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*SIAM Journal on Applied Mathematics*, Vol. 45, No. 2. (Apr., 1985), pp. 289-311.

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*SIAM Journal on Applied Mathematics* is currently published by Society for Industrial and Applied Mathematics.

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## STOCHASTIC FUNCTIONAL EXPANSION FOR RANDOM MEDIA WITH PERFECTLY DISORDERED CONSTITUTION\*

C. I. CHRISTOV† AND K. Z. MARKOV‡

**Abstract.** The random field created by a system of random points which are centers of perfectly disordered equi-sized spheres of a finite radius is introduced and named the PDS-field. The notion of Perfect Disorder is defined statistically correctly taking the cumulants instead of moments to be  $\delta$ -functions. The Volterra–Wiener functional expansion with the PDS-field as a basis function is considered and a system of orthogonal Wiener functionals is explicitly constructed. The expansion is employed to the problem of specifying the overall conductivity for a random medium containing an array of perfectly disordered spherical inclusions. An infinite hierarchy of coupled equations for the kernels of the Wiener functionals is derived. It is shown that the Wiener expansion with respect to the PDS-field is also a virial one, i.e. the  $n$ th order term contributes quantities of order  $c^n$ , where  $c$  is the volume fraction of the inclusions. This allows the full stochastic solution to the problem of heat conduction through the random medium in question to be broken into consecutive steps. The first-order approximation yields a formula for the overall conductivity which coincides with the linear part of the known Maxwell relation. A heuristic approach in which only the first term in the Wiener expansion is retained is shown to yield, within the frame of the so-called singular approximation, the full Maxwell relation for the overall conductivity. The equations for the second-order approximation are investigated and the kernels of the respective Wiener functionals are related to the temperature fields in an unbounded material, containing one or two spherical inclusions.

**Introduction.** In recent years, development of adequate statistical techniques for stochastic systems has constantly been in the focus of scientific attention because of its outstanding importance for a number of physical applications such as turbulence, heterogeneous flows and materials, random noises, etc. All these phenomena are modelled by systems of differential equations which are stochastically nonlinear: suspensions and composite materials—due to random coefficients; turbulence and noises—due to intrinsic nonlinearity of the governing equations [1]. In other words, in all the above cases a nonlinear operator  $T$  is implicitly defined which transforms the random field of coefficients (or disturbances) into the random field of solution. A very general and powerful method for a successful attack of stochastic systems consists then in expanding the operator  $T$  into Volterra series and truncating this series afterward in order to get approximate solutions. The underlying aim of this paper is to be seen as an illustration of the latter statement.

Wiener [26] was first to employ Volterra series for a stochastic problem. He chose the Gaussian white random noise as a basis function for the series and used the method for identification of nonlinear systems with Gaussian input. Due to Cameron and Martin [2], who built the rigorous basis beneath the Wiener method and introduced the Hermite polynomials into the Wiener functionals in order to orthogonalize them, the method was named the Wiener–Hermite expansion and has been since applied to a number of stochastic problems (cf. [21] for a detailed survey). The very heart of the Wiener method—employment of Gaussian white noise as a basis function—makes it, however, not quite fitted to the most popular physical systems. For instance, in the above mentioned cases—turbulence, heterogeneous flows and materials, etc.—the Gaussianity implies a very dense population of “point-like” inhomogeneities thus not

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\* Received by the editors October 4, 1983, and in revised form June 27, 1984.

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being able to account for their finite size. As a result the Wiener-Hermite expansion turned out in many cases to be too formal an approach, which led to troubles when estimating the higher order kernels and raised the need of a physically unwarranted renormalization, cf. [19].

The most natural development of the Wiener approach is to include more a priori given physical information when choosing the basis function. The first thing to be noted along this line is that all random fields above listed are created by systems of random points in the sense that they are superpositions of equi- (or similarly-) shaped inhomogeneities situated at random points (spatially or temporarily). In turn, the systems of random points which occur in physics and mechanics are often statistically independent or can be well approximated by such ones [23], [24]. This means that in case we have no detailed information about the constitution of a random medium (and this is the typical case we are faced with in reality), the most suited basis function should unequivocally be the Poisson random function.

First to replace the Gaussian white random function by the Poisson white random function in Volterra expansions was Ogura [20], who constructed the orthogonal Wiener functionals and named the respective series Poisson-Wiener expansion. He did so, however, only on a formal basis rather than on the basis of the above reasoning and without any physical applications. Christov [3], [4], [5] only recently revealed the spectacular performance of the Poisson-Wiener expansion for nonlinear stochastic systems and developed the necessary technique with application to the model case of Burgers turbulence. Further the problem of application of this method to the case of dilute suspensions has been posed [6]. The Poisson-Wiener expansion has also been successfully employed to the bifurcation and emerging of a stochastic solution to a problem for the plane Poiseuille flow [7].

It should be noted here that if more detailed information about the inclusion distribution in a heterogeneous medium is available, a natural extension of the Wiener method is to take the given random field of inclusion centers as a basis function in the Volterra-Wiener series [15].

When higher-order approximation is concerned, the application of the functional series based on the plain Poissonian field is faced with certain hard and well-acknowledged difficulties stemming from the fact that the inclusions possess finite size. The latter contradicts the possibility for the random points in a Poissonian field to appear arbitrarily close one to another. This raises the need to construct a Poisson-like system of random points for which the points are only correlated in such a manner that their appearance arbitrarily close one to another is forbidden in order to prevent overlapping of the inhomogeneities.

In the present paper such a Poissonian-like system is developed for the case when the inhomogeneities are equi-sized spheres, i.e. when the system comprises centers of identical spheres which are chaotically distributed in the three-dimensional space  $\mathbb{R}^3$  and there is no intersection among the spheres. This random field is named, borrowing a Kröner's coinage [12], Perfect Disorder of Spheres or PDS-field. The plain Poisson system will be referred to as perfect disorder of points or the PDP-field.

The newly constructed system—the PDS-field—is introduced into the Volterra-Wiener series replacing the PDP-field. It turns out that the Charlier polynomials which are employed in the Poisson-Wiener method can be kept for the PDS expansion if only some fully natural limitations are imposed on the kernels of functionals. To demonstrate the performance of the PDS expansion the problem of specifying the temperature field in a random medium containing a system of perfectly disordered spherical inclusions is considered in detail. A special attention is paid to the determination of the overall conductivity for such a random medium.

**1. Volterra–Wiener functional expansions.** Consider a general functional relation

$$(1.1) \quad u(\mathbf{x}) = \mathbf{T}[v(\mathbf{x})],$$

where  $u, v$  are, for definiteness, functions of the vectors  $\mathbf{x} \in \mathbb{R}^3$  and  $\mathbf{T}$  is an operator. Such relations always appear when dealing with physical systems—to any system we attribute an operator  $\mathbf{T}$  which transforms the “input”  $v$  into the “output”  $u$ ; the interpretation of  $u$  and  $v$  depends on the physical system under consideration.

According to the general idea of Volterra [25] the operator  $\mathbf{T}$  can be represented as a functional series—the Volterra series—

$$(1.2) \quad u(\mathbf{x}) = \sum_{n=0}^{\infty} G_v^{(n)}[u],$$

$$G_v^{(n)}[u] = \int \cdots \int K_u^{(n)}(\mathbf{x}-\mathbf{y}_1, \cdots, \mathbf{x}-\mathbf{y}_n) v(\mathbf{y}_1) \cdots v(\mathbf{y}_n) d^3\mathbf{y}_1 \cdots d^3\mathbf{y}_n;$$

we assume tacitly that the operator  $\mathbf{T}$  is homogeneous, i.e.  $u(\mathbf{x}+\mathbf{y}) = \mathbf{T}[v(\mathbf{x}+\mathbf{y})]$ ,  $\forall \mathbf{y} \in \mathbb{R}^3$ . Hereafter, if the integration domain is not explicitly indicated, the integrals are taken over the whole  $\mathbb{R}^3$ .

It was Wiener [26] who introduced Volterra series in the theory of stochastic processes developing an arbitrary stochastic function into a functional series with respect to the so-called white Gaussian noise (the derivative of the Brownian motion process). Wiener also offered the idea to orthogonalize the functionals  $G^{(n)}$ , which proved crucial when separating the influence of different members of the series (1.2) on the investigated function  $u(\mathbf{x})$ . Cameron and Martin [2] underlaid the rigorous mathematical basis beneath the Wiener theory and showed that the orthogonal Wiener functionals are generated by the multivariate stochastic Hermite polynomials of the Gaussian white noise, cf. [2], [11], [21] for details.

