NUMERICAL GRID GENERATION
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ORTHOGONAL COORDINATE MESHES WITH MANAGEABLE JACOBIAN

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INTRODUCTION

Recently, the problem of numerical generating curvilinear coordinate meshes has received a vast exploration because of its outstanding importance in solving the partial differential equations of continuous mechanics. The major advantage of this method is that the boundary of the region becomes a coordinate line which decidedly simplifies the numerical schemes for approximate integration of boundary value problems. In some sence the method of adapted coordinates is an alternative to the method of finite elements.

In two dimension the most natural way to create curvilinear meshes was, perhaps, the inversion of conformal mapping[1,2]. This approach was generalized by means of variational principle[3]. The coordinates obtained in this way, however, were not orthogonal in general and the Jacobian assumed in some cases incomfortable values approaching zero or infinity. It was due to the rigid prescription of the boundary points. The orthogonality has been restored only after reducing the conditions on the boundary points to the natural ones for a conformal mapping[4,5]. In present note an other approach ensuring the Jacobian to be a priory prescribed function is attempted.

GOVERNING EQUATIONS

Consider a region D in the plane Oxy with boundary δD. The transformation

(1) $\quad x = x(\xi,\eta) \quad$ and $\quad y = y(\xi,\eta)$

relates D to a region D' of the plane $O\xi\eta$ (see fig.1). Respectively ξ = const and η = const represent two families of curves in Oxy which are desired to be orthogonal. Then

(2) $\quad x_\xi x_\eta + y_\xi y_\eta = 0$

where subscribing denotes a partial differentiation.

Since eq.(2) is not enough to define the two functions x and y one more relation is needed. In present work it is chosen to impose a condition on Jacobian, namely

(3) $$x_\xi y_\eta - x_\eta y_\xi = f^{-1}(x,y) = F^{-1}(\xi,\eta)$$

where f(x,y) is certain arbitrary function. Respectively $F(\xi,\eta) = f(x(\xi,\eta),y(\xi,\eta))$. In the case when $f(x,y)=1$ the coordinates obtained can be named "uniform".

The boundary conditions for the system (2),(3) are:

(4) $$\Phi(x,y) = 0 \quad \text{at} \quad (x,y)\in \delta D, \text{ i.e. at } \quad (\xi,\eta)\in \delta D'$$

where $\Phi(x,y)$ is the analytical representation of the curves which comprise δD. For complete definiteness it is necessary to prescribe the rule of correspondence between the corners A,B,C,E and A',B', C',E' (see fig.1).

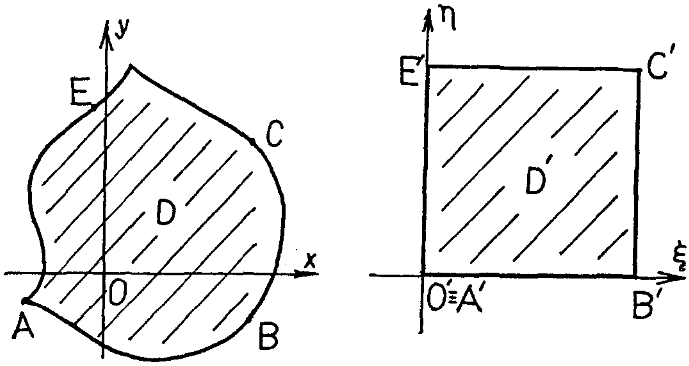

Fig.1. Geometry of the problem

The boundary value problem (2),(3) and (4) in its present form is very inconvenient for direct numerical integration. In addition its correctness is not obvious. However, (2) and (3) are readily transformed to

(5) $$f(x,y)x_\xi(x_\eta^2 + y_\eta^2) = y_\eta \quad \text{and} \quad f(x,y)y_\xi(x_\eta^2 + y_\eta^2) = -x_\eta$$

which is certain nonlinear generalization of the Cauchy-Riemann problem. This is which assures one that the problem is of elliptic type and could be expected to possess a solution under the boundary conditions (4).

After obvious manipulations one finds

$$(6) \qquad f(x,y)(x_\xi^2 + y_\xi^2)(x_\eta^2 + y_\eta^2) = 1$$

and then eqs.(5) are easily transformed into following

$$(7) \qquad f(x,y)y_\eta(x_\xi^2 + y_\xi^2) = x_\xi \quad \text{and} \quad f(x,y)x_\eta(x_\xi^2 + y_\xi^2) = -y_\xi$$

The last set of equations is not independent of (5) and it is outlined only for further convenience.

