On a Bifurcation and Emerging of a Stochastic Solution in a Variational Problem for Poiseuille Flows

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SUMMARY

For the pressure-driven viscous flow between two flat plates and in tubes (Poiseuille flows) the principle of minimal dissipation is applied. Minimum is sought in the class of point random functions. For kernels of stochastic integrals involved a boundary value problem is derived and appropriate numerical method is proposed. Calculations show that beyond certain critical magnitude of Reynolds number a bifurcation takes place and non-trivial stochastic solution emerges. Various characteristics of this solution are calculated and most of them are in good quantitative or qualitative agreement with the experimental data concerning turbulent Poiseuille flows. The kernels of stochastic integrals are interpreted as large eddies (coherent structures).

Introduction

Transition to turbulence in Poiseuille flows has always been one of the major problems of hydrodynamic instability. All principal theoretical approaches to instability has been checked on these flows. Reynolds called unstable those regimes for which a disturbance of increasing energy can be traced [1]. This led him to the variational problem of minimizing of the time derivative of full energy of a perturbation. After in [2] Sommerfeld turned directly to the Navier-Stokes equations when investigating the evolution of disturbances and posed the respective boundary value problem his method became the most popular approach to instability and now an extensive literature is availble (see for instance survey [3]).

Recently, the variational approach has enjoyed some revived interest. Malkus [4] introduced the principle of maximal dissipation of disturbances in turbulent flow. Goldshtik [5]

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stated the principle of maximal stability of the mean-averaged profile of the turbulent flow. Christov discussed in [6] the existence of a kind of least rate of dissipation principle. It turned out that the principle of minimal dissipation predicts the lower critical Reynolds number of transition fairly well. Advantage of variational methods is the opportunity to derive somewhat simpler equations for disturbances. In turn, the results are to be thought of as bounds for the full-scale solutions and are more rough as a rule. In the present work the class of random point functions is considered when minimizing the dissipation functional which is a step ahead of the previous author's paper [7] where the Gaussian random functions are considered. The existence of random solutions is named "stochastic bifurcation".

Principle of minimal dissipation

In parallel flows there exists only one non-trivial component of the averaged velocity - the longitudinal velocity $\bar{u}_x$ and and hydrodynamic characteristics depend only on transverse coordinate y (see Fig.1), the latter being either cartesian coordinate or polar radius for channels and tubes respectively. The averaged flow is governed by the Reynolds equation:

$$\nu \frac{d\bar{u}_x}{dy} = -Ay + \overline{u'_x u'_y} \tag{1}$$

where A is the magnitude of the average pressure gradient and $\overline{u'_x u'_y}$ is the so-called Reynolds stress. The last equation is not closed. One of the ways to that is to summon the principle of minimal dissipation as an additional empirical information. The total rate of dissipation in the half of the channel is given by (for pipe flow a is the radius)

$$\int_0^a \left[\nu \left(\frac{d\bar{u}_x}{dy} \right)^2 + \varepsilon \right] y^{\alpha} dy = min \tag{2}$$

where ε is the density of the dissipation rate of turbulent pulsations, $\alpha=0$ refers to channel flow and $\alpha=1$ - to pipe one.

The class of random functions for minimization

Variational problem (2) with constraint (1) is still not defined unless a kind of connection between the Reynolds stress and turbulent dissipation is specified. In [6] those two are connected through semi-empirical considerations. Here we shall follow the idea of [7] where the dissiparion functional is minimized over the class of Gaussian random functions. However, in the present work the class of random point functions is considered instead of that of Gaussian random functions. The random point functions are better model for the situation at the threshold of instability where the disturbances occur seldom and the random process resulted can be considered as a random point function. Let us also assume that velocity field depends on time t as on parameter. Then velocities of a disturbance can be expressed as follows (for brevity only component u_x' is displayed):

$$u_x' = \int_{-\infty}^{\infty} K_x(x-\xi;y)[g(\xi)-\gamma]\,d\xi\ , \tag{3}$$

where K_x is non-random function and $g(\xi)$ is the so-called random density function (for definition see [8],[9]) with mean γ.