Later on the Wiener–Hermite expansion has been used in various physical problems such as system identification, turbulence, etc., cf. [11], [19], [21], and proved to be a powerful tool when attacking linear and certain nonlinear problems. The main disadvantage of the method lies, as it turned out, in the employment of the Gaussian white noise as a basis function in the expansion (1.2). As it was argued in the Introduction, in most of the problems concerned, e.g., with heterogeneous materials, the random fields appear to be closer to point random functions than to the continuous in stochastic sense Brownian-motion process. (This causes, e.g., the non-Gaussian terms  $G^{(n)}$ ,  $n \geq 2$ , in the Wiener–Hermite expansion to increase to infinity temporarily or spatially, especially in nonlinear problems, cf. [19].) That is why in the present work we shall employ the Poisson random function or a certain its generalization as a basis function in the expansion (1.2).

**2. Poisson–Wiener expansion.** Let  $\mathbf{x}_j$  be a random system of points in  $\mathbb{R}^3$  and let  $W$  be, for simplicity, a deterministic function. Consider the random field

$$(2.1) \quad \eta(\mathbf{x}) = \sum_j W(\mathbf{x}, \mathbf{x}_j),$$

i.e.  $\eta$  is the superposition of equi-shaped disturbances located at the random centers  $\mathbf{x}_j$ .

A particular case of the random field (2.1) is the so-called random-density function [24], [23]

$$(2.2) \quad \varphi(\mathbf{x}) = \sum_j \delta(\mathbf{x} - \mathbf{x}_j).$$

(When  $\mathbf{x}$ ,  $\mathbf{x}_j$  are temporal coordinates,  $\varphi$  is the shot-effect random process [24].) The function  $\varphi$  plays a major role in the theory of point random fields or processes. The underlying reason is that any random field (2.1) is obviously a linear transformation of  $\varphi$

$$(2.3) \quad \eta(\mathbf{x}) = \int W(\mathbf{x}, \mathbf{y}) \varphi(\mathbf{y}) d^3\mathbf{y}.$$

Moreover, the stochastic properties of  $\varphi(\mathbf{x})$  are fully determined by the probability distribution of the random system  $\mathbf{x}_j$  and vice versa [24]. That is why, when dealing with random systems  $\mathbf{x}_j$ , we shall hereafter focus our attention on the respective functions  $\varphi(\mathbf{x})$ .

Let the points  $\mathbf{x}_j$  be distributed according to a Poissonian law [23], [24]. This means, in particular, that there is no order in the random set  $\mathbf{x}_j$ —the location of any point  $\mathbf{x}_k$  is statistically independent of the location of the rest of the points  $\mathbf{x}_m$ ,  $m \neq k$ . In this sense, the Poissonian random set can be called Perfect Disorder of Points (PDP), or a perfect white noise as it was named in [3].

The cumulants of the random density function (2.2) are called by Stratonovich [24] generalized correlation functions for the random system  $\mathbf{x}_j$ . In [24] it is shown that the PDP-field is characterized by the fact that its generalized correlation functions are Dirac's delta functions

$$(2.4) \quad \begin{aligned} \kappa_1^\varphi(\mathbf{y}) &= \gamma, & \kappa_2^\varphi(\mathbf{y}_1, \mathbf{y}_2) &= \gamma \delta(\mathbf{y}_1 - \mathbf{y}_2), \dots, \\ \kappa_n^\varphi(\mathbf{y}_1, \dots, \mathbf{y}_n) &= \gamma \delta(\mathbf{y}_1 - \mathbf{y}_2) \dots \delta(\mathbf{y}_1 - \mathbf{y}_n), \dots, \end{aligned}$$

where  $\kappa_n^\varphi(\mathbf{y}_1, \dots, \mathbf{y}_n) \equiv \text{cml}[\varphi(\mathbf{y}_1), \dots, \varphi(\mathbf{y}_n)]$  denote the cumulants for the set of random variables  $\varphi(\mathbf{y}_i)$ ,  $i = 1, \dots, n$ , and  $\gamma$  is the mean number of points per unit volume.

As is known the cumulants, and not the moments, are the measure of statistical dependence between random variables [13], [24]. Thus the relations (2.4) represent a formal expression of the mentioned fact that there is no statistical connections between the points in a PDP-field. It should be noted that the name Perfect Disorder was coined by Kröner [12] who, however, took the respective moments instead of cumulants to be  $\delta$ -functions. This definition was later on criticised, e.g., by Hori and Yonezawa [10] (see also [17]) who, starting from Kröner's definition and speaking of a certain "exclusion effect", arrived, as it should have been expected, at the conclusion that moments should be replaced by cumulants in the said definition.

Let us take now the random density function  $\varphi(\mathbf{x})$  for a PDP-field as a basis function in the Volterra series (1.2). As shown by Ogura [20], the orthogonal Wiener functionals have then the form

$$(2.5) \quad G_\varphi^{(n)} = \int \dots \int K^{(n)}(\mathbf{x} - \mathbf{y}_1, \dots, \mathbf{x} - \mathbf{y}_n) C_\varphi^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) d^3\mathbf{y}_1 \dots d^3\mathbf{y}_n,$$

where  $C_\varphi^{(n)}$  are the so-called multivariate Chartier polynomials of order  $n$

$$(2.6) \quad \begin{aligned} & C_\varphi^{(n)}(\mathbf{y}_{i_1}, \dots, \mathbf{y}_{i_n}) \\ &= \varphi(\mathbf{y}_{i_1}) [\varphi(\mathbf{y}_{i_2}) - \delta(\mathbf{y}_{i_1} - \mathbf{y}_{i_2})] \dots [\varphi(\mathbf{y}_{i_n}) - \delta(\mathbf{y}_{i_1} - \mathbf{y}_{i_n}) - \dots - \delta(\mathbf{y}_{i_{n-1}} - \mathbf{y}_{i_n})] \\ & \quad - \gamma \left\{ \sum_{(n, n-1)} \varphi(\mathbf{y}_{j_1}) [\varphi(\mathbf{y}_{j_2}) - \delta(\mathbf{y}_{j_1} - \mathbf{y}_{j_2})] \dots [\varphi(\mathbf{y}_{j_{n-1}}) - \dots - \delta(\mathbf{y}_{j_{n-2}} - \mathbf{y}_{j_{n-1}})] \right\} \end{aligned}$$

$$\begin{aligned}
 & + \dots + (-\gamma)^{n-2} \sum_{(n,2)} \varphi(\mathbf{y}_{j_1})[\varphi(\mathbf{y}_{j_2}) - \delta(\mathbf{y}_{j_1} - \mathbf{y}_{j_2})] \\
 & + (-\gamma)^{n-1} \sum_{\nu=1}^n \varphi(\mathbf{y}_{j_\nu}) + (-\gamma)^n;
 \end{aligned}$$

cf. [20] for more details. The first three polynomials are

$$\begin{aligned}
 (2.7) \quad C_\varphi^{(0)} &= 1, \quad C_\varphi^{(1)}(\mathbf{y}_1) = \varphi(\mathbf{y}_1) - \gamma, \\
 C_\varphi^{(2)}(\mathbf{y}_1, \mathbf{y}_2) &= \varphi(\mathbf{y}_1)[\varphi(\mathbf{y}_2) - \delta(\mathbf{y}_1 - \mathbf{y}_2)] - \gamma[\varphi(\mathbf{y}_1) + \varphi(\mathbf{y}_2)] + \gamma^2, \\
 C_\varphi^{(3)}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= \varphi(\mathbf{y}_1)[\varphi(\mathbf{y}_2) - \delta(\mathbf{y}_1 - \mathbf{y}_2)][\varphi(\mathbf{y}_3) - \delta(\mathbf{y}_1 - \mathbf{y}_3) - \delta(\mathbf{y}_2 - \mathbf{y}_3)] \\
 &\quad - \gamma\{\varphi(\mathbf{y}_1)[\varphi(\mathbf{y}_2) - \delta(\mathbf{y}_1 - \mathbf{y}_2)] + \varphi(\mathbf{y}_2)[\varphi(\mathbf{y}_3) - \delta(\mathbf{y}_2 - \mathbf{y}_3)] \\
 &\quad\quad\quad + \varphi(\mathbf{y}_3)[\varphi(\mathbf{y}_1) - \delta(\mathbf{y}_3 - \mathbf{y}_1)]\} \\
 &\quad + \gamma^2[\varphi(\mathbf{y}_1) + \varphi(\mathbf{y}_2) + \varphi(\mathbf{y}_3)] - \gamma^3.
 \end{aligned}$$

