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PERTURBATION OF A LINEAR TEMPERATURE FIELD IN AN UNBOUNDED MATRIX DUE TO THE PRESENCE OF TWO UNEQUAL NON-OVERLAPPING SPHERES

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Христо Иванов Христов. **Возмущение линейного поля температуры в неограниченном пространстве двумя непересекающимися разными сферами.** Рассматривается задача о распространении тепла в неограниченном пространстве с заданным коэффициентом теплопроводности, которое содержит две различные непересекающиеся сферы из материала с другим коэффициентом теплопроводности. Принято, что на бесконечности поле температуры является линейным, т. е. градиент температуры — постоянен. Задача сводится к граничной задаче для уравнения Лапласа с прерывным коэффициентом, которая в бисферической координатной системе приводит к трем граничным задачам, решения которых сопрягаются специальным образом на границах сфер (на поверхностях прерывания коэффициента). Решение ищется в ряды Фурье по полиномам Лежандра. Применяя метод производящей функции, из граничных условий получена линейная алгебраическая система для коэффициентов ряда Фурье — Лежандра. Показано, что эта система является в некотором смысле трехдиагональной, если рассматривать неизвестные как двумерные векторы, и предложен эффективный алгоритм ее решения. Далее, известное решение для одиночной сферы тоже представлено в бисферических координатах, и, вычитая его из решения для двух сфер, получено выражение для возмущения к линейному полю температуры, которое обязано чистому взаимодействию двух сфер.

Christo Ivanov Christov. **Perturbation of a linear temperature field in an unbounded matrix due to the presence of two unequal non-overlapping spheres.** The problem of heat conduction in an unbounded media of a given conductivity, containing two non-overlapping spherical inclusions of different material, is considered when at infinity the temperature field approaches linear field of constant gradient. The obtained boundary value problem for the Laplace equation with discontinuous coefficient is rendered in terms of bi-spherical coordinates to three boundary value problems whose solutions match in an appropriate manner at the spherical surfaces (surfaces of discontinuity of coefficient). The solution is sought in Fourier-Legendre series. A system of linear algebraic equation is derived for the Fourier-Legendre coefficients from the boundary conditions (matching conditions) by means of generating function. It is shown that this system is in a sense a three-diagonal one, provided that the unknowns are thought of as two-dimensional vectors, and an effective procedure for numerical solution is outlined. Further, the known solution of the single-sphere problem is rendered into Legendre series, too, and upon substracting it from the two-sphere solution an expression for the perturbation to the temperature field due to the sole pair interaction is derived.

Recently the micro-level studies of properties of multi-component media have attracted a considerable attention in connection with the rigorous derivation of effective properties of heterogeneous materials and flows. The most typical example of such media are, perhaps, the particulate materials (suspensions) for which the second (the particulate) phase is comprised by particles randomly dispersed throughout the first (the continucus) phase. The latter is usually called matrix and the former — filler. It is broadly accepted now that for dilute suspensions one can obtain asymptotically correct results for averaged characteristics with respect to the volume fraction (concentration) c of inclusions, provided that the respective problems for one, two, etc. inclusions are solved, cf e. g. [1, 2]. Validity of this supposition has been recently verified by the authors for the case of perfectly disordered suspensions of equi-sized spheres [3, 4]; polydisperse spherical suspensions [5]; arbitrarily shaped non-overlapping perfectly disordered inclusions [6]; certain not perfectly disordered materials [7, 8].

Studies confined to order of approximation $o(c)$ require only knowledge about the solution for the field created by the presence of a single inclusion in an unbounded matrix when at infinity a constant heat flux (or temperature gradient) is prescribed. This solution is now well known for all practical fields of interest, e. g., strain, velocity, temperature, etc. The situation with the two-particle problem is much more complicated because of technical difficulties connected with the solution. A couple of approximate methods was developed, e. g., the method of reflections [9], the singular approximation method [10], and the multipole-expansion technique [11], but all of them fail, as a rule, to account properly for the interaction of two inclusions (spheres) when the last are close to each other. At the time, as shown in [3, 4], the solution has to be valid for arbitrary separation of the spheres in order to get accurate results. So we prefer the approach which makes use of bi-spherical (bi-cylindrical) coordinates, cf. [2, 12, 13]. This approach gives the solution in closed form though in infinite series with respect to Legendre polynomials and it is hoped that it will turn out that the required integrals of the solution can be evaluated in closed form, too.

The problem with the above-mentioned solutions is that neither of them is presented in full detail. Rather, various integrals of solution are the main objective of those papers. So that the present paper is devoted to obtaining an analytical solution to the problem of two spherical inclusions of different radii in linear temperature field. We resort to the problem of heat conductivity since the latter is a bit more tractable in the sense of the technique involved. At the time it displays all the specific traits of two-particle problems and there is no reason at this stage to consider elastic or viscous problems.

1. POSING THE PROBLEM

In [14] the following equation is derived for the perturbation $S\ (\mathbf{x}, \mathbf{z}; a, b)$ to the temperature field of a continuum containing two spheres which is due to the interaction between them:

$$(1) \qquad \nabla\cdot\Big\{\varkappa_m \nabla S\ (\mathbf{x},\mathbf{z};\ a, b) + \frac{1}{2}[\varkappa]\,[h\ (\mathbf{x}; a)\nabla T_1\ (\mathbf{x}-\mathbf{z};\ b) + h\ (\mathbf{x}-\mathbf{z};\ b)\nabla T_1\ (\mathbf{x}; a)] + [\varkappa]\,[h\ (\mathbf{x}; a) + h\ (\mathbf{x}-\mathbf{z};\ b)]\nabla S\ (\mathbf{x},\mathbf{z};\ a, b)\Big\} = 0,$$

where z is the vector pointed from the centre of sphere of radius a to the centre of sphere of radius b, $\varkappa_m$ — heat conductivity of the matrix, $\varkappa_f$ — heat conductivity of the filler, and $[\varkappa] = \varkappa_f - \varkappa_m$. Respectively

$$\nabla \equiv \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

is the gradient with respect to cartesian coordinates $\boldsymbol{x}$, and

$$h(\boldsymbol{x}; a) = \begin{cases} 0 & \text{for } |\boldsymbol{x}| > a, \\ 1 & \text{for } |\boldsymbol{x}| \leqq a \end{cases}$$

is the characteristic function of a sphere of radius a. In eq. (1) $T_1(\xi, \alpha)$ is the perturbation of the temperature field introduced by the presence of a single sphere of radius a centered at point ξ, namely

$$(2) \qquad T_1(\boldsymbol{x}; a) = \begin{cases} -\beta\, \boldsymbol{G} \cdot \boldsymbol{x} & \text{for } |\boldsymbol{x}| \leqq a, \\ -\beta \dfrac{a^3}{|\boldsymbol{x}|^3} \boldsymbol{G} \cdot x & \text{for } |\boldsymbol{x}| > a, \quad \beta = \dfrac{[\varkappa]}{\varkappa_f + 2\varkappa_m}. \end{cases}$$

