;\ a)\vec{n}\right. \tag{60}$$

$$\left.+\mu_0\frac{\partial}{\partial n}K_0^{(1)}(x;\ a)\right]_{\substack{|x|\to a \\ |x|>a,}}$$

$$Q_0^{(1)},\ K_0^{(1)}\to 0,\ |x|\to\infty. \tag{61}$$

The inertial terms in the left-hand sides of the momentum equations in (57) and (58) are discarded because of the assumption of slow drift velocity of particles, and more specificly — of small drift Reynolds number. As a result the problem (57) —(61) reduces to the well known problem of slow motion of a drop of viscosity μ_1 through a resting liquid of viscosity μ_0 under the action of a driving force f_1—f_0. Kernels $K_0^{(1)}$, $Q_0^{(1)}$ are the relative velocity and pressure in the moving coordinate system originated at the drop centre.

The solution of the problem (57)—(61) is known (see in this instance [14, 15] and one can evaluate (56) as follows:

$$F_{c-p}=-\gamma\int P(a)\left[6\pi\mu_0\, a\,(v_p-u)\,\frac{\mu_1+{}^2/_3\mu_0}{\mu_1+\mu_0}\right]da \tag{62}$$

or

$$F_{c-p}=-6\pi\mu_0\gamma\,\frac{\mu_1+{}^2/_3\mu_0}{\mu_1+\mu_0}\,[\int P(a)\,a v_p(a)\,da-\langle a\rangle u], \tag{63}$$

where $\langle a\rangle$ is the mean value of the radius of a sphere. Here is to be noted that $\frac{4\pi}{3}\langle a\rangle^3$ is not equal to volume fraction c. In the terms of concentration (63) renders to

(64) $$F_{c-p} = -6\pi\mu_0 c \frac{\mu_1 + {}^2/_3\mu_0}{\mu_1+\mu_0} \frac{\int a v_p(a) P(a)\, da - u\langle a\rangle}{\frac{4}{3}\pi\langle a^3\rangle}.$$

It is instructive to calculate the weighted velocity of particulate phase from (30) and (32). It is easily shown that

(65) $$v_1 = u(x, t) + \frac{\int V_a (v_p - u) P(a)\, da}{V} = \frac{\int a^2 v_p P(a)\, da}{\langle a^3\rangle}.$$

Only for a monodisperse suspension when probability distribution of the radius is delta function $P(a)=\delta(a-a_0)$ (64) yields

(66) $$F_{c-p} = -6\pi\mu_0 \frac{c}{a_0^2} [v_1 - u] \frac{\mu_1 + {}^2/_3\mu_0}{\mu_1+\mu_0},$$

which is usually employed in phenemenological theories.

CONCLUSION

On the basis of the Volterra — Wiener functional expansion the flow of dilute polydisperse emulsion is treated. It is found that the force of interaction between the particulate and continuous phase is given by (64) which depends solely on the probability distribution $P(a)$.

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