It is readily seen that the functionals  $G_\varphi^{(n)}$  are centered stochastic functions,  $\langle G_\varphi^{(n)} \rangle = 0$ , since

$$(2.8) \quad \langle C_\varphi^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) \rangle = 0,$$

cf. [20]. Hereafter the brackets  $\langle \cdot \rangle$  denote statistical averaging. Further Ogura [20] showed that  $G_\varphi^{(n)}$  are orthogonal because of the stochastic orthogonality of  $C_\varphi^{(n)}$  which reads

$$(2.9) \quad \langle C_\varphi^{(n)}(\mathbf{y}_{i_1}, \dots, \mathbf{y}_{i_n}) C_\varphi^{(m)}(\mathbf{y}_{j_1}, \dots, \mathbf{y}_{j_m}) \rangle = \gamma^n \delta_{nm} \Delta^n(\mathbf{i}, \mathbf{j}).$$

Here  $\delta_{nm}$  is the Kronecker delta,  $\Delta^n(\mathbf{i}, \mathbf{j})$  equals the sum of all distinct products of  $n$  Dirac deltas of the form  $\delta(\mathbf{y}_{i_\nu} - \mathbf{y}_{j_\mu})$ ,  $\mathbf{i} = (i_1, \dots, i_n)$ ,  $\mathbf{j} = (j_1, \dots, j_n)$ ,  $i_\nu$  and  $j_\mu$  appear only once in each product. Hence  $\Delta^n(\mathbf{i}, \mathbf{j})$  contains  $n!$  terms. For instance

$$(2.10) \quad \Delta^2(\mathbf{i}, \mathbf{j}) = \delta(\mathbf{y}_{i_1} - \mathbf{y}_{j_1})\delta(\mathbf{y}_{i_2} - \mathbf{y}_{j_2}) + \delta(\mathbf{y}_{i_1} - \mathbf{y}_{j_2})\delta(\mathbf{y}_{i_2} - \mathbf{y}_{j_1}).$$

Up to this point the difference between the Wiener-Hermite and Poisson-Wiener expansion is not evident. The latter, however, exhibits third moments which are the respective multidimensional Dirac deltas while the third moments of the former vanish. A way of deriving the formula for the third moments of Charlier polynomials is outlined in [5]. This formula will not be needed in its general form; that is why we recite the respective relation from [5] without going in much detail, namely, for  $n \leq m \leq l$ , one has

$$\begin{aligned}
 (2.11) \quad \langle C_\varphi^{(n)}(\mathbf{y}_{i_1}, \dots, \mathbf{y}_{i_n}) C_\varphi^{(m)}(\mathbf{y}_{j_1}, \dots, \mathbf{y}_{j_m}) C_\varphi^{(l)}(\mathbf{y}_{k_1}, \dots, \mathbf{y}_{k_l}) \rangle &= \begin{cases} A^{nml}, & m+n \geq l \\ 0, & m+n < l, \end{cases} \\
 A^{nml} &= \gamma^l \sum \Delta^{m+n-l}(\mathbf{i}, \mathbf{j}, \mathbf{k}) \Delta^{2l-m-n}(\mathbf{j}', \mathbf{k}') \\
 &\quad + \gamma^{l+1} \sum \Delta^{m+n-l-2}(\mathbf{i}'', \mathbf{j}'', \mathbf{k}'') \Delta^{2l-n-m-3}(\mathbf{j}''', \mathbf{k}''') + \dots, \\
 \mathbf{i} &= \begin{cases} (i_{n+m-l+1}, \dots, i_n) & \text{for } n \geq n+m-l+1, \\ \text{vanishes} & \text{for } n < n+m-l+1, \end{cases} \\
 \mathbf{j} &= (j_1, \dots, j_{n+m-l}), \quad \mathbf{j}' = (i_{m+n-l+1}, \dots, i_1, \dots, i_{l-m}), \\
 \mathbf{k} &= (k_{l-1}, \dots, k_{n+m-l}), \quad \mathbf{k}' = \begin{cases} (k_{m+n-l+1}, \dots, k_l) & \text{for } l \geq n+m-l+1, \\ \text{vanishes} & \text{for } l < n+m-l+1. \end{cases}
 \end{aligned}$$

In the same manner  $\mathbf{i}'', \mathbf{j}'', \mathbf{k}''$ ,  $\mathbf{i}''', \mathbf{j}''', \mathbf{k}'''$ , etc. are specified.

The equality (2.11) makes the Poisson–Wiener expansion a powerful tool when attacking nonlinear stochastic systems because it enables one to derive intrinsically nonlinear equations for the kernels, as it will be seen below when dealing with applications. This is not the case with the Wiener–Hermite expansion for which the nonlinearity of the original system of stochastic differential equations is modelled in the system for kernels only through the higher-order kernels since the respective third moments vanish.

Another pleasant feature of the Poisson–Wiener expansion is that in a sense it is a virial expansion with respect to the volume fraction  $c = \gamma V$  of the inclusions (particles, eddies, etc.);  $V$  is the volume of a single inclusion. This is a corollary of (2.9) and (2.11) which yield that the contribution of a functional of order  $n$  into the second and third moments is of order  $c^n$  when  $c$  is small compared to unity. Neglecting all the terms of order  $p \geq n$  in the Poisson–Wiener expansion, one thus makes a mistake of order  $c^n$  or less. Nothing like that holds for the Wiener–Hermite expansion for which the higher order terms are responsible for deviation from Gaussianity and have no clear physical meaning.

The straight application of the Poisson–Wiener expansion to particulate media is possible, however, only for dilute concentration  $c \ll 1$ , i.e. within the accuracy  $o(c)$ . The reason is that the inclusions have finite dimension and if their centers are perfectly disordered there is nothing to prevent their overlapping at nondilute concentration  $c$ . That is why a certain modification of the PDP-field will be introduced in the next section for which overlapping of the inclusions would be impossible.

**3. Random field created by a system of perfectly disordered spheres (PDS-field).** Let us consider an arbitrary statistically homogeneous system of points  $\mathbf{x}_j \in \mathbb{R}^3$ . After Stratonovich [24], we introduce the multipoint distribution functions  $f_n(\mathbf{y}_1, \dots, \mathbf{y}_n)$ ,  $n = 1, 2, \dots$ , which give the probability  $dP$  of simultaneous finding of a point from  $\mathbf{x}_j$  per each of the infinitesimal volumes  $\mathbf{y}_i < \mathbf{y} < \mathbf{y}_i + d^3\mathbf{y}_i$ ,  $i = 1, \dots, n$ , to be

$$(3.1) \quad dP = f_n(\mathbf{y}_1, \dots, \mathbf{y}_n) d^3\mathbf{y}_1 \cdots d^3\mathbf{y}_n.$$

The functions  $f_n$  are to satisfy, in particular, the conditions

$$\frac{1}{|D|^n} \int_D \cdots \int_D f_n(\mathbf{y}_1, \dots, \mathbf{y}_n) d^3\mathbf{y}_1 \cdots d^3\mathbf{y}_n \xrightarrow{D \rightarrow \mathbb{R}^3} \gamma^n,$$

$n = 1, 2, \dots$ , where  $\gamma$ , let be repeated, is the average number of points  $\mathbf{x}_j$  per unit volume,  $|D| = \text{volume}(D)$ .

Along with the distribution functions  $f_n$  it is convenient to have also the so-called correlation functions of distribution  $g_n$  for the system  $\mathbf{x}_j$  [24], which are related to  $f_n$  in the same manner as cumulants are related to moments, namely,

$$(3.2) \quad \begin{aligned} f_1(\mathbf{y}_1) &= g_1(\mathbf{y}_1) = \gamma, \\ f_2(\mathbf{y}_1, \mathbf{y}_2) &= g_2(\mathbf{y}_1, \mathbf{y}_2) + g_1(\mathbf{y}_1)g_1(\mathbf{y}_2), \\ f_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= g_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) + 3\{g_1(\mathbf{y}_1)g_2(\mathbf{y}_2, \mathbf{y}_3)\}_s + g_1(\mathbf{y}_1)g_1(\mathbf{y}_2)g_1(\mathbf{y}_3). \end{aligned}$$

Here  $\{\cdot\}_s$  denotes a symmetrization with respect to all the arguments of the function in the brackets.

As shown in [24], the functions  $g_n$  furnish full information about statistical dependence in the excerpts comprising  $n$  points, arbitrary taken out of the system  $\mathbf{x}_j$ . The Poisson system of points—the PDP-field—corresponds to the particular case for

which

$$(3.3) \quad g_1 = \gamma, \quad g_n = 0, \quad n \geq 2,$$

so that  $f_n = \gamma^n$ ,  $n = 1, 2, \dots$ .