Here $\boldsymbol{G}$ is the constant gradient of the temperature field at infinity. The full expression for the temperature field $T(\alpha, z; a, b)$ in an unbounded matrix, containing two spherical inclusions of radii a and b, which are separated on distance z, is simply given by

$$T(\boldsymbol{x}, z; a, b) = \boldsymbol{x} \,.\, \boldsymbol{G} + T_1(\boldsymbol{x}; a) + T_1(\boldsymbol{x} - z; b) + S(\boldsymbol{x}, z; a, b).$$

The last function must be continuous since no point heat sources are allowed. Another continuous function which can be considered is the total perturbation to the linear temperature field, namely

$$(3) \qquad T_2(\boldsymbol{x}, z; a, b) = T_1(\boldsymbol{x}; a) + T_1(\boldsymbol{x} - z; b) + S(\boldsymbol{x}, z; a, b).$$

Note that S is the perturbation due only to the pair interaction between the spheres.

Making use of the governing equations for $T_1(\boldsymbol{x}; a)$, the following equation for T_2 is derived in [14]:

$$(4) \qquad \nabla \cdot \left\{ \varkappa_m \nabla T_2 + [\varkappa] \left[b(\boldsymbol{x}; a) + h(\boldsymbol{x} - z; b) \right] (\boldsymbol{G} + \nabla T_2) \right\} = 0,$$

which we call hereafter an equation for the perturbation T_2 to the linear temperature field due to the presence of two spheres. Being an equation with discontinuous coefficient, the last one is equivalent to three different Laplace equations (see Fig. 1):

$$(5) \qquad \varkappa_f \nabla^2 T_2^{(1)} = 0 \qquad \text{for} \quad |\boldsymbol{x}| \leqq a,$$

$$(6) \qquad \varkappa_f \nabla^2 T_2^{(2)} = 0 \qquad \text{for} \quad |\boldsymbol{x} - \boldsymbol{z}| \leqq b,$$

$$(7) \qquad \varkappa_m \nabla^2 T_2^{(0)} = 0 \qquad \text{for} \quad |\boldsymbol{x}| > a \wedge |\boldsymbol{x} - \boldsymbol{z}| > b.$$

The solutions of these three equations have to match continuously at the common boundaries of their regions. From the integral heat balance equation, which can be obtained from (1.4), one derives the conditions for the so-called normal "jump" of the heat flux. On balance one has the following conditions at the boundaries of the two inclusions:

$$(8)\qquad T_2^{(0)} = T_2^{(1)},\ \varkappa_m \frac{\partial T^{(0)}}{\partial n} = \varkappa_f \left[\frac{\partial T^{(1)}}{\partial n} + \boldsymbol{G}\,.\,\boldsymbol{n} \right] \text{ for } |\,\boldsymbol{x}\,| = a,$$

$$(9)\qquad T_2^{(0)} = T_2^{(2)},\ \varkappa_m \frac{\partial T^{(0)}}{\partial n} = \varkappa_j \left[\frac{\partial T^{(2)}}{\partial n} + \boldsymbol{G}\,.\,\boldsymbol{n} \right] \text{ for } |\,\boldsymbol{x} - \boldsymbol{z}\,| = b,$$

where n stands for the outward normal vector to a sphere and denotes derivative in normal direction.

Since T_2 bears the meaning of a perturbation, then it must vanish at infinity, i. e.,

$$(10)\qquad T_2^{(0)} \to 0 \text{ at } |\,\boldsymbol{x}\,| \to \infty \text{ and } \boldsymbol{z} \text{ arbitrary.}$$

2. COORDINATE TRANSFORMATION

The appropriate coordinate system for which both boundaries $|\,\boldsymbol{x}\,| = a$ and $|\,\boldsymbol{x} - \boldsymbol{z}\,| = b$ are coordinate surfaces is the bispherical one. In order to introduce the bispherical coordinates, one must at the beginning to change the Cartesian

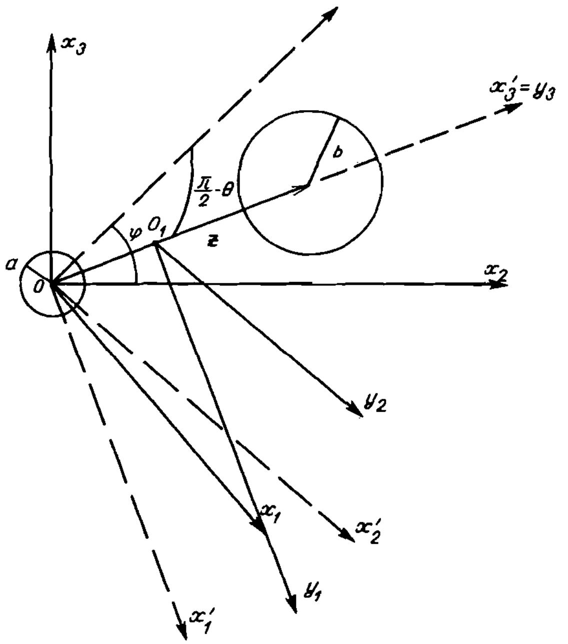

Fig. 1.

coordinates. In Fig. 1 are plotted the new coordinate system $O_1\, y_1\, y_2\, y_3$ and the auxilliary system $O\, x_1'\, x_2'\, x_3'$. The latter is obtained from the original system $O\, x_1\, x_2\, x_3$ after rotating about axis $O\, x_1$ on angle $(\pi/2 - \varphi)$ in negative direction and consequent rotation about axis $O\, x_2'$ on angle $(\pi/2 - \theta)$ in negative direction.

