Bifurcation and the appearance of a stochastic solution in a single variational problem connected with plane Poiseuille flow

C. I. Christov (Bulgaria) and V. P. Nartov

Institute of Theoretical and Applied Mechanics, Siberian Branch of the Academy of Sciences of the USSR, Novosibirsk, and Institute of Mechanics and Biomechanics, Bulgarian Academy of Sciences, Sofia

(Presented by Academician N. N. Yanenko, September 9, 1983)
(Submitted October 25, 1983)
Dokl. Akad. Nauk SSSR **277**, 825–828 (August 1984)

The principle of least dissipation is applied to the flow of a viscous liquid between two parallel plates (plane Poiseuille flow). The minimum is found in the class of random point functions. A boundary-value problem is set up for the kernels of the stochastic integrals, and the problem is solved numerically. The kernels are interpreted as a velocity field created by a unified structure. Calculations are made for several values of the Reynolds number, constructed from the dynamic velocity. The calculated flow characteristics are compared with experimental data.

1. Instability and the transition to turbulence in plane Poiseuille flow is an important problem of the theory of hydrodynamic stability. The first approach to the investigation of instability was the energy approach,[1,2] and only then did the evolution of disturbances begin to be investigated on the basis of the Navier–Stokes equations.[3] Such an approach experienced extensive development (see the review Ref. 4) and led to a large number of results in the field of the linear and nonlinear instability of various kinds of flows.

Interest in the variational approach to stability has recently redeveloped. Malkus[5] suggested the principle of maximum dissipation of the energy of disturbances. Gol'dshtik[6] formulated the principle of the maximum stability of the averaged flow, and on its basis he calculated the Kármán constant for plane Poiseuille flow. Christov[7] justified, on a semiempirical level, the possibility of the existence of the principle of minimum dissipation for plane Poiseuille flow, while it was shown in Ref. 4 that the lower critical Reynolds number for the transition to turbulence in this flow is well predicted on the basis of this principle.

In the present article the principle of least dissipation is taken as the variational principle, while the class of functions for the perturbations is the set of random flows.

It is assumed that in plane Poiseuille flow only that regime (laminar or turbulent) occurs which provides the minimum of the dissipation function,

$$\int_0^H \left[\nu\left(\frac{d\bar{u}_x}{dy}\right)^2 + \epsilon\right]dy = \min, \tag{1}$$

where $\bar{u}_x$ is the averaged velocity in the direction of the channel axis, ν is the kinematic viscosity coefficient, H is the channel half-width, and ε is the density of the rate of dissipation of turbulent pulsations. The functional (1) should be minimized under the restriction

$$\nu\frac{d\bar{u}_x}{dy} = -Ay + \overline{u'_x u'_y}, \tag{2}$$

which is the Reynolds equation for the corresponding velocity component. Here A is the modulus of the pressure gradient which generates the flow and $-\overline{u'_x u'_y}$ is the Reynolds stress.

2. To use the principle of minimum dissipation, we must have the connection between the dissipation of pulsations and the Reynolds stress. In Ref. 7 the problem was closed semiempirically. Now the minimum is sought in the class of random flows. The latter is characterized by the fact that it consists of impulses identical in shape but randomly scattered along the length of the channel. Moreover, let the averaged quantities depend only on the transverse variable y. Then, following Wiener,[8] we have

$$u'_x = \int_{-\infty}^{\infty} K_x(y, x-\xi)f(\xi)d\xi, \quad u'_y = \int_{-\infty}^{\infty} K_y(y, x-\xi)f(\xi)d\xi, \tag{3}$$

where $K_x, K_y \in L^2_\xi(-\infty, \infty)$, while f is white noise, i.e.,

$$\langle f(x)\rangle = 0, \quad \langle f(\xi)f(x+\xi)\rangle = L^2\delta(x), \tag{4}$$

where $\delta(x)$ is the Dirac delta function, $\langle\cdot\rangle$ denotes averaging over the ensemble, and L^2 has the dimensionality of length. Substituting (3) in the equations for dissipation and for turbulent stress, and also allowing for (4), we have

$$\epsilon = 2\nu L^2\int_{-\infty}^{\infty}\left[\left(\frac{\partial K_x}{\partial\xi}\right)^2 + \left(\frac{\partial K_y}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial K_x}{\partial y} + \frac{\partial K_y}{\partial\xi}\right)^2\right]d\xi, \tag{5}$$

and

$$\overline{u'_x u'_y} = L^2\int_{-\infty}^{\infty} K_x(y,\xi)K_y(y,\xi)d\xi. \tag{6}$$

Accordingly, from the continuity equation it is easy to obtain

$$\frac{\partial K_x}{\partial\xi} + \frac{\partial K_y}{\partial y} = 0. \tag{7}$$

And the variational problem (1), (2), (5), (6), (7) is the stochastic realization of the principle of least dissipation.

3. The Euler–Lagrange equations for the minimization of the functional (1) under the restrictions (2) and (7), with allowance for (5) and (6), have the form

$$2\nu\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial y^2}\right)K_x + \frac{1}{L^2}\frac{\partial\lambda_2}{\partial\xi} + \lambda_1 K_y = 0,$$

$$2\nu\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial y^2}\right)K_y + \frac{1}{L^2}\frac{\partial\lambda_2}{\partial y} + \lambda_1 K_x = 0,$$

$$\lambda_1 = -2\frac{d\bar{u}_x}{dy}$$

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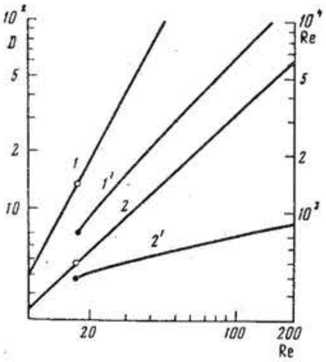

FIG. 1. Average-velocity Reynolds number $\overline{\mathrm{Re}}$ (1, 1') and total dissipation D (2, 2') in mid-channel as functions of Re. 1, 2) laminar regime; 1', 2') turbulent regime.