Consider the random density function  $\varphi(\mathbf{x})$ , cf. (2.2), generated by the random system  $\mathbf{x}_j$  under discussion. Its cumulants, the Stratonovich generalized correlation functions, cf. § 2, are connected with the correlation functions  $g_n$  as follows

$$(3.4) \quad \begin{aligned} \kappa_1^\varphi(\mathbf{y}_1) &= \langle \varphi(\mathbf{y}_1) \rangle = g_1(\mathbf{y}_1) = \gamma, \\ \kappa_2^\varphi(\mathbf{y}_1, \mathbf{y}_2) &= g_1(\mathbf{y}_1)\delta(\mathbf{y}_1 - \mathbf{y}_2) + g_2(\mathbf{y}_1, \mathbf{y}_2), \\ \kappa_3^\varphi(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) &= g_1(\mathbf{y}_1)\delta(\mathbf{y}_1 - \mathbf{y}_2)\delta(\mathbf{y}_1 - \mathbf{y}_3) \\ &\quad + 3\{\delta(\mathbf{y}_1 - \mathbf{y}_2)g_2(\mathbf{y}_1, \mathbf{y}_3)\}_s + g_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3), \\ &\quad \dots \end{aligned}$$

[24]. Obviously, for the PDP-field for which (3.3) holds, (3.4) reduces to (2.4).

Making use of the known relations between moments and cumulants [13], [24], it is easy to obtain the moments of the random density function  $\varphi(\mathbf{x})$  through the multipoint distribution functions

$$(3.5) \quad \begin{aligned} \langle \varphi(\mathbf{y}) \rangle &= f_1(\mathbf{y}) = \gamma, \\ \langle \varphi(\mathbf{y}_1)\varphi(\mathbf{y}_2) \rangle &= f_1(\mathbf{y}_1)\delta(\mathbf{y}_1 - \mathbf{y}_2) + f_2(\mathbf{y}_1, \mathbf{y}_2), \\ \langle \varphi(\mathbf{y}_1)\varphi(\mathbf{y}_2)\varphi(\mathbf{y}_3) \rangle &= f_1(\mathbf{y}_1)\delta(\mathbf{y}_1 - \mathbf{y}_2)\delta(\mathbf{y}_1 - \mathbf{y}_3) \\ &\quad + 3\{\delta(\mathbf{y}_1 - \mathbf{y}_2)f_2(\mathbf{y}_1, \mathbf{y}_3)\}_s + f_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3), \\ &\quad \dots \end{aligned}$$

It is time now to see how the functions  $f_n$  and  $g_n$  can be engaged in constructing the required set of random points. Hereafter, we confine ourselves to the case of equi-sized spheres of a radius  $a$ . Obviously, their centers cannot appear closer than  $2a$ . Let us assume that outside this distance of  $2a$  the spheres can appear at arbitrary positions which are statistically independent. So, the probability of finding two points (spheres centers) closer than  $2a$  is zero while separating them at distances larger than  $2a$  makes the points statistically independent, i.e.

$$f_2(\mathbf{y}_1, \mathbf{y}_2) = \begin{cases} 0 & \text{for } |\mathbf{y}_1 - \mathbf{y}_2| \leq 2a, \\ f_1(\mathbf{y}_1)f_1(\mathbf{y}_2) = \gamma^2 & \text{for } |\mathbf{y}_1 - \mathbf{y}_2| > 2a. \end{cases}$$

(A similar form of the two-point distribution function for nonoverlapping statistically independent system of spheres was adopted, e.g., in [8].) In a more concise form the latter formula reads

$$(3.6) \quad f_2(\mathbf{y}_1, \mathbf{y}_2) = \gamma^2 Q(\mathbf{y}_1 - \mathbf{y}_2),$$

where

$$Q(\mathbf{x}) = 1 - R(\mathbf{x}), \quad R(\mathbf{x}) = \begin{cases} 1, & |\mathbf{x}| \leq 2a, \\ 0, & |\mathbf{x}| > 2a. \end{cases}$$

Let us note in passing that up to this point the results can be rendered compatible to the Stratonovich definition of "not-drawing-close" points [24] but the higher-order distribution functions will differ from those of Stratonovich. Guided by the above

considerations, we write

$$(3.7) \quad f_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \gamma^3 Q(\mathbf{y}_1 - \mathbf{y}_2) Q(\mathbf{y}_1 - \mathbf{y}_3) Q(\mathbf{y}_2 - \mathbf{y}_3).$$

The last formula secures that in the case when at least two points are closer than  $2a$  the probability of simultaneous occurrence equals zero. It is interesting to summon here the respective formula of Stratonovich which reads

$$f_3(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \frac{2}{3} \gamma^3 Q(\mathbf{y}_1 - \mathbf{y}_2) Q(\mathbf{y}_1 - \mathbf{y}_3).$$

Obviously, there exists a case when two points are closer than  $2a$  and  $f_3 \neq 0$ . In our opinion this fact spoils completely the definition of "not-drawing-close" points if we try to employ it to the random media under discussion.

Going further in interpreting (3.7), one sees that if the three points are separated at distances greater than  $2a$  the probability density  $f_3$  is just the product of the "one-dimensional" probabilities  $f_1$ , i.e. the points become independent on the third level of statistical connections.

The general formula for  $f_n$  is easily derived on the basis of the above considerations

$$(3.8) \quad \begin{aligned} f_n(\mathbf{y}_1, \dots, \mathbf{y}_n) \\ = \gamma^n Q(\mathbf{y}_1 - \mathbf{y}_2) \cdots Q(\mathbf{y}_1 - \mathbf{y}_n) Q(\mathbf{y}_2 - \mathbf{y}_3) \cdots Q(\mathbf{y}_2 - \mathbf{y}_n) \cdots Q(\mathbf{y}_{n-1} - \mathbf{y}_n). \end{aligned}$$

A random field with multipoint distribution functions (3.8) will be called perfect disorder of spheres (PDS-field). If the radius  $a$  of the spheres tends to zero, the PDS-field tends to the PDP-field.

It should be pointed out that the PDS-field seems close to the random system of spheres introduced by Herczynski [9]. However, this author was primarily interested in nearest neighbour distribution, which makes the straightforward comparison between the PDS-field and Herczynski's system difficult and goes beyond the scope of this work.

As it follows from (3.3) and (3.8), the correlation functions of distribution  $g_n$  for the PDS-field vanish if  $|\mathbf{y}_i - \mathbf{y}_j| > 2a$ ,  $i, j = 1, \dots, n$ ,  $i \neq j$ . This corroborates the statement that there is no correlation between the random points  $\mathbf{x}_j$  in the PDS-field provided they are duly separated and once again justifies the name Perfect Disorder of Spheres.

The PDS-field introduced here should play the same role in heterogeneous media with spherical inclusions as the Poisson random field does for the problems with point sources of disturbances. Physically the PDS-field is, of course, only an approximation to the real random fields of inclusions in composites and suspensions since some kind of correlation always exists even outside the distance of  $2a$ . Nevertheless it seems reasonable enough to approximate these random fields by the PDS-field in a great number of applications, especially when we have no detailed knowledge of the internal constitution of the media.

**4. Functional expansion with respect to the PDS-field.** Having introduced the PDS-field, a natural question arises—what are its orthogonal Wiener functionals. As the PDS-field is akin to the PDP-field, the first thing to be checked along this line is whether the Charlier polynomials can again be employed in constructing the needed Wiener functionals. With this aim in view, let us introduce the random density function  $\psi$  for the PDS-field and let us insert it into the Charlier polynomials (2.6) instead of the random density function  $\varphi$  for the PDP-field. As it follows from (2.6), (3.5) and

(3.8), we have

$$\begin{aligned}
 \langle C_\psi^{(0)} \rangle &= \gamma, & \langle C_\psi^{(1)}(\mathbf{y}) \rangle &= 0, \\
 \langle C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2) \rangle &= -\gamma^2 R(\mathbf{y}_1 - \mathbf{y}_2), \\
 \langle C_\psi^{(3)}(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) \rangle & \\
 (4.1) \quad &= \gamma^3 \{ R(\mathbf{y}_1 - \mathbf{y}_2) R(\mathbf{y}_2 - \mathbf{y}_3) + R(\mathbf{y}_2 - \mathbf{y}_3) R(\mathbf{y}_3 - \mathbf{y}_1) + R(\mathbf{y}_3 - \mathbf{y}_1) R(\mathbf{y}_1 - \mathbf{y}_2) \\
 &\quad - R(\mathbf{y}_1 - \mathbf{y}_2) R(\mathbf{y}_2 - \mathbf{y}_3) R(\mathbf{y}_3 - \mathbf{y}_1) \}, \\
 &\dots \\
 \langle C_\psi^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) \rangle &= \gamma^n P^{n(n-1)/2} [R(\mathbf{y}_1 - \mathbf{y}_2), \dots, R(\mathbf{y}_{n-1} - \mathbf{y}_n)],
 \end{aligned}$$

where  $P^{n(n-1)/2}$  is a multivariate polynomial of degree  $n(n-1)/2$  whose arguments are the  $n(n-1)/2$  functions  $R(\mathbf{y}_i - \mathbf{y}_j)$ . The least power of this polynomial is just  $n-1$ . The latter means that the averaged value  $\langle C_\psi^{(n)} \rangle = 0$  only if all of arguments  $\mathbf{y}_j$  are well separated, i.e.  $|\mathbf{y}_i - \mathbf{y}_j| > 2a$ . Respectively, a nonzero average value  $\langle C_\psi^{(n)} \rangle$  is to be expected if at least  $n-1$  pairs of arguments are closer than  $2a$ . For instance  $\langle C_\psi^{(3)} \rangle \neq 0$  when  $|\mathbf{y}_1 - \mathbf{y}_2| \leq 2a$  and  $|\mathbf{y}_2 - \mathbf{y}_3| \leq 2a$  regardless to the relationship between  $\mathbf{y}_1$  and  $\mathbf{y}_3$ .