The new coordinate system $O_1\, y_1\, y_2\, y_3$ is obtained by $O\, x_1'\, x_2'\, x_3'$ after a translation on distance $OO_1 = \alpha \mid \boldsymbol{z} \mid = \alpha\, \boldsymbol{z}$ along the positive direction of the axis $O\, x_3'$. Here $z = \mid \boldsymbol{z} \mid$ and θ, φ are its directory angles measured from the original coordinate system, namely

$$(11) \qquad z_1 = z \cos\theta,\ \ z_2 = z \sin\theta \cos\varphi,\ \ z{=}3 = z \sin\theta \sin\varphi.$$

Respectively, the relation between the old and new Cartesian systems, whose formal expression is $\boldsymbol{x} = A\boldsymbol{y} + \alpha\, \boldsymbol{z}$, adopts the form

$$(12) \qquad \left| \begin{array}{l} x_1 = y_1 \sin\theta + (y_3 + \alpha\, z) \cos\theta, \\ x_2 = -\, y_1 \cos\theta \cos\varphi + y_2 \sin\varphi + (y_3 + \alpha\, z) \sin\theta \cos\varphi, \\ x_3 = -\, y_1 \cos\theta \sin\varphi - y_2 \cos\varphi + (y_3 + \alpha\, z) \sin\theta \sin\varphi, \end{array} \right.$$

where the value of α is specified in what follows.

Now the connection between the Cartesian coordinates $O_1\, y_1\, y_2\, y_3$ and bi-spherical ones ξ, η, ζ may be expressed (see, e. g., [9, 15]) as follows:

$$(13) \qquad y_1 = c\, \frac{\sin\xi}{\mathrm{ch}\eta - \cos\xi} \cos\zeta,\ y_2 = c\, \frac{\sin\xi}{\mathrm{ch}\eta - \cos\xi} \sin\zeta,\ y_3 = c\, \frac{\mathrm{sh}\eta}{\mathrm{ch}\eta - \cos\xi},$$

where c is called focal distance and it, along with α and the numbers η_a and η_b of coordinate lines representing the spheres, are functions of z and sphere radii a and b. Values η_a and η_b are of different signs if the spheres do not intersect each other and of the same sign if one of the spheres encircles the other. We consider here the former case and, therefore, we must choose one of these numbers negative. Without loosing the generality, we select $\eta_a < 0$. Then

$$(14) \qquad c = \frac{\sqrt{z^4 + b^4 + a^4 - 2z^2\, b^2 - 2z^2\, a^2 - 2a^2\, b^2}}{2z} = \sqrt{\alpha^2\, z^2 - a^2}\ ,$$

$$(15) \qquad \eta_a = -\ln \left| \frac{c}{a} + \sqrt{1 + \frac{c^2}{a^2}} \right| \equiv -\, \mathrm{arcsh}\, \frac{c}{a}\ ,$$

$$(16) \qquad \eta_b = \ln \left| \frac{c}{b} + \sqrt{1 + \frac{c^2}{b^2}} \right| \equiv \mathrm{arcsh}\, \frac{c}{b}\ ,$$

and

$$(17) \qquad \alpha = \frac{z^2 - b^2 + a^2}{2\, z^2}.$$

The last quantity is positive since $z > a + b$ is the expression of the fact that spheres do not intersect. each other.

Here is to be noted that the constant gradient G transforms in the new coordinate system as follows:

$$(18) \qquad \boldsymbol{E} = \boldsymbol{A} \cdot \boldsymbol{G},$$

where $\boldsymbol{A}$ is the matrix of coordinate transformation (see (12)). The latter means that in (7) — (10) nothing changes when introducing the new Cartesian system $O_1 y_1 y_2 y_3$, provided that $\boldsymbol{G}$ is replaced by $\boldsymbol{E}$.

The Laplace equations (5) — (7) have the following form in terms of bi-spherical coordnates

$$(19)\quad \Delta \Phi \equiv \frac{(\mathrm{ch}\eta - \cos\xi)^3}{c^2 \sin\xi}\left[\frac{\partial}{\partial\eta}\left(\frac{\sin\xi}{\mathrm{ch}\eta - \cos\xi}\frac{\partial\Phi}{\partial\eta}\right) + \frac{\partial}{\partial\xi}\left(\frac{\sin\xi}{\mathrm{ch}\eta - \cos\xi}\frac{\partial\Phi}{\partial\xi}\right) + \frac{1}{\sin\xi(\mathrm{ch}\eta - \cos\xi)}\frac{\partial^2\Phi}{\partial\zeta^2}\right],$$

where Φ stands for the functions $T_2^{(0)}$, $T_2^{(1)}$ or $T_2^{(2)}$. It is well seen that the variable ζ does not enter the coefficients of (19), i. e., it is a cyclic variable of the Laplace operator in bi-spherical coordinates. The dependence of the solution on ζ shows up only through the boundary conditions. Transforming the latter into terms of bi-spherical coordinates, one notes that the outward normal derivative $\frac{\partial}{\partial n}$ is in fact a partial derivative with respect to η at $\eta = \eta_a$ and with respect to $(-\eta)$ at $\eta = \eta_b$. Respectively, the outward normal vector $\boldsymbol{n}$ is equal to $\boldsymbol{e}_\eta$ or — $\boldsymbol{e}_\eta$, where $\boldsymbol{e}_\eta$ is the unit vector tangential to η-coordinate line. Unit vector $\boldsymbol{e}_\eta$ is easily expressed in terms of unit vectors $\boldsymbol{y}_1^0$, $\boldsymbol{y}_2^0$, $\boldsymbol{y}_3^0$ of the auxilliary Cartesian coordinate system as follows:

$$(20)\qquad \boldsymbol{e}_\eta = -\frac{\sin\xi\,\mathrm{sh}\eta\cos\zeta}{\mathrm{ch}\eta - \cos\xi}\boldsymbol{y}_1^0 - \frac{\sin\xi\,\mathrm{sh}\eta\sin\zeta}{\mathrm{ch}\eta - \cos\xi}\boldsymbol{y}_2^0 - \frac{\mathrm{ch}\eta\cos\xi - 1}{\mathrm{ch}\eta - \cos\xi}\boldsymbol{y}_3^0.$$