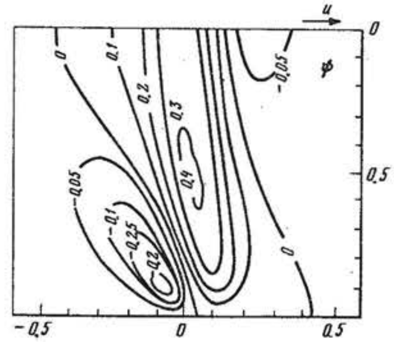

FIG. 3. Isolines of the function ψ (coherent structures) for Re = 2340.

To these equations we add (2) and (7) and we obtain a closed system. By introducing the stream functions

$$K_x = \frac{\partial\psi}{\partial y}, \quad K_y = -\frac{\partial\psi}{\partial\xi} \tag{8}$$

and the dimensionless quantities

$$\psi = \nu\sqrt{\frac{\nu}{L^2 u^*}}\,\psi', \quad y = Hy', \quad \xi = H\xi', \quad \mathrm{Re} = \frac{Hu^*}{\nu},$$
$$\bar{u}_x = u^* u' \quad (u^* = \sqrt{|A|H}),$$

we obtain the system

$$-\left(2\frac{du}{dy}\frac{\partial^2\psi}{\partial\xi\,\partial y} + \frac{d^2u}{dy^2}\frac{\partial\psi}{\partial\xi}\right) = \frac{1}{\mathrm{Re}}\nabla\nabla\psi, \tag{9}$$

$$-y - \int_{-\infty}^{\infty}\frac{\partial\psi}{\partial y}\frac{\partial\psi}{\partial\xi}\,d\xi = \frac{1}{\mathrm{Re}}\frac{du}{dy} \tag{10}$$

with the boundary conditions

$$\frac{\partial\psi}{\partial\xi} = \frac{\partial\psi}{\partial y} = 0 \quad \text{for} \quad y = \pm 1, \tag{11}$$

$$\int_{-\infty}^{\infty}\left[\left(\frac{\partial\psi}{\partial\xi}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right]d\xi < +\infty, \tag{12}$$

which signify that the pulsations die out at the channel walls and that the energy of the disturbances is finite.

4. The boundary-value problem (9)-(12) was solved numerically by introducing transformations for the ξ coordinate, which "compresses" the infinite interval $(-\infty, \infty)$ into the interval $[-1, 1]$. A nonuniform grid was set up along the y axis, bunching up near the solid walls. After seeking the function ψ, on the basis of the properties of the expansion (3), (4) one can calculate all the stochastic characteristics of the flow. The calculations allowed us to obtain the lower and upper critical Reynolds numbers, which proved to be 10 and 17, respectively, and the lower critical number agrees well with Orr's result[2] in a conversion to the Reynolds number for the maximum velocity. Between the upper and lower numbers lies a region in which a nontrivial solution develops only for an initial disturbance of sufficiently large amplitude. The dissipation D and the average-velocity Reynolds number $\overline{\mathrm{Re}}$ as functions of Re are presented in Fig. 1. It is seen that the dissipation of stochastic flow becomes less than that of laminar flow at the Reynolds number Re = 10, which coincides with the lower critical Reynolds number, i.e., the appearance of a stochastic solution is consistent with the principle of least dissipation. The representation (3) limits the application of the principle to the region of moderately supercritical Reynolds numbers for the description of turbulence; nevertheless, we calculated the evolution of the stochastic characteristics of the flow for Re = 20, 40, 100, and 2340, the latter value corresponding to $\overline{\mathrm{Re}}$ = 57,000, allowing us to make a comparison with Comte-Bellot's experimental data.[9]

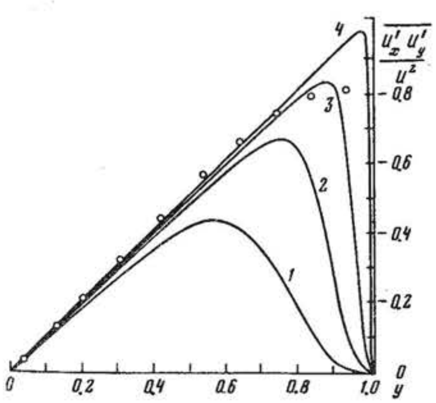

FIG. 2. Reynolds stress: Re = 20 (1), 40 (2), 100 (3), and 2340 (4); points: Comte-Bellot's experiment for Re = 2340.

It turned out that the calculated and experimental characteristics of the turbulence are in qualitative or in quantitative agreement. In Fig. 2 it is seen that for the turbulent stresses, in particular, even the quantitative agreement with experiment is very good.

We note that the functions K_x and K_y are interpreted as the velocity components of a single surge (coherent structure), i.e., ψ is the stream function of a single vortex. The shape of the surge is presented in Fig. 3 for the Reynolds number $\overline{\mathrm{Re}}$ = 2340. It is similar in shape to structures from Ref. 10 for the boundary layer at a flat plate.

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Translated by Edward U. Oldham