Since sufficiently fast numerical method for direct integration of Cauchy-Riemann problem is not known it is convenient to render (5) and (7) into more standart form, namely:

$$(8) \qquad \frac{\partial}{\partial\xi}f(x,y)H_\eta^2 \frac{\partial x}{\partial\xi} + \frac{\partial}{\partial\eta}f(x,y)H_\xi^2 \frac{\partial x}{\partial\eta} = 0,$$

$$(9) \qquad \frac{\partial}{\partial\xi}f(x,y)H_\eta^2 \frac{\partial y}{\partial\xi} + \frac{\partial}{\partial\eta}f(x,y)H_\xi^2 \frac{\partial y}{\partial\eta} = 0,$$

where $H^2_\xi = x_\xi^2 + y_\xi^2$ and $H_\eta^2 = x_\eta^2 + y_\eta^2$ and the squares of the coordinate scale factors. This system of two second-order differential equations is equivalent to a boundary value problem of Cauchy-Riemann type only when one of the first-order equations is satisfied at the boundary δD. In present work it is assumed to employ the orthogonality condition (2) as a boundary condition for (8),(9) along with (4).

METHOD OF SOLUTION

In order to solve the set of nonlinear elliptic equations (8), (9) with the boundary conditions (2),(4) the method of convergence[6] is chosen. Therefore derivatives of x and y with respect to some fictitious time t are added into (8) and (9) respectively,

$$(10) \qquad -\frac{\partial x}{\partial t} + \frac{\partial}{\partial\xi}fH_\eta^2 \frac{\partial x}{\partial\xi} + \frac{\partial}{\partial\eta}fH_\xi^2 \frac{\partial x}{\partial\eta} = 0,$$

$$(11) \qquad -\frac{\partial y}{\partial t} + \frac{\partial}{\partial\xi}fH_\eta^2 \frac{\partial y}{\partial\xi} + \frac{\partial}{\partial\eta}fH_\xi^2 \frac{\partial y}{\partial\eta} = 0.$$

The finite-difference scheme is constructed on the basis of the alternating-direction method[6]:

$$(12)\qquad \frac{\bar{x}_{ij}-x^{n}_{ij}}{0.5\tau} = \Lambda^{n}_{\xi}\bar{x} + \Lambda^{n}_{\eta}x^{n} \quad , \quad \frac{\bar{y}_{ij}-y^{n}_{ij}}{0.5\tau} = \Lambda^{n}_{\xi}\bar{y} + \Lambda^{n}_{\eta}y^{n} \;;$$

$$(13)\qquad \frac{x^{n+1}_{ij}-\bar{x}_{ij}}{0.5\tau} = \Lambda^{n}_{\xi}\bar{x} + \Lambda^{n}_{\eta}x^{n+1} \quad , \quad \frac{y^{n+1}_{ij}-\bar{y}_{ij}}{0.5\tau} = \Lambda^{n}_{\xi}\bar{y} + \Lambda^{n}_{\eta}y^{n+1} \;.$$

Here $x_{ij}=x(\xi_i,\eta_j)$, $\xi_i=(i-1)h_\xi-0.5h_\xi$, $\eta_j=(j-1)h_\eta-0.5h_\eta$, where h_ξ and h_η are the uniform grid spacings in and directions respectively. The differential operators are approximated with second order of approximation as follows:

$$h^2_\xi\Lambda^n_\xi\varphi = (fH^2_\eta)^n_{i+\frac{1}{2},j}\,\varphi_{i+1,j} + (fH^2_\eta)^n_{i-\frac{1}{2},j}\,\varphi_{i-1,j}$$

$$- \left[(fH^2_\eta)^n_{i+\frac{1}{2},j} + (fH^2_\eta)^n_{i-\frac{1}{2},j}\right]\varphi_{i,j}$$

$$h^2_\eta\Lambda^n_\eta\varphi = (fH^2_\xi)^n_{i,j+\frac{1}{2}}\,\varphi_{i,j+1} + (fH^2_\xi)^n_{i,j-\frac{1}{2}}\,\varphi_{i,j-1}$$

$$- \left[(fH^2_\xi)^n_{i,j+\frac{1}{2}} + (fH^2_\xi)^n_{i,j-\frac{1}{2}}\right]\varphi_{i,j}$$

where $\varphi_{i,j}$ is either x or y.