Then

$$(21)\qquad \boldsymbol{G}\cdot\boldsymbol{n} = \mathrm{sgn}\,(\eta)\,\frac{E_1\sin\xi\,\mathrm{sh}\,\eta\cos\xi + E_2\sin\xi\,\mathrm{sh}\,\eta\sin\zeta + E_3(\mathrm{ch}\,\eta\cos\xi - 1)}{\mathrm{ch}\,\eta - \cos\xi} \equiv \mathrm{sgn}\,(\eta)\,E_\eta\,,$$

where E_j are the components of vector $\boldsymbol{E}$ defined in (18).

Let us now denote

$$(22)\qquad R^{(i)}(\xi, \eta, \zeta;\ c, \theta, \varphi) \equiv T_2^{(i)}(y_1, y_2, y_3;\ z_1, z_2, z_3),\ i = 0, 1, 2.$$

Obviously, each $R^{(i)}$ satisfies eq. (19). The boundary conditions adopt the form

$$(23)\qquad R^{(0)} \to 0 \quad \text{at} \quad \xi, \eta \to 0.$$

$$(24)\qquad \left|\begin{array}{l} R^{(0)} = R^{(2)} \\ \varkappa_m \dfrac{\partial R^{(0)}}{\partial\eta} = \varkappa_f \dfrac{\partial R^{(2)}}{\partial\eta} - \dfrac{c}{\mathrm{ch}\,\eta - \cos\xi}E_\eta \quad \text{for} \quad \eta = \eta_a \end{array}\right.$$

and

$$(25)\quad \left| \begin{array}{l} R^{(0)} = R^{(2)} \\ \varkappa_m \dfrac{\partial R^{(2)}}{\partial \eta} = \varkappa_j \dfrac{\partial R^{(2)}}{\partial \eta} - \dfrac{c}{\text{ch } \eta - \cos \xi} E_\eta \quad \text{at} \quad \eta = \eta_b. \end{array} \right.$$

For the sake of reducing the complexity of indices we denote

$$(26)\quad U(\xi, \eta) = \frac{c}{\text{ch } \eta_a - \cos \xi} E_{\eta_a} \quad \text{and} \quad V(\xi, \eta) = \frac{c}{\text{ch } \eta_b - \cos \xi} E_{\eta_b}.$$

3. EXCLUDING THE CYCLIC VARIABLE

As it has been mentioned in the above, ζ appears to be a cyclic variable and it enters the picture only through the boundary conditions. Formulae (24) and (25) hint the idea that one can seek for the solution of the 3-D problem in the form of following linear combinations:

$$(27)\quad R^{(i)} = R_0^{(i)} + R_1^{(i)} \cos \zeta + R_2^{(i)} \sin \zeta \quad \text{for} \quad i = 0, 1, 2,$$

where $R_j^{(i)}$ are functions of ξ and η only.

The equation for the newly introduced functions are

$$(28)\quad \frac{\partial}{\partial \eta} \left(\frac{\sin \xi}{\text{ch}\eta - \cos \xi} \frac{\partial R_j^{(i)}}{\partial \eta} \right) + \frac{\partial}{\partial \xi} \left(\frac{\sin \xi}{\text{ch}\eta - \cos \xi} \frac{\partial R_j^{(i)}}{\partial \xi} \right) - a_j \frac{R_j^{(i)}}{\sin \xi (\text{ch } \eta - \cos \xi)} = 0,$$

where $a_0 = 0$, $a_1 = a_2 = 1$.

In the same manner the boundary conditions are manipulated. Upon assuming that

$$U(\xi, \eta) = U_0(\xi, \eta) + U_1(\xi, \eta) \cos \zeta + U_2(\xi, \eta) \sin \zeta,$$

$$V(\xi, \eta) = V_0(\xi, \eta) + V_1(\xi, \eta) \cos \zeta + V_2(\xi, \eta) \sin \zeta,$$

one derives

$$(29)\quad U_0 = c\, E_3 \frac{\text{ch } \eta_a \cos \xi - 1}{(\text{ch } \eta_a - \cos \xi)^2}, \quad U_j = c\, E_j \frac{\sin \xi \text{ sh } \eta_a}{(\text{ch } \eta_a - \cos \xi)^2}, \quad i = 1, 2,$$

and

$$(30)\quad V_0 = c\, E_3 \frac{\text{ch } \eta_b \cos \xi - 1}{(\text{ch } \eta_b - \cos \xi)^2}, \quad V_j = c\, E_j \frac{\sin \xi \text{ sh } \eta_b}{(\text{ch } \eta_b - \cos \xi)^2}, \quad i = 1, 2,$$

and then

$$(31)\quad \left| \begin{array}{l} R_j^{(0)} = R_j^{(1)} \\ \varkappa_m \dfrac{\partial R_i^{(0)}}{\partial \eta} = \varkappa_f \dfrac{\partial R_i^{(1)}}{\partial \eta} - \varkappa_f U_j \qquad \text{for} \quad \eta = \eta_a,\ j = 0, 1, 2, \end{array} \right.$$

$$
\text{(32)}\qquad \left|\begin{array}{l} R_j^{(0)} = R_j^{(2)} \\ \varkappa_m \dfrac{\partial R_j^{(0)}}{\partial \eta} = \varkappa_f \dfrac{\partial R_j^{(2)}}{\partial \eta} - \varkappa_f V_j \end{array}\right. \qquad \text{for } \eta = \eta_b,\ j = 0, 1, 2.
$$

Respectively the boundary condition at infinity (23) yields

$$
\text{(33)}\qquad R_j^{(0)} \to 0 \quad \text{at} \quad \xi, \eta \to 0, \ \ j = 0, 1, 2.
$$

So far, for i=0, 1, 2 and j=0, 1, 2 we have nine linear boundary value problems for evaluating the nine functions $R_j^{(i)}$.