Thus, it is easily seen that the stochastic functions  $C_\psi^{(n)}$  are not centered. It can be also shown that they are not orthogonal for arbitrary values of their arguments. So, the Charlier polynomials cannot form Wiener functionals for the PDS-field unless some additional requirements are set on their arguments.

Consider closer the functionals

$$(4.2) \quad G_\psi^{(n)}[F] = \int \dots \int K_F^{(n)}(\mathbf{x} - \mathbf{y}_1, \dots, \mathbf{x} - \mathbf{y}_n) C_\psi^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) d^3\mathbf{y}_1 \dots d^3\mathbf{y}_n$$

generated by the Charlier polynomials of the random function  $\psi$ . Due to (4.1), these functionals can be rendered centered stochastic variables if one assumes that

$$(4.3) \quad K_F^{(n)}(\mathbf{z}_1, \dots, \mathbf{z}_n) = 0 \quad \text{if there exists at least one pair of indexes } i, j \text{ for which } |\mathbf{z}_i - \mathbf{z}_j| \leq 2a.$$

This requirement is completely natural in the case under consideration and leads to no loss of generality since in a perfectly disordered system of spheres one never has two points situated closer than  $2a$ . Thus the values of the kernels  $K_F^{(n)}$  have no physical meaning in the regions for which  $|\mathbf{z}_i - \mathbf{z}_j| \leq 2a$  at least for a pair  $\mathbf{z}_i, \mathbf{z}_j$ . This allows one to continue  $K_F^{(n)}$  by zeros in these regions, as it was done in (4.3). It is readily seen now that in the regions where  $\langle C_\psi^{(n)} \rangle \neq 0$  the kernels  $K_F^{(n)}$ , as a consequence of (4.3), vanish so that

$$(4.4) \quad \langle G_\psi^{(n)}[F] \rangle = 0, \quad n = 1, 2, \dots$$

Another important corollary of the assumption (4.3) is that it makes the functionals (4.2) orthogonal in stochastic sense. In order to prove that we make use of the following property of the Charlier polynomials

$$(4.5) \quad C_\psi^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) = C_\psi^{(1)}(\mathbf{y}_1) \dots C_\psi^{(1)}(\mathbf{y}_n) \quad \text{if } \mathbf{y}_i \neq \mathbf{y}_j, \quad i \neq j,$$

which is easily derived from (2.6) and has been effectively engaged by Ogura [20] and Christov [5] when obtaining explicit relations for the second and third moments of the Charlier polynomials respectively.

As far as only functionals, whose kernels obey (4.3), are considered in what follows, the relation (4.5) always holds. Moreover the quantities  $C_\psi^{(1)}(\mathbf{y}_i)$  are indepen-

dent random variables under these conditions. Then, being reminded of (4.3) and (4.5), we write

$$\begin{aligned} & \langle G_\psi^{(n)}[F]G_\psi^{(m)}[H] \rangle \\ &= \int \cdots \int K_F^{(n)}(\mathbf{y}-\mathbf{y}_1, \dots, \mathbf{y}-\mathbf{y}_n) K_H^{(m)}(\mathbf{z}-\mathbf{z}_1, \dots, \mathbf{z}-\mathbf{z}_m) \\ & \quad \cdot \langle C_\psi^{(1)}(\mathbf{y}_1) \cdots C_\psi^{(1)}(\mathbf{y}_n) C_\psi^{(1)}(\mathbf{z}_1) \cdots C_\psi^{(1)}(\mathbf{z}_m) \rangle d^3\mathbf{y}_1 \cdots d^3\mathbf{y}_n d^3\mathbf{z}_1 \cdots d^3\mathbf{z}_m. \end{aligned}$$

The only way to obtain a nonzero value for the above quantity is to have all pairs  $C_\psi^{(1)}(\mathbf{y}_i), C_\psi^{(1)}(\mathbf{z}_j)$  correlated. This is possible solely for  $n = m$ , otherwise we have "free" terms of the type  $\langle C_\psi^{(1)}(\mathbf{y}_i) \rangle$  or  $\langle C_\psi^{(1)}(\mathbf{z}_j) \rangle$  which equals zero, cf. (4.4), and bring the zero value for the entire complex. If  $n = m$ , one has

$$\begin{aligned} & \langle C_\psi^{(1)}(\mathbf{y}_1) \cdots C_\psi^{(1)}(\mathbf{y}_n) C_\psi^{(1)}(\mathbf{z}_1) \cdots C_\psi^{(1)}(\mathbf{z}_n) \rangle \\ &= \sum_1^{n!} \langle C_\psi^{(1)}(\mathbf{y}_{i_1}) C_\psi^{(1)}(\mathbf{z}_{j_1}) \rangle \cdots \langle C_\psi^{(1)}(\mathbf{y}_{i_n}) C_\psi^{(1)}(\mathbf{z}_{j_n}) \rangle; \end{aligned}$$

the sum is taken over all distinct combinations with  $i_\nu$  and  $j_\mu$  occurring only once.

Adopting the notation

$$(4.6) \quad M_\psi^{(2)}(\mathbf{y}, \mathbf{z}) = \langle C_\psi^{(1)}(\mathbf{y}) C_\psi^{(1)}(\mathbf{z}) \rangle = \gamma \delta(\mathbf{y} - \mathbf{z}) - \gamma^2 R(\mathbf{y} - \mathbf{z}),$$

one can write now that

$$\begin{aligned} & \langle G_\psi^{(n)}[F]G_\psi^{(n)}[H] \rangle \\ (4.7) \quad &= \int \cdots \int K_F^{(n)}(\mathbf{y}-\mathbf{y}_1, \dots, \mathbf{y}-\mathbf{y}_n) K_H^{(n)}(\mathbf{z}-\mathbf{z}_1, \dots, \mathbf{z}-\mathbf{z}_n) \\ & \quad \cdot \left[ \sum_1^{n!} M_\psi^{(2)}(\mathbf{y}_{i_1}, \mathbf{z}_{j_1}) \cdots M_\psi^{(2)}(\mathbf{y}_{i_n}, \mathbf{z}_{j_n}) \right] d^3\mathbf{y}_1 \cdots d^3\mathbf{y}_n d^3\mathbf{z}_1 \cdots d^3\mathbf{z}_n \end{aligned}$$

and

$$(4.8) \quad \langle G_\psi^{(n)}[F]G_\psi^{(m)}[H] \rangle = 0, \quad n \neq m,$$

which proves the stochastic orthogonality of the functionals (4.2) whose kernels obey (4.3).

The obtained relations (4.4) and (4.8) yield the important conclusion that the functionals (4.2), generated by the multivariate Charlier polynomials of the PDS-random density function  $\psi(\mathbf{x})$  constitute Wiener orthogonal functionals for the PDS-field, as they did for the PDP-field. For the PDS-field, however, the kernels should always comply with the requirement (4.3).