The above scheme is completed with a second-order approximation of the boundary conditions (2) and (4):

$$(x^n_{1,j+1} - x^n_{1,j-1} + x^n_{2,j+1} - x^n_{2,j-1})(\bar{x}_{2,j} - \bar{x}_{1,j})$$

$$+ (y^n_{1,j+1} - y^n_{1,j-1} + y^n_{2,j+1} - y^n_{2,j-1})(\bar{y}_{2,j} - \bar{y}_{1,j}) = 0 \;,$$

(12^a)

$$\frac{\bar{x}_{1,j} + \bar{x}_{2,j}}{2}\frac{\partial F^n}{\partial x} + \frac{\bar{y}_{1,j} + \bar{y}_{2,j}}{2}\frac{\partial F^n}{\partial y} = - F^n$$

$$+ \frac{x^n_{1,j} + x^n_{2,j}}{2}\frac{\partial F^n}{\partial x} + \frac{y^n_{1,j} + y^n_{2,j}}{2}\frac{\partial F^n}{\partial y} \;.$$

$$(x^n_{M,j+1} - x^n_{M,j-1} + x^n_{M+1,j+1} - x^n_{M+1,j-1})(\bar{x}_{M+1,j} - \bar{x}_{M,j})$$

$$+ (y^n_{M,j+1} - y^n_{M,j-1} + y^n_{M+1,j+1} - y^n_{M+1,j-1})(\bar{y}_{M+1,j} - \bar{y}_{M,j}) = 0 \, ,$$

(12[b])

$$\frac{\bar{x}_{M,j} + \bar{x}_{M+1,j}}{2} \frac{\partial F^n}{\partial x} + \frac{\bar{y}_{M,j} + \bar{y}_{M+1,j}}{2} \frac{\partial F^n}{\partial y} = - F^n$$

$$+ \frac{x^n_{M,j} + x^n_{M+1,j}}{2} \frac{\partial F^n}{\partial x} + \frac{y^n_{M,j} + y^n_{M+1,j}}{2} \frac{\partial F^n}{\partial y} \, .$$

$$(x^n_{i+1,1} - x^n_{i-1,1} + x^n_{i+1,2} - x^n_{i-1,2})(x^{n+1}_{i,2} - x^{n+1}_{i,1})$$

$$+ (y^n_{i+1,1} - y^n_{i-1,1} + y^n_{i+1,2} - y^n_{i-1,2})(y^{n+1}_{i,2} - y^{n+1}_{i,1}) = 0 \, ,$$

(13[a])

$$\frac{x^{n+1}_{i,1} + x^{n+1}_{i,2}}{2} \frac{\partial F^n}{\partial x} + \frac{y^{n+1}_{i,1} + y^{n+1}_{i,2}}{2} \frac{\partial F^n}{\partial y} = - F^n$$

$$+ \frac{x^n_{i,1} + x^n_{i,2}}{2} \frac{\partial F^n}{\partial x} + \frac{y^n_{i,1} + y^n_{i,2}}{2} \frac{\partial F^n}{\partial y} \, .$$

$$(x^n_{i+1,N} - x^n_{i-1,N} + x^n_{i+1,N+1} - x^n_{i-1,N+1})(x^{n+1}_{i,N+1} - x^{n+1}_{i,N})$$

$$+ (y^n_{i+1,N} - y^n_{i-1,N} + y^n_{i+1,N+1} - y^n_{i-1,N+1})(y^{n+1}_{i,N+1} - y^{n+1}_{i,N}) = 0 \, ,$$

(13[b])

$$\frac{x^{n+1}_{i,N} + x^{n+1}_{i,N+1}}{2} \frac{\partial F^n}{\partial x} + \frac{y^{n+1}_{i,N} + y^{n+1}_{i,N+1}}{2} \frac{\partial F^n}{\partial y} = - F^n$$

$$+ \frac{x^n_{i,N} + x^n_{i,N+1}}{2} \frac{\partial F^n}{\partial x} + \frac{y^n_{i,N} + y^n_{i,N+1}}{2} \frac{\partial F^n}{\partial y} \, .$$

The major significance of the scheme proposed here is that the functions x and y are being computed together as a vector at each fractional step. This appear to be crucial because of the structure of the boundary conditions containing both the functions. In fact, on each half-time step an algebraic system of following type is solved:

$$(14)\quad A\binom{x_{i-1}}{y_{i-1}} - C\binom{x_i}{y_i} + B\binom{x_{i+1}}{y_{i+1}} = \binom{f_i}{g_i} .$$

Here A,B,C are natrices 2x2. It is easily proven that they satisfy the sufficient conditions for stability of the Gaussian-elimination -type method proposed[7] for system with the above structure.