4. SEPARATING THE VARIABLES

It is well known (see, e. g., [16]) that eqs (28) can be transformed into equations with separating variables by means of the substitution

$$
\text{(34)}\qquad R_j^{(i)} = \sqrt{2\,(\operatorname{ch}\eta - \cos\xi)}\; A_j^{(i)}.
$$

Then (28) adopts the form

$$
\text{(35)}\qquad \frac{\partial^2 A_j^{(i)}}{\partial \eta^2} + \frac{\partial^2 A_j^{(i)}}{\partial \xi^2} + \operatorname{ctg}\xi \frac{\partial A_j^{(i)}}{\partial \xi} - \frac{1}{4} A_j^{(i)} - \frac{a_j}{\sin^2\xi} A_j^{(i)} = 0.
$$

Let us now seek for the solution of (35) in the form

$$
\text{(36)}\qquad A_j^{(i)}(\xi, \eta) = B_{ij}(\xi)\, C_{ij}(\eta).
$$

Then each of eqs (35) breakes into two independent equations

$$
\text{(37)}\qquad \frac{d^2}{d\eta^2} C_{ij} = \lambda^2 C_{ij},
$$

and

$$
\text{(38)}\qquad \frac{d^2}{d\xi^2} B_{ij} + \operatorname{ctg}\xi\, B_{ij} + \left(\lambda^2 - \frac{1}{4}\right) B_{ij} - \frac{a_j}{\sin^2\xi} B_{ij} = 0.
$$

Consider first the last equation. Let us denote

$$
\text{(39)}\qquad B_{ij} = D_{ij}(\mu), \quad \text{where} \quad \mu = \cos\xi.
$$

Then (38) immediately transforms to the following:

$$
\text{(40)}\qquad (1-\mu^2)\frac{d^2}{d\mu^2} D_{ij} - 2\mu \frac{d}{d\mu} D_{ij} + \left[\left(\lambda^2 - \frac{1}{4}\right) - \frac{a_j}{1-\mu^2}\right] D_{ij} = 0,
$$

which is the well-known Legendre equation. Its solutions that do not possess any singularities are the Legendre polynomials $P^{\sqrt{a_j}}_{\lambda-1/2}$ (see, e. g., [17]), namely

$$D_{ij} = P^{a_j}_{\lambda-1/2}(\mu), \tag{41}$$

and it is already acknowledged that $\sqrt{a_j}=a_j$. For $a_j = 0$ these are the Legendre polynomials and for $a_j = 1$ — the associated Legendre polynomials. Both kind of polynomials are defined for real $\lambda > {}^1/_2$ and, therefore, the fundamental solutions of (37) are the exponential functions. So, the general solution for $A^{(i)}_j$ is given by

$$A^{(i)}_j = \sum_{n=0}^{\infty} \left[L^{(n)}_{ij}\, e^{(n+1/2)\eta} + M^{(n)}_{ij}\, e^{-(n+1/2)\eta} \right] P^{a_j}_n(\mu). \tag{42}$$

Here is to be mentioned that in order not to have singularities at the focal points the coefficients $L^{(n)}_{2j}$ and $M^{(n)}_{1j}$ must be equal to zero. This reduces the total number of unknown constants to 12 per each n, which is exactly equal to the number of boundary conditions derived from (24), (25). The boundary condition at infinity (23) is automatically satisfied due to the specific form of the substitution (34) and to the fact that $A^{(i)}_j$ exhibit no singularities.

In terms of functions $A^{(i)}_j$ the boundary conditions (31), (32) adopt the form

$$\left| \begin{aligned} & \varkappa_m \frac{\partial A^{(0)}_j}{\partial \eta} + \frac{1}{2} \varkappa_m \frac{\operatorname{sh}\eta}{\operatorname{ch}\eta-\mu} A^{(0)}_j \\ & = \varkappa_f \frac{\partial A^{(1)}_j}{\partial \eta} + \frac{1}{2} \varkappa_f \frac{\operatorname{sh}\eta}{\operatorname{ch}\eta-\cos\xi} A^{(i)}_j - \varkappa_f U_j \left[2(\operatorname{ch}\eta-\mu)\right]^{-1/2}, \\ & A^{(0)}_j = A^{(1)}_j \quad \text{at} \quad \eta=\eta_a, \quad j=0, 1, 2, \end{aligned} \right. \tag{43}$$

and

$$\left| \begin{aligned} & \varkappa_m (\operatorname{ch}\eta - \mu) \frac{\partial A^{(0)}_j}{\partial \eta} + \frac{1}{2} \varkappa_m \operatorname{sh}\eta\, A^{(0)}_j \\ & = \varkappa_f (\operatorname{ch}\eta - \mu) \frac{\partial A^{(2)}_j}{\partial \eta} + \frac{1}{2} \varkappa_f \operatorname{sh}\eta\, A^{(2)}_j - \frac{1}{2} \varkappa_f V_j \sqrt{2(\operatorname{ch}\eta-\mu)}, \\ & A^{(0)}_j = A^{32)}_j \quad \text{at} \quad \eta=\eta_b, \quad j=0, 1, 2. \end{aligned} \right. \tag{44}$$

5. EXPANDING THE BOUNDARY CONDITIONS INTO LEGENDRE SERIES. METHOD OF GENERATING FUNCTION

Taking a more close look at (43) and (44), one sees that these boundary conditions can be effectively satisfied only if the functions

$$\overline{U}_j = \sqrt{2(\operatorname{ch}\eta-\cos\xi)}\, U_j \quad \text{and} \quad \overline{V}_j = \sqrt{2(\operatorname{ch}\eta-\cos\xi)}\, V_j \tag{45}$$

are developed into Legendre series.