The formula for the third moment of the functionals (4.2) can be obtained similarly to (4.7). Let  $l \geq m \geq n$ , then

$$(4.9) \quad \langle G_\psi^{(n)}[F]G_\psi^{(m)}[H]G_\psi^{(l)}[P] \rangle = \begin{cases} 0 & \text{for } m+n < l, \\ B^{nml} & \text{for } m+n \geq l, \end{cases}$$

where

$$\begin{aligned} B^{nml} &= \int \cdots \int K_F^{(n)}(\mathbf{y}-\mathbf{y}_1, \dots, \mathbf{y}-\mathbf{y}_n) K_H^{(m)}(\mathbf{z}-\mathbf{z}_1, \dots, \mathbf{z}-\mathbf{z}_m) K_P^{(l)}(\mathbf{w}-\mathbf{w}_1, \dots, \mathbf{w}-\mathbf{w}_l) \\ & \quad \cdot \{ \gamma^l \sum M_\psi^{(3)}(\mathbf{y}_{i_1}, \mathbf{z}_{j_1}, \mathbf{w}_{k_1}) M_\psi^{(2)}(\mathbf{y}_{i_2}, \mathbf{z}_{j_2}) + \gamma^{l+1} \sum \Pi \cdots \} \\ & \quad d^3\mathbf{y}_1 \cdots d^3\mathbf{y}_n d^3\mathbf{z}_1 \cdots d^3\mathbf{z}_m d^3\mathbf{w}_1 \cdots d^3\mathbf{w}_l. \end{aligned}$$

Here

$$\begin{aligned}
 M_{\psi}^{(3)}(\mathbf{y}, \mathbf{z}, \mathbf{w}) &= \langle C_{\psi}^{(1)}(\mathbf{y}) C_{\psi}^{(1)}(\mathbf{z}) C_{\psi}^{(1)}(\mathbf{w}) \rangle \\
 &= \gamma \delta(\mathbf{y} - \mathbf{z}) \delta(\mathbf{y} - \mathbf{w}) - 3\gamma^2 \{ \delta(\mathbf{y} - \mathbf{z}) R(\mathbf{z} - \mathbf{w}) \}_s \\
 &\quad + \gamma^3 [ 3 \{ R(\mathbf{y} - \mathbf{z}) R(\mathbf{z} - \mathbf{w}) \}_s - R(\mathbf{y} - \mathbf{z}) R(\mathbf{z} - \mathbf{w}) R(\mathbf{w} - \mathbf{y}) ].
 \end{aligned}$$

For details see the derivation of (2.11) in [5].

The higher order moments of the multivariate Charlier polynomials can be obtained in a similar way making use of the respective relations for the first-order Charlier polynomials of a single variable. They can be developed if necessary.

Having the functionals (4.2) one can expand every random function into functional series with respect to the PDS-field

$$\begin{aligned}
 (4.10) \quad F(\mathbf{x}) &= G_F^{(0)} + \int K_F^{(1)}(\mathbf{x} - \mathbf{y}) C_{\psi}^{(1)}(\mathbf{y}) d^3\mathbf{y} + \dots \\
 &\quad + \int \dots \int K_F^{(n)}(\mathbf{x} - \mathbf{y}_1, \dots, \mathbf{x} - \mathbf{y}_n) C_{\psi}^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) d^3\mathbf{y}_1 \dots d^3\mathbf{y}_n + \dots.
 \end{aligned}$$

It is natural to name (4.10) the PDS-Wiener expansion.

The expansion (4.10) contains all the stochastic information about the random function  $F(\mathbf{x})$  in the sense that one can calculate various of its characteristics such as correlation functions, spectra, higher-order moments, etc. through the kernels  $K_F^{(n)}$ , making use of the above obtained relations (4.4), (4.7), (4.8).

Taking a closer look at formulas (4.6) to (4.9), one sees that each higher order moment of the multivariate Charlier polynomials consists of terms whose lowest power is  $\gamma^n$ , where  $n$  is the highest power of the polynomials entering the respective formula. Thus, the PDS-Wiener expansion (4.10) is a virial one, like the Poisson-Wiener expansion, cf. § 2,  $n$ th order functional contributes quantities of order  $c^n$ , where  $c$  was the volume fraction of the spheres.

The virial quality of the PDS-Wiener functional expansion is of critical importance when heterogeneous media are considered since it allows one to obtain asymptotically correct results with respect to any prescribed power of the concentration  $c$ .

**5. Temperature field in random suspension of equi-sized spheres.** Consider a two-phase medium which consists of an array of perfectly disordered equi-sized spheres of thermal conductivity  $\kappa_f$  distributed in an unbounded matrix of conductivity  $\kappa_m$ . The random field of conductivity for such a medium has the form

$$(5.1) \quad \kappa(\mathbf{x}) = \langle \kappa \rangle + [\kappa] \int h(\mathbf{x} - \mathbf{y}) C_{\psi}^{(1)}(\mathbf{y}) d^3\mathbf{y} = \langle \kappa \rangle + [\kappa] G_{\psi}^{(1)}[\kappa],$$

as  $C_{\psi}^{(1)} = \psi - \gamma$ , cf. (2.2), (2.6). Here

$$\langle \kappa \rangle = c\kappa_f + (1 - c)\kappa_m = \kappa_m + c[\kappa]$$

is the average conductivity of the medium,  $[\kappa] = \kappa_f - \kappa_m$ ,  $c = \gamma V_a$  is the volume fraction of the inclusions ( $V_a = \frac{4}{3}\pi a^3$  is the volume of a single inclusion) and

$$(5.2) \quad h(\mathbf{x}) = \begin{cases} 1 & \text{for } |\mathbf{x}| \leq a, \\ 0 & \text{for } |\mathbf{x}| > a \end{cases}$$

is the characteristic function for a single inclusion  $V_a$  located at the origin. (The double use of the symbol  $V_a$  will not bring confusion.)

Let the medium undergo a constant temperature gradient

$$(5.3) \quad \langle \nabla T \rangle = \mathbf{G},$$

where  $T = T(\mathbf{x})$  is the random temperature field. The condition (5.3) will play the role of a boundary one in what follows.

The very form of the conductivity coefficient (5.1) hints the idea to seek the temperature field in the medium as a PDS-Wiener expansion

$$(5.4) \quad T(\mathbf{x}) = T_0(\mathbf{x}) + \int T_1(\mathbf{x}-\mathbf{y}) C_\psi^{(1)}(\mathbf{y}) d^3\mathbf{y} + \dots \\ + \int \dots \int T_n(\mathbf{x}-\mathbf{y}_1, \dots, \mathbf{x}-\mathbf{y}_n) C_\psi^{(n)}(\mathbf{y}_1, \dots, \mathbf{y}_n) d^3\mathbf{y}_1 \dots d^3\mathbf{y}_n + \dots$$

Here is to be stressed that the kernels  $T_n$  are supposed to satisfy the condition (4.3).

In a sense, the single-integral term in the expansion (5.4) accounts for the influence of single noninteracting inclusions on the temperature field, while the double-integral term accounts for the influence of binary interactions among the inclusions and so forth. Such an interpretation will be corroborated below, in the sense that  $T_1$  will be connected with the temperature field in a body containing a single inclusion,  $T_2$  will be connected with that field in a body containing a pair of inclusions, etc.

The functional expansion for the temperature gradient is

$$(5.5) \quad \nabla T(\mathbf{x}) = \sum_{n=0}^{\infty} G_\psi^{(n)}[\nabla T] = \mathbf{Q}_0 + \int \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) C_\psi^{(1)}(\mathbf{y}) d^3\mathbf{y} \\ + \int \int \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2) C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2) d^3\mathbf{y}_1 d^3\mathbf{y}_2 + \dots,$$

where

$$\mathbf{Q}_n(\mathbf{x}-\mathbf{y}_1, \dots, \mathbf{x}-\mathbf{y}_n) = \nabla_{\mathbf{x}} T_n(\mathbf{x}-\mathbf{y}_1, \dots, \mathbf{x}-\mathbf{y}_n) \\ = (\nabla_{\mathbf{z}_1} + \dots + \nabla_{\mathbf{z}_n}) T_n(\mathbf{z}_1, \dots, \mathbf{z}_n), \quad \mathbf{z}_i = \mathbf{x}_i - \mathbf{y}_i.$$

The kernels  $\mathbf{Q}_i$  as well as  $T_i$  are symmetric functions of their arguments, due to the symmetry properties of the Charlier polynomials  $C^{(n)}$ , cf. (2.6).

For the zeroth-order kernels in (5.4), (5.5) we have

$$(5.6) \quad T_0(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x}, \quad \mathbf{Q}_0 = \mathbf{G},$$

as a consequence of (4.4) and (5.3).