RESULTS AND DISCUSSION

The method outlined here can be used in two major ways. The first is the construction of regular orthogonal meshes for domains with curved boundaries. The simpliest and, perhaps, the most natural definition of regularity is the requirement Jacobian to be equal to unity, i.e. f=1. The meshes related to this condition can be informally named "orthonormal".

First of all, it has been checked out wether cartesian coordinates appear to be among these uniform coordinates. Indeed, when the above boundary value problem has been solved for the case when the region D is simply the unit square, then the mesh obtained has resulted into cartesian set with very good accuracy.

The next test has been to generate an uniform orthogonal mesh for the curvilinear rectangular domain shown on fig.2. It is specially selected to possess right angles in order to avoid some difficulties connected with the breaking of conformity at the corner points when the angles are not right. It is not a restriction in general, but the case with arbitrary angles requires a special care when spacing the grid points near the corners. The latter goes beyond The frame of present short note. In addition three straight rims are selected and only one curved boundary is allowed according to the formula $y=1.5 - .5\cos(\pi x)$. This is fully enough to display the method. The grid is chosen to be uniformly spaced in both directions with space steps h_ξ and h_η respectively. In order to obtain higher accuracy as well as to check and verify the computations two different meshes are employed with number of grid cells 21x21 and 41x41 respectively. Results turn out to vary slightly with the reduce of the grid size. On the basis of the two solutions and by means of Richardson extrapolation a solution with order of approximation $O(h_\xi^4+h_\eta^4)$ on the mesh 21x21 is constructed. On fig.2 are shown several characteristic coordinate lines of this solution, while on fig.3 are plotted the coordinates obtained simply from the conformal mapping. It is well seen the spoiled behaviour of the Jacobian in the last case as well as the significant improving attained after the method of the present work is applied.

The second way of application of the proposed method is in construction of optimal meshes. One of the possible usefull definitions of optimality is to seek for a mesh which is more dense in the regions where the profile of certain given function is more steep. A similar idea was employed[8] but on the basis of a variational principle. In present work the instrument for governing the mesh appears to be function f. It is assumed that f(x,y) is nothing but the two-dimensional slope of a given surface

$$(15) \quad f(x,y) = \sqrt{1 + u_x'^2 + u_y'^2}$$

where z = u(x,y) is the equation of that surface. Eq.(3) obviously yields:

$$dxdy = (1 + u_x'^2 + u_y'^2)^{-\frac{1}{2}} d\xi d\eta \ .$$

Tha latter relation asserts that if one tekes a regularly spaced grid in the region D' one obtains coordinate lines in D which are more dense in the regions where the two-dimensional slope of function u(x,y) is greater, i.e. where function u(x,y) is steeper.

To avoid the unnecessary complications and to demonstrate the idea of optimality in its pure form it is considered here a square domain in the plane Oxy. It should be mentioned that several different "leading functions" u(x,y) has been used in calculations. To give a better feeling of the results the simplest function $u(x,y) = 1 + x^2 + y^2$ has been chosen among others to expose the method. The optimal mesh obtained with this function is shown on fig.4. The shape of function u along with the coordinate mesh is plotted on fig.5. There can be seen the uniform portions in which the surface area is divided by the coordinate lines.

In the end it should be mentioned that the rate of convergence of the method for the optimal meshes has been much greater than that for the uniform ones with curved boundaries.