Let us begin with $\overline{U}_0$ which is expressed as

$$(46)\qquad \overline{U}_0 = \sqrt{2}\, c\, E_3 \frac{\operatorname{ch} \eta_a\, \mu - 1}{(\operatorname{ch} \eta_a - \mu)^{3/2}} = 4\, c\, E_3\, e^{3/2\, \eta_a} \frac{\mu \operatorname{ch} \eta_a - 1}{(1 + e^{2\, \eta_a} - 2\, e^{\eta_a}\, \mu)^{3/2}}\,.$$

The direct attack on (46) is a tedious task and we shall persue another way. It is known (cf. [17, § 3.6]) that the generating function of Legendre polynomials reads

$$(47)\qquad G\,(t, \mu) = (1 - 2\, t\, \mu + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n\, P_n\,(\mu) \qquad \text{for} \quad t < 1.$$

Then we have

$$(48)\qquad \frac{\partial\, G}{\partial t} = \frac{(t - \mu)}{(1 - 2\, t\, \mu + t^2)^{3/2}} = \sum_{n=0}^{\infty} n t^{n-1}\, P_n\,(\mu).$$

It is clearly seen now that $\overline{U}_0$ from (46) can be represented as a linear combination of G and G_t, provided that t is selected as follows:

$$t = e^{\eta_a} < 1.$$

Then

$$(49)\qquad \overline{U}_0 = 2\, c\, E_3\, e^{3/2\, \eta_a} \sum_{n=0}^{\infty} \left[(n-1)\, e^{n\, \eta_a} - n\, e^{(n-2)\, \eta_a}\right] P_n\,(\mu).$$

Absolutely in the same manner it is derived that

$$(50)\qquad \overline{V}_0 = 2\, c\, E_3\, e^{-3/2\, \eta_b} \sum_{n=0}^{\infty} \left[(n-1)\, e^{-n\eta_b} - n\, e^{-(n-2)\, \eta_b}\right] P_n\,(\mu).$$

A bit more complicated is the problem with the rest of functions $\overline{U}_j$ and $\overline{V}_j$, which are to be expanded into series with respect to associated Legendre polynomials of first order $P_n^1\,(\mu)$. Once again the respective generating function shall be employed:

$$(51)\qquad G^1\,(t, \mu) \equiv \frac{t\,(1 - \mu^2)^{1/2}}{(1 - 2\, t\, \mu + t^2)^{3/2}} = \sum_{n=0}^{\infty} t^n\, P_n^1\,(\mu).$$

Then

$$(52)\qquad \overline{U}_j = 4\, c\, E_j\, e^{1/2\, \eta_a} \operatorname{sh} \eta_a \sum_{n=0}^{\infty} e^{n\, \eta_a}\, P_n^1\,(\mu), \qquad j=1, 2,$$

$$(53)\qquad \overline{V}_j = 4\, c\, E_j\, e^{-1/2\, \eta_b} \operatorname{sh} \eta_b \sum_{n=0}^{\infty} e^{-n\eta_b}\, P_n^1\,(\mu), \qquad j=1, 2.$$

In the end we shall specify the terms of the type (ch $\eta-\mu$) $P_n^{a_j}$ which enter (43) and (44) if the respective functions are developed into Legendre series. Making use of the relation (see [17], § 3.9)

$$(54)\qquad (\mathrm{ch}\,\eta-\mu)\,P_n^{a_j}(\mu)=-\frac{n+a_j}{2\,n+1}\,P_{n-1}^{a_j}+\mathrm{ch}\,\eta\,P_n^{a_j}-\frac{n+1-a_j}{2\,n+1}\,P_{n+1}^{a_j},$$

we arrive, e. g., for $A_j^{(0)}$ to the following:

$$(55)\quad (\mathrm{ch}\eta-\mu)\frac{\partial A_j^{(0)}}{\partial\eta}=\sum_{n=0}^{\infty}\left\{\left(n+\frac{1}{2}\right)e^{(n+1/2)\,\eta}\left[-\frac{n+1-a_j}{2\,n+3}\,e^{\eta}\,L_{0j}^{(n+1)}+\mathrm{ch}\eta\,L_{0j}^{(n)}\right.\right.$$

$$\left.-\frac{n+a_j}{2n-1}\,e^{-\eta}L_{0j}^{(n-1)}\right]-\left(n+\frac{1}{2}\right)e^{-(n+1/2)\,\eta}\left[-\frac{n+1-a_j}{2n+3}\,e^{-\eta}\,M_{0j}^{(n+1)}\right.$$

$$\left.\left.+\,\mathrm{ch}\,\eta\,M_{0j}^{(n)}-\frac{m+a_j}{2n-1}\,e^{-\eta}\,M_{0j}^{(n-1)}\right]\right\}P_n^{a_j}(\mu).$$

It should be noted here that the coefficients of $L_{ij}^{(-1)}$, $M_{ij}^{(-1)}$ vanish identically, which leaves the system coupled.

6. COMPLETING THE SOLUTION

As mentioned in Section 4, the requirement of no singularities at focal points yields

$$(56)\qquad L_{2j}^{(n)}=0,\quad M_{1j}^{(n)}=0,\quad j=0,\ 1,\ 2,\quad n=0,\ 1,\ldots\ldots$$

Making use of the first of equations (43), one can recast the second of them as follows:

$$\varkappa_m\,(\mathrm{ch}\,\eta-\mu)\,\frac{\partial A_j^{(0)}}{\partial\eta}-\frac{1}{2}\,[\varkappa]\,\mathrm{sh}\,\eta\,A_j^{(0)}$$

$$=\varkappa_j\,(\mathrm{ch}\,\eta-\mu)\,\frac{\partial A_j^{(1)}}{\partial\eta}-\frac{1}{2}\,\varkappa_f\,\overline{U}_j,\qquad \eta=\eta_a$$

and being reminded of (42), (56), the coefficients $L_{1j}^{(n)}$ can be excluded from the last equation, namely

$$(57)\qquad [\varkappa]\left\{-(n+1+a_j)\,e^{\eta_a}\,L_{0j}^{(n+1)}+[(2\,n+1)\,\mathrm{ch}\,\eta_a+\mathrm{sh}\,\eta_a]\,L_{0j}^{(n)}\right.$$

$$\left.-\,(n-a_j)\,e^{-\eta_a}\,L_{0j}^{(n-1)}\right\}$$

$$+\,(\varkappa_m+\varkappa_f)\left\{-(n+1+a_j)\,e^{-\eta_a}\,M_{0j}^{(n+1)}+[(2\,n+1)\,\mathrm{ch}\,\eta_a]\,M_{0j}^{(n)}\right.$$

$$- (n-a_j)\, e^{\eta_a} M_{0j}^{(n-1)} \Big\} e^{-(2n+1)\eta_a}$$

$$+ [\varkappa] \operatorname{sh} \eta_a\, e^{-(2n+1)\eta_a} M_{0j}^{(n)} = \begin{cases} 4c\varkappa_f E_3 \left[n \operatorname{sh} \eta_a - \dfrac{1}{2} e^{\eta_a} \right], & j=0, \\ 4c\varkappa_f E_j \operatorname{sh} \eta_a, & j=1, 2. \end{cases}$$