Making use of (5.1), (5.5), one finds the following expression for the heat flux

$$(5.7) \quad \kappa(\mathbf{x}) \nabla T(\mathbf{x}) = \langle \kappa \rangle \sum_{n=0}^{\infty} G_\psi^{(n)}[\nabla T] + [\kappa] \sum G_\psi^{(1)}[\kappa] G_\psi[\nabla T] \\ = \langle \kappa \rangle \left\{ \mathbf{Q}_0 + \int \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) C_\psi^{(1)}(\mathbf{y}) d^3\mathbf{y} \right. \\ \left. + \int \int \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2) C_\psi^{(2)}(\mathbf{y}_1, \mathbf{y}_2) d^3\mathbf{y}_1 d^3\mathbf{y}_2 + \dots \right\} \\ + [\kappa] \left\{ \mathbf{Q}_0 \int h(\mathbf{x}-\mathbf{y}) C_\psi^{(1)}(\mathbf{y}) d^3\mathbf{y} \right. \\ \left. + \int \int h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_1(\mathbf{x}-\mathbf{y}_2) C_\psi^{(1)}(\mathbf{y}_1) C_\psi^{(1)}(\mathbf{y}_2) d^3\mathbf{y}_1 d^3\mathbf{y}_2 + \dots \right\},$$

so that the average heat flux is simply

$$(5.8) \quad \langle \kappa \nabla T \rangle = \kappa^* \langle \nabla T \rangle = \kappa^* \mathbf{G} = \langle \kappa \rangle \mathbf{Q}_0 + [\kappa] \langle G_\psi^{(1)}[\kappa] G_\psi^{(1)}[\nabla T] \rangle$$

—the remainder vanishes due to the orthogonality (4.8) of the functionals (4.2);  $\kappa^*$  stands for the overall heat conductivity of the medium. Together with (4.7), (5.8) yields

$$(5.9) \quad \kappa^* \mathbf{G} = \langle \kappa \rangle \mathbf{G} + [\kappa] \int \int h(\mathbf{x}) \mathbf{Q}_1(\mathbf{y}) [\gamma \delta(\mathbf{x} - \mathbf{y}) - \gamma^2 R(\mathbf{x} - \mathbf{y})] d^3 \mathbf{x} d^3 \mathbf{y}.$$

It is amazing that the overall conductivity  $\kappa^*$  is thus specified by means of the kernel  $\mathbf{Q}$  only.

Let us introduce the function

$$(5.10) \quad S(\mathbf{x}) = T_1(\mathbf{x}) - \gamma \int R(\mathbf{x} - \mathbf{y}) T_1(\mathbf{y}) d^3 \mathbf{y}.$$

The relation (5.9) then simplifies

$$(5.11) \quad \kappa^* \mathbf{G} = \langle \kappa \rangle \mathbf{G} + \gamma [\kappa] \int_{V_a} \nabla S(\mathbf{x}) d^3 \mathbf{x},$$

where it is acknowledged that  $h(\mathbf{x}) = 1$  within the sphere  $V_a$ . In other words, one needs to know the values of  $S(\mathbf{x})$  only within  $V_a$  in order to calculate the overall conductivity  $\kappa^*$  of the medium under discussion. These values, however, depend in general on the whole solution of the stochastic problem in question.

**6. Governing equations for kernels.** The basic problem we are faced with when employing functional expansions is the identification of the respective kernels. In our case it will be done by making use of the differential equation of heat conductivity

$$(6.1) \quad \nabla \cdot [\kappa(\mathbf{x}) \nabla T(\mathbf{x})] = 0$$

with the random field of coefficients  $\kappa(\mathbf{x})$ , given in (5.1).

Let us first average (6.1). The obtained equation will then automatically be satisfied due to (5.6) (since the left-hand side of (5.9) is constant).

When dealing with a Poisson-Wiener expansion, Christov [4], [5] derived equations for the higher-order kernels through multiplying (6.1) by

$$C_\varphi^{(1)}(0), C_\varphi^{(2)}(0, \mathbf{z}), \dots, C_\varphi^{(n)}(0, \mathbf{z}_1, \dots, \mathbf{z}_{n-1})$$

and taking the average. In our case the technique needs some development since the Charlier polynomials of the PDS-field do not form an orthogonal set and orthogonality is achieved only on the level of the functionals (4.2) whose kernels obey (4.3). As a result we should multiply (6.1) by certain functions which satisfy (4.3). In order to keep the manipulations as simple as possible, the following sequence of such functions seems preferable

$$(6.2) \quad \begin{aligned} \tilde{g}_1 &= C_\psi^{(1)}(0), & \tilde{g}_2 &= [1 - R(\mathbf{z})] C_\psi^{(2)}(0, \mathbf{z}), \dots, \\ \tilde{g}_n &= C_\psi^{(n)}(0, \mathbf{z}_1, \dots, \mathbf{z}_{n-1}) \prod_{i,j} [1 - R(\mathbf{z}_i - \mathbf{z}_j)], \dots \end{aligned}$$

Having multiplied (6.1) by  $\tilde{g}_1$  and averaged the result, one finds, after some algebra based on (4.6) to (4.9), the following equation for  $\mathbf{Q}_1$  and  $\mathbf{Q}_2$ :

$$\begin{aligned}
 \nabla \cdot \left\{ \langle \kappa \rangle \left[ \gamma \mathbf{Q}_1(\mathbf{x}) - \gamma^2 \int \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \right] \right. \\
 + [\kappa] \mathbf{G} \left[ \gamma h(\mathbf{x}) - \gamma^2 \int h(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \right] \\
 + [\kappa] \left[ \gamma h(\mathbf{x}) \mathbf{Q}_1(\mathbf{x}) - \gamma^2 \int h(\mathbf{x}-\mathbf{y}) \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \right] \\
 - \gamma^2 \mathbf{Q}_1(\mathbf{x}) \int h(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} - \gamma^2 h(\mathbf{x}) \int \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \\
 + \gamma^3 \iint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_1(\mathbf{x}-\mathbf{y}_2) [R(\mathbf{y}_2-\mathbf{y}_1) R(\mathbf{y}_1) + R(\mathbf{y}_1) R(\mathbf{y}_2) \\
 + R(\mathbf{y}_2) R(\mathbf{y}_2-\mathbf{y}_1) - R(\mathbf{y}_2-\mathbf{y}_1) R(\mathbf{y}_1) R(\mathbf{y}_2)] d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
 + 2[\kappa] \left[ \gamma^2 \int h(\mathbf{x}-\mathbf{y}) \mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x}) d^3\mathbf{y} \right. \\
 - \gamma^3 \iint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}) R(\mathbf{y}_1-\mathbf{y}_2) d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
 - \gamma^3 \iint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2) R(\mathbf{y}_2) d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
 \left. \left. + \gamma^4 \iiint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3) R(\mathbf{y}_2) R(\mathbf{y}_1-\mathbf{y}_3) d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3 \right] \right\} = 0.
 \end{aligned}
 \tag{6.3}$$

Having multiplied (6.1) by the function  $\tilde{g}_2$  from (6.2), one finds similarly the following equation for  $\mathbf{Q}_1$ ,  $\mathbf{Q}_2$ , and  $\mathbf{Q}_3$ :

$$\begin{aligned}
 \nabla \cdot \left\{ 2\gamma^2 \langle \kappa \rangle \mathbf{Q}_2(\mathbf{x}, \mathbf{x}-\mathbf{z}) - 2\gamma^3 \langle \kappa \rangle \right. \\
 \cdot \left[ \int \mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z}) R(\mathbf{y}) d^3\mathbf{y} + \int \mathbf{Q}_2(\mathbf{x}, \mathbf{x}-\mathbf{y}) R(\mathbf{y}-\mathbf{z}) d^3\mathbf{y} \right] \\
 + 2\gamma^4 \langle \kappa \rangle \iint \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2) R(\mathbf{y}_1) R(\mathbf{y}_2-\mathbf{z}) d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
 + \gamma^2 [\kappa] [h(\mathbf{x}) \mathbf{Q}_1(\mathbf{x}-\mathbf{z}) + h(\mathbf{x}-\mathbf{z}) \mathbf{Q}_1(\mathbf{x})] \\
 - \gamma^3 [\kappa] \left[ \mathbf{Q}_1(\mathbf{x}-\mathbf{z}) \int h(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} + \mathbf{Q}_1(\mathbf{x}) \int h(\mathbf{x}-\mathbf{y}) R(\mathbf{y}-\mathbf{z}) d^3\mathbf{y} \right. \\
 \left. + h(\mathbf{x}) \int \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) R(\mathbf{y}-\mathbf{z}) d^3\mathbf{y} + h(\mathbf{x}-\mathbf{z}) \int \mathbf{Q}_1(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \right] \\
 + \gamma^4 [\kappa] \iint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_1(\mathbf{x}-\mathbf{y}_2) [R(\mathbf{y}_1) R(\mathbf{y}_2-\mathbf{z}) + R(\mathbf{y}_2) R(\mathbf{y}_1-\mathbf{z})] d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
 \left. + 2\gamma^2 [\kappa] [h(\mathbf{x}) + h(\mathbf{x}-\mathbf{z})] \mathbf{Q}_2(\mathbf{x}, \mathbf{x}-\mathbf{z}) \right\}
 \end{aligned}$$