GENERALIZATION FOR THREE DIMENSIONS

It is important to note that the present method is easily generalized in three dimensions, i.e. when three cartesian coordinates x,yz are transformed to three curvilinear coordinates ξ,η,ζ. In this case there exist three conditions of orthogonality:

$$x_\xi x_\eta + y_\xi y_\eta + z_\xi z_\eta = 0 ,$$

$$(16) \quad x_\eta x_\zeta + y_\eta y_\zeta + z_\eta z_\zeta = 0 ,$$

$$x_\zeta x_\xi + y_\zeta y_\xi + z_\zeta z_\xi = 0 ,$$

but only two of them are independent. The condition on Jacobian is

$$(17) \quad x_\xi y_\eta z_\zeta + x_\eta y_\zeta z_\xi + x_\zeta y_\xi z_\eta - x_\zeta y_\eta z_\xi - x_\eta y_\xi z_\zeta - x_\xi y_\zeta z_\eta = f^{-1} .$$

Once again, it is easy to show that

$$(18) \quad f^2(x,y,z)\, H_\xi^2\, H_\eta^2\, H_\zeta^2 = 1$$

and to derive the following set of equations

$$(19) \quad \frac{\partial}{\partial\xi} fH_\eta^2 H_\zeta^2 \frac{\partial\varphi}{\partial\xi} + \frac{\partial}{\partial\eta} fH_\zeta^2 H_\xi^2 \frac{\partial\varphi}{\partial\eta} + \frac{\partial}{\partial\zeta} fH_\xi^2\, H_\eta^2 \frac{\partial\varphi}{\partial\zeta} = 0 ,$$

where φ is either x,y or z.

The boundary conditions for this system of equations are the equation of boundary surface:

$$(20) \quad F(x,y,z) = 0 \text{ at } (x,y,z) \in \delta D$$

on one hand and two of the othogonality conditions (16) on the other. At each side of the unit cube in D' have to be chosen those two of (16) which are normal to this side.

CONCLUSIONS

In present work a method for constructing orthogonal coordinates when their Jacobian satisfies certain condition is outlined. The respective equation of this condition along with the orthogonality condition yield a system of equations which is a nonlinear analog to the Cauchy-Riemann problem. The role of a boundary condition is played by the equation of the boundary. This system is rendered to a pair of coupled second-order elliptic equations which is solved by means of a kind of splitting method.

Two general ways of application are displayed: generation of "uniform" meshes with unit Jacobian, and "optimal" meshes for which the Jacobian is governed by the magnitude of the two-dimensional slope of a given function. The latter assures one that the mesh is more dense in the regions where computed function is steeper.

Generalization of the method for the case with three dimensions is sketched.

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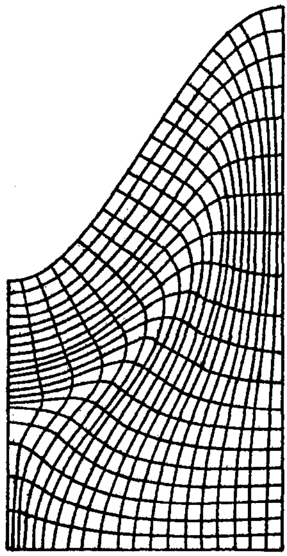

Fig. 2.

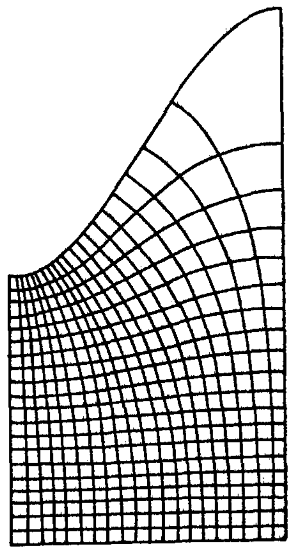

Fig. 3.

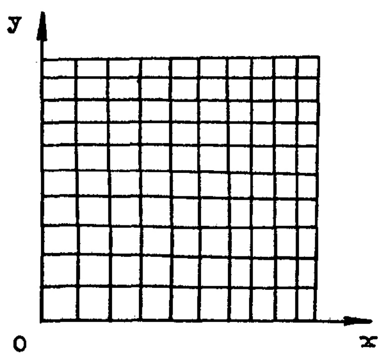

Fig. 4.

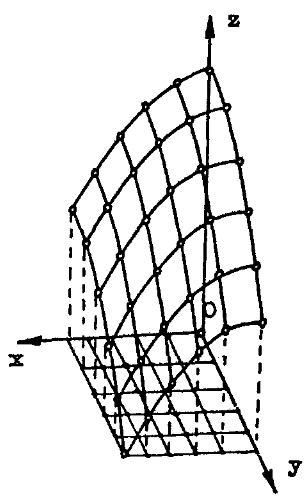

Fig. 5.