Absolutely in the same manner it is derived from (44) that

$$(58) \qquad (\varkappa_m+\varkappa_j) \Big\{ -(n+1+a_j)\, e^{\eta_b} L_{0j}^{(n+1)} + (2n+1) \operatorname{ch} \eta_b L_{0j}^{(n)}$$

$$- (n-a_j)\, e^{-\eta_b} L_{0j}^{(n-1)} \Big\} e^{(2n+1)\eta_b} - [\varkappa] \operatorname{sh} \eta_b\, e^{(2n+1)\eta_b} L_{0j}^{(n)}$$

$$+ [\varkappa] \Big\{ -(n+1+a_j)\, e^{-\eta_b} M_{0j}^{(n+1)} + [(2n+1) \operatorname{ch} \eta_b - \operatorname{sh} \eta_b] M_{0j}^{(n)}$$

$$- (n-a_j)\, e^{\eta_b} M_{0j}^{(n-1)} \Big\}$$

$$= \begin{cases} +4c\varkappa_f E_3 \left[n \operatorname{sh} \eta_b + \dfrac{1}{2} e^{-\eta_b} \right], & j=0, \\ -4c\varkappa_j E_j \operatorname{sh} \eta_b, & j=1, 2. \end{cases}$$

System (57), (58) is now closed for evaluating the coefficients $L_{0j}^{(n)}$, $M_{0j}^{(n)}$. Its most significant feature is that it is a tridiagonal system for the vector $\left\{ L_{0j}^{(n)}, M_{0j}^{(n)} \right\}$, i. e.,

$$(59) \qquad -\Gamma \cdot \begin{Bmatrix} L_{0j}^{(n-1)} \\ M_{0j} \end{Bmatrix} + \Delta \cdot \begin{Bmatrix} L_{0j}^{(n)} \\ M_{0j} \end{Bmatrix} - \Pi \cdot \begin{Bmatrix} L_{0j}^{(n+1)} \\ M_{0j}^{(n+1)} \end{Bmatrix} = \boldsymbol{K},$$

where the 2×2 matrices Γ, Δ, and Π, as well as the two-dimensional vector $\boldsymbol{K}$, are composed from the coefficients of system (57), (58)

$$\Pi = \begin{Bmatrix} [\varkappa](n+1+a_j)\, e^{\eta_a} & (\varkappa_m+\varkappa_f)(n+1+a_j)\, e^{-(2n+2)\eta_a} \\ (\varkappa_m+\varkappa_f)(n+1+a_j)\, e^{(2n+2)\eta_b} & [\varkappa](n+1+a_j)\, e^{-\eta_b} \end{Bmatrix},$$

$$\Gamma = \begin{Bmatrix} [\varkappa](n-a_j)\, e^{-\eta_a} & [\varkappa_m+\varkappa_f)(n-a_j)\, e^{-2n\eta_b} \\ (\varkappa_m+\varkappa_f)(n-a_j)\, e^{2n\eta_b} & [\varkappa](n-a_j)\, e^{\eta_b} \end{Bmatrix},$$

$$\Delta = \begin{Bmatrix} [\varkappa][(2n+1) \operatorname{ch} \eta_a + \operatorname{sh} \eta_a] & [(\varkappa_m+\varkappa_f)(2n+1) \operatorname{ch} \eta_a + [\varkappa] \operatorname{sh} \eta_a]\, e^{-(2n+1)\eta_a} \\ [(\varkappa_m+\varkappa_f)(2n+1) \operatorname{ch} \eta_b - [\varkappa] \operatorname{sh} \eta_b]\, e^{(2n+1)\eta_b} & [\varkappa][(2n+1(\operatorname{ch} \eta_b - \operatorname{sh} \eta_b] \end{Bmatrix}$$

and $n=0, 1, 2, \ldots$ for $j=0$; $n=1, 2, \ldots$ for $j=1, 2$.

The last system is solved numerically assuming that after certain suffuciently large number $n=N$ the unknowns vanish. The specific form of the system allows employing a special kind of Gaussian elimination method called "progonka" (see [18]), namely its vector generalization (for details of implementation of the latter see [19]). It is easily shown that each of the components of the matrix Δ is greater or equal than the sum of the respective components of the matrices Π, Γ. The latter ascertains one that computations will be stable. So that we can think of this system as solved and then the rest of unknown coefficients are defined explicitly:

$$L_{1j}^{(n)} = L_{0j}^{(n)} + e^{-(2n+1)\eta_a} M_{0j}^{(n)}, \tag{60}$$

$$M_{2j}^{(n)} = e^{(2n+1)\eta_b} L_{0j}^{(n)} + M_{0j}^{(n)} \tag{61}$$

and thus the problem of evaluating the functions $R_{j}^{(i)}$, or which is the same — the solution T_2 (see (22)), is reduced to an algorithm and therefore solved. As the purpose of the present work is the solution for function S (see (13)), we need to represent also T_1 into series.

7. REPRESENTING THE SINGLE-SPHERE SOLUTION INTO LEGENDRE SERIES

Being consistent with the formulae for T_2, the following expressions for $T_1(x;\,a)$ and $T_1(\boldsymbol{x}-z;\,b)$ are to be introduced:

$$T_1(\boldsymbol{x};\,a) = \sqrt{2(\operatorname{ch}\eta-\mu)}\,[Q_{i0}+Q_{i1}\cos\zeta + Q_{i2}\sin\zeta], \tag{62}$$

where Q_{ij} are functions of μ, η only and the index i has the same meaning as in Section 1. The difference, however, is that for function T we have

$$Q_{0j} \equiv Q_{2j}, \tag{63}$$

since it is expressed by the same formula (2) for all $|x| > a$.