Here is to be mentioned that the class of random functions is oversimplified and the results which will be obtained will bear only a distant resemblance to the real developed turbulence, but the solution to the presented problem is of profound physical meaning since it answers the question whether a random flow can bring the value of total dissipation to a level lesser than that for the laminar flow.

Now the dissipation and Reynolds stress are readily obtained:

$$\int_0^a \left\{ \nu\left(\frac{d\bar{u}_x}{dy}\right)^2 + 2\nu\gamma\int_{-\infty}^{\infty}\left[\left(\frac{\partial K_x}{\partial \xi}\right)^2 + \left(\frac{\partial K_y}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial K_x}{\partial y} + \frac{\partial K_y}{\partial \xi}\right)^2\right]d\xi\right\}dy\ , \tag{4}$$

$$\nu\frac{d\bar{u}_x}{dy} = -Ay + \gamma\int_{-\infty}^{\infty} K_x(\xi;y)\,K_y(\xi;y)\,d\xi\ . \tag{5}$$

On the other hand equation of continuity yields

$$\frac{\partial K_x}{\partial \xi} + \frac{\partial K_y}{\partial y} = 0\ . \tag{6}$$

So far the variational problem is completed and (4)-(6) appear to be a stochastic implementation of the principle of minimal dissipation.

Euler-Lagrange equations

Introducing Lagrange factors for each of the constraints (5) and (6) the functional (4) can be recast and the problem of minimization without constraints can be considered. Without going in details we can display here only the final dimensionless form of the respective Euler-Lagrange equations

$$Re^{-1}\,\Delta\Delta\psi = -\left[2\frac{du}{dy}\cdot\frac{\partial\psi}{\partial y\partial\xi} + \frac{\alpha}{y}\frac{du}{dy}\cdot\frac{\partial\psi}{\partial\xi} + \frac{d^2u}{dy^2}\cdot\frac{\partial\psi}{\partial\xi}\right], \quad (7)$$

$$Re^{-1}\frac{du}{dy} = -y - \int_{-\infty}^{\infty}\frac{1}{y^{\alpha}}\left(\frac{\partial}{\partial y}y^{\alpha}\psi\right)\frac{\partial\psi}{\partial\xi}d\xi\,, \quad (8)$$

which are solved with the following boundary conditions

$$\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial\xi} = 0 \quad \text{at} \quad y = \pm 1\,, \quad \int_{-\infty}^{\infty}\left[\left(\frac{\partial\psi}{\partial y}\right)^2 + \left(\frac{\partial\psi}{\partial\xi}\right)^2\right] < +\infty\,. \quad (9)$$

In the above formulae $Re = \sqrt{Aa^3/\nu^{2\alpha}}$ and $\Delta = \frac{\partial}{\partial y}y^{-\alpha}\frac{\partial}{\partial y}y^{\alpha} + \frac{\partial^2}{\partial\xi^2}$.

Due to its complexity the above boundary value problem is treated numerically by a method similar to that of [7]. The infinite interval is "compressed" into the finite interval [-1,1] by means of appropriate coordinate transformation.

Results and discussion

In the previous work [7] we have already presented the results for plane Poiseuille flow and here we concentrate on pipe flow. Our theoretical predictions are compared with the work of Laufer [10], so all the experimental data plotted in figures below are taken from that work.

In Fig.2 is presented the total dissipation as a function of frictional Reynolds number. It is well seen that the total dissipation of the stochastic flow becomes lesser than that of the laminar one for Reynolds numbers greater than 15. It does mean that we have indeed obtained a solution which brings a minimum to the dissipation functional. So that it stands clear that the stochastic solution has a right of existence and its occurrence is consistent with the least-dissipation principle. This flow, however, can be employed also to predict to some extent even the specific quantitative characteristics of the turbulent flow. Fig.3 presents the evolution of Reynolds stress with Reynolds number and comparison with experiments [10]. It is seen that the agreement is quantitatively good both in the core of the flow and in the viscous sublayer. In Fig.4 is plotted the energy "component" $\sqrt{\overline{u_x'^2}}/u^*$ and compared with experiment. On Fig.5 are presented the averaged values of derivatives of longitudinal pulsation. Agreement with experiment is qualitatively good. In the end Fig.6 presents the shape of a structure which is in qualitative agreement with the structures observed in [11] for the flow in turbulent boundary layer.