Now, making use of the method of generating function, we derive

$$\left|\begin{aligned} Q_{10} &= -\frac{\beta}{\sqrt{2}}\,c\,E_3\,\frac{\operatorname{sh}\eta}{(\operatorname{ch}\eta-\mu)^{3/2}} = -\beta\,c\,E_3\sum_{n=0}^{\infty}(2n-1)\,e^{(n+1/2)\eta}P_n(\mu),\\ Q_{1j} &= -\frac{\beta}{\sqrt{2}}\,c\,E_j\,\frac{\sqrt{1-\mu^2}}{(\operatorname{ch}\eta-\mu)^{3/2}} = -2\beta\,c\,E_j\sum_{n=1}^{\infty}e^{(n+1/2)\eta}P_n(\mu), \quad j=1,2.\end{aligned}\right. \tag{64}$$

A bit more complicated is the problem with Q_{1j}, since they are expressed by more complex formulae. Let us begin with Q_{00}:

$$Q_{00} = -\frac{\beta}{\sqrt{2}}\,\frac{a^3}{c^2}\,\frac{E_3\operatorname{sh}\eta + (\alpha z/c)\,E_3(\operatorname{ch}\eta-\mu)}{\left[\operatorname{ch}\eta+\mu+\dfrac{2\alpha z}{c}\operatorname{sh}\eta+\dfrac{\alpha^2 z^2}{c^2}(\operatorname{ch}\eta-\mu)\right]^{3/2}}. \tag{65}$$

The last expression is valid for $\eta > \eta_a$. Once again the use of method of generating function yields

$$(66) \qquad Q_{00} = -\beta \frac{a^2}{c} E_3 \sum_{n=0}^{\infty} e^{(n+1/2)(2\eta_a-\eta)} (2n + e^{-2\eta_a}) P_n(\mu).$$

Concerning the rest of functions, we have

$$(67) \quad Q_{0j} = Q_{2j} = -\frac{\beta}{\sqrt{2}} \frac{a^3}{c^2} \frac{E_j \sqrt{1-\mu^2}}{\left[\text{ch}\,\eta+\mu + \frac{2\alpha z}{c}\,\text{sh}\,\eta + \frac{\alpha^2 z^2}{c^2}(\text{ch}\,\eta-\mu)\right]^{3/2}},$$

which are immediately developed into Legendre series as follows:

$$(68) \qquad Q_{0j} = -2\beta a E_j \sum_{n=1}^{\infty} e^{(n+1/2)(2\eta_a-\eta)} P_n^1(\mu), \quad j=1, 2.$$

The same procedure is repeated for $T_1(\boldsymbol{x}-\boldsymbol{z}; b)$ assuming that

$$(69) \qquad T_1(\boldsymbol{x}-\boldsymbol{z}; b) = \sqrt{2(\text{ch}\,\eta-\mu)}[P_{i0} + P_{i1}\cos\zeta + P_{iz}\sin\zeta].$$

In this case we have

$$(70) \qquad P_{0j} = P_{1j}.$$

For the solution inside the second sphere it is easily derived that

$$(71) \quad \left| \begin{aligned} P_{20} &= -\frac{\beta}{\sqrt{2}} c E_3 \frac{\text{sh}\,\eta}{(\text{ch}\,\eta-\mu)^{3/2}} = -\beta c E_3 \sum_{n=0}^{\infty} (2n-1) e^{-(n+1/2)\eta} P_n(\mu), \\ P_{2j} &= -\frac{\beta}{\sqrt{2}} c E_j \frac{\sqrt{1-\mu^2}}{(\text{ch}\,\eta-\mu)^{3/2}} = -2\beta c E_j \sum_{n=1}^{\infty} e^{-(n+1/2)\eta} P_n^1(\mu), \quad j=1, 2. \end{aligned} \right.$$

Further for

$$(72) \quad P_{00} = -\beta \frac{a^3}{c^2\sqrt{2}} \frac{E_3\,\text{sh}\,\eta + (\alpha-1)(z/c) E_3(\text{ch}\,\eta-\mu)}{\left[\text{ch}\,\eta+\mu + \frac{2(\alpha-1)z}{c}\,\text{sh}\,\eta + \frac{(\alpha-1)^2 z^2}{c^2}(\text{ch}\,\eta-\mu)\right]^{3/2}}$$

we obtain

$$(73) \qquad P_{00} = -\beta \frac{a^2}{c} E_3 \sum_{n=0}^{\infty} e^{(n+1/2)(\eta-2\eta_b)} (2n + e^{2\eta_b}) P_n(\mu).$$

And finally for

$$(74)\quad P_{0j} = P_{1j} = -\frac{\beta}{\sqrt{2}}\,\frac{a^3}{c^2}\,\frac{E_j\sqrt{1-\mu^2}}{\left[\operatorname{ch}\eta+\mu + \frac{2(\alpha-1)\,z}{c}\operatorname{sh}\eta + \frac{(\alpha-1)^2 z^2}{c^2}(\operatorname{ch}\eta-\mu)\right]^{3/2}},$$

we derive

$$(75)\quad P_{0j} = -2\,\beta_a\,E_j \sum_{n=1}^{\infty} e^{(n+1/2)\,(\eta-2\,\eta_b)}\,P_n^1(\mu),\quad j=1,2.$$

In the end it is convenient to write down the general forn of the solution for S from (3) in terms of bi-spherical coordinates:

$$(76)\quad S(\boldsymbol{x}, z; a, b) \equiv \tilde{S}(\xi, \eta, \zeta; \theta, \varphi;\ a, b)$$

$$= \sqrt{2(\operatorname{ch}\eta-\mu)} \sum_{n=a_j}^{\infty} \left\{ L_{ij}^{(n)} e^{(n+1/2)\,\eta} + M_{ij}^{(n)} e^{-(n+1/2)\,\eta} - Q_{ij} - P_{ij}\right\} P_n^{a_j}(\mu).$$

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