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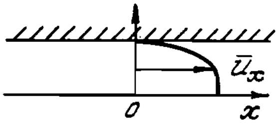

Fig. 1

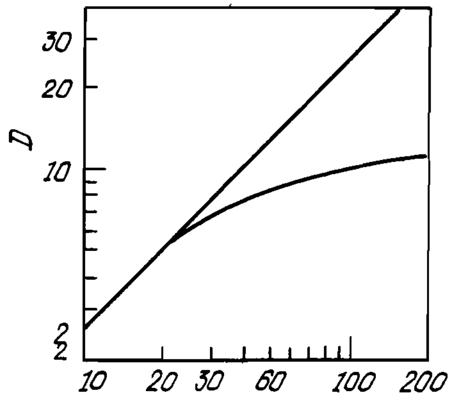

Fig. 2

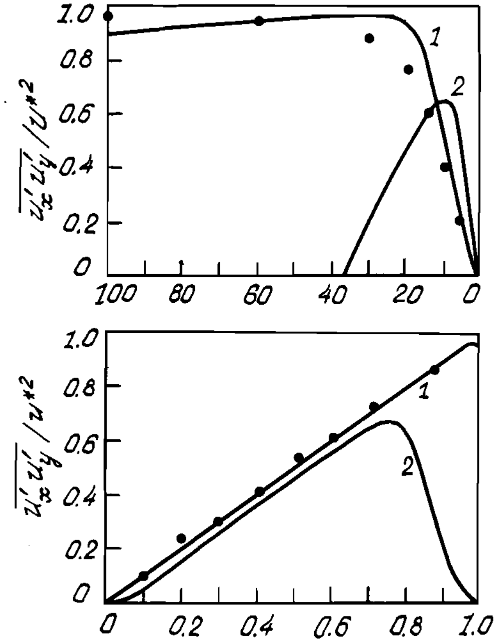

Fig. 3

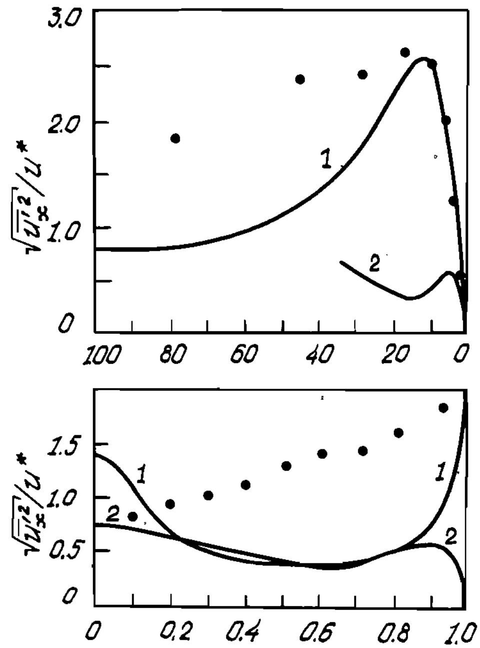

Fig. 4

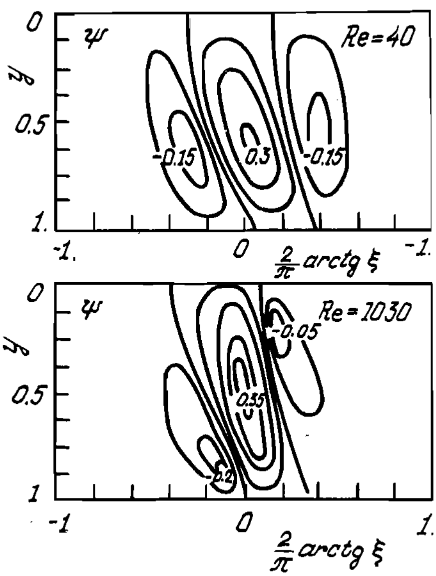

Fig. 5

LEGEND:
1 - Re = 1030, 2 - Re = 40,
● - experiments [10] for Re = 1030