$$\begin{aligned}
& -\gamma^3[\kappa] \left[ \int h(\mathbf{x}-\mathbf{y}) \{ \mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z}) + \mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x}) \} [R(\mathbf{y}) + R(\mathbf{y}-\mathbf{z})] d^3\mathbf{y} \right. \\
& \quad + 2[h(\mathbf{x}) + h(\mathbf{x}-\mathbf{z})] \int [\mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z})R(\mathbf{y}) + \mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x})R(\mathbf{y}-\mathbf{z})] d^3\mathbf{y} \\
& \quad \left. + 2\mathbf{Q}_2(\mathbf{x}, \mathbf{x}-\mathbf{z}) \int h(\mathbf{x}-\mathbf{y}) [R(\mathbf{y}) + R(\mathbf{y}-\mathbf{z})] d^3\mathbf{y} \right] \\
(6.4) \quad & + 2\gamma^4[\kappa] \left[ \int h(\mathbf{x}-\mathbf{y}_1) [R(\mathbf{y}_1) + R(\mathbf{y}_1-\mathbf{z})] d^3\mathbf{y}_1 \right. \\
& \quad \cdot \int [\mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{z})R(\mathbf{y}_2) + \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x})R(\mathbf{y}_2-\mathbf{z})] d^3\mathbf{y}_2 \\
& \quad + [h(\mathbf{x}) + h(\mathbf{x}-\mathbf{z})] \iint \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2)R(\mathbf{y}_1)R(\mathbf{y}_2-\mathbf{z}) d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
& \quad - \iint h(\mathbf{x}-\mathbf{y}_1) [\mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{z}) + \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x})] R(\mathbf{y}_1)R(\mathbf{y}_2)R(\mathbf{y}_1-\mathbf{y}_2) d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
& \quad + \iint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2) [R(\mathbf{y}_1)R(\mathbf{y}_2-\mathbf{z}) + R(\mathbf{y}_2)R(\mathbf{y}_1-\mathbf{z})] d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
& \quad + \iint h(\mathbf{x}-\mathbf{y}_1) R(\mathbf{y}_1-\mathbf{y}_2) \{ [R(\mathbf{y}_1) + R(\mathbf{y}_2)] \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{z}) \\
& \quad \quad \quad + [R(\mathbf{y}_1-\mathbf{z}) + R(\mathbf{y}_2-\mathbf{z})] \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}) \} d^3\mathbf{y}_1 d^3\mathbf{y}_2 \left. \right] \\
& - 2\gamma^5[\kappa] \iiint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_2(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3) \\
& \quad \cdot [R(\mathbf{y}_1)R(\mathbf{y}_2)R(\mathbf{y}_3-\mathbf{z}) \\
& \quad \quad + R(\mathbf{y}_1)R(\mathbf{y}_1-\mathbf{y}_2)R(\mathbf{y}_3-\mathbf{z}) + R(\mathbf{y}_1-\mathbf{y}_2)R(\mathbf{y}_2)R(\mathbf{y}_3-\mathbf{z}) \\
& \quad \quad + 2R(\mathbf{y}_1-\mathbf{z})R(\mathbf{y}_2)R(\mathbf{y}_1-\mathbf{y}_3) + R(\mathbf{y}_2-\mathbf{y}_1)R(\mathbf{y}_2-\mathbf{z})R(\mathbf{y}_3) \\
& \quad \quad - R(\mathbf{y}_2-\mathbf{z})R(\mathbf{y}_1-\mathbf{y}_3)R(\mathbf{y}_1)R(\mathbf{y}_3) \\
& \quad \quad \quad + R(\mathbf{y}_2)R(\mathbf{y}_1-\mathbf{y}_3)R(\mathbf{y}_3-\mathbf{z})R(\mathbf{y}_1-\mathbf{z})] d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3 \\
& + 6[\kappa]\gamma^3 \int h(\mathbf{x}-\mathbf{y}) \mathbf{Q}_3(\mathbf{x}-\mathbf{y}, \mathbf{x}-\mathbf{z}, \mathbf{x}) d^3\mathbf{y} \\
& + 6[\kappa]\gamma^4 \iint h(\mathbf{x}-\mathbf{y}_1) [\mathbf{Q}_3(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{z})R(\mathbf{y}_2) + \mathbf{Q}_3(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2, \mathbf{x})R(\mathbf{y}_2-\mathbf{z}) \\
& \quad \quad \quad + \mathbf{Q}_3(\mathbf{x}-\mathbf{y}_2, \mathbf{x}, \mathbf{x}-\mathbf{z})R(\mathbf{y}_1-\mathbf{y}_2)] d^3\mathbf{y}_1 d^3\mathbf{y}_2 \\
& + 6[\kappa]\gamma^5 \iiint h(\mathbf{x}-\mathbf{y}_1) [\mathbf{Q}_3(\mathbf{x}-\mathbf{y}_1, \mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3)R(\mathbf{y}_2)R(\mathbf{y}_3-\mathbf{z}) \\
& \quad \quad \quad + \mathbf{Q}_3(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3, \mathbf{x}-\mathbf{z})R(\mathbf{y}_3)R(\mathbf{y}_1-\mathbf{y}_2) \\
& \quad \quad \quad + \mathbf{Q}_3(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3, \mathbf{x})R(\mathbf{y}_3-\mathbf{z})R(\mathbf{y}_1-\mathbf{y}_2)] d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3 \\
& - 6[\kappa]\gamma^6 \iiint h(\mathbf{x}-\mathbf{y}_1) \mathbf{Q}_3(\mathbf{x}-\mathbf{y}_2, \mathbf{x}-\mathbf{y}_3, \mathbf{x}-\mathbf{y}_4)R(\mathbf{y}_2) \\
& \quad \quad \quad \cdot R(\mathbf{y}_3-\mathbf{z})R(\mathbf{y}_1-\mathbf{y}_4) d^3\mathbf{y}_1 d^3\mathbf{y}_2 d^3\mathbf{y}_3 d^3\mathbf{y}_4.
\end{aligned}$$

Proceeding similarly—multiplying (6.1) by the functions  $\tilde{g}_n$ , cf. (6.2), and averaging the result—one will obtain an infinite hierarchy of conjugated equations for the kernels  $Q_i$  in the PDS-Wiener expansion (5.5), each of which will contain three consecutive kernels  $Q_{n-1}, Q_n, Q_{n+1}, n = 1, 2, \dots$ . The first two equations of this hierarchy are displayed above in detail, cf. (6.3), (6.4).

**7. Uncoupling of the hierarchy of equations for kernels.** Guided by the virial property of the PDS-Wiener expansion, cf. § 4, we try to split the hierarchy of conjugated equations (6.3), (6.4), etc. by expanding the unknown kernels  $T_n$  and  $Q_n$  into power series with respect to  $\gamma$  (the mean number of spheres per unit volume)

$$(7.1) \quad \begin{aligned} T_n(\mathbf{x}) &= \sum_{k=0}^{\infty} \gamma^k T_{nk}(\mathbf{x}), & Q_n(\mathbf{x}) &= \sum_{k=0}^{\infty} \gamma^k Q_{nk}(\mathbf{x}); \\ Q_{nk}(\mathbf{x}) &= \nabla T_{nk}(\mathbf{x}), & n, k &= 0, 1, \dots \end{aligned}$$

(The dimensional character of  $\gamma$  is automatically taken care of in what follows—everywhere it will appear multiplied by  $V_a$ , so that in the final results powers of the volume inclusion fraction  $c = \gamma V_a$  will only enter the respective relations.)

It is clearly seen from the equations (6.3) and (6.4) that the kernels of higher order ( $Q_2$  in (6.3),  $Q_3$  in (6.4) and so on) multiply higher degrees of  $\gamma$ . That is why, having substituted (7.1) into the hierarchy of equations (6.3), (6.4), etc., and having equated the coefficients of the degrees of  $\gamma$  to zero, we obtain consecutive equations which specify  $Q_{nk}$  one after another. Thus no closure procedure is to be devised once the hierarchy (6.3), (6.4), etc., is coupled with asymptotic expansions (7.1). In the result, we are able to determine the kernels within an order of accuracy  $\gamma^n$  for each  $n$  prescribed. This will be illustrated when considering in more detail the first and second order approximations in §§ 8 and 10 respectively.

**8. First-order approximation or theory of noninteracting inclusions.** It should be noted first of all that, due to (5.6),

$$(8.1) \quad Q_{00} = G, \quad Q_{0k} = 0, \quad k = 1, 2, \dots$$

In the linear with respect to  $\gamma$  approximation we have

$$(8.2) \quad Q_1 = Q_{10}$$

since, e.g., in the formula (5.11) for the overall conductivity  $Q_1$  is scaled by  $\gamma$ . Thus, there is only one unknown function,  $Q_{10}$ , in the first order approximation under consideration.

To get an equation for  $Q_{10}(\mathbf{x})$  we insert (8.2) into (6.3) and neglect all terms of order  $o(\gamma)$

$$(8.3) \quad \nabla \cdot \{ \kappa_m \nabla T_{10}(\mathbf{x}) + [\kappa] h(\mathbf{x}) [G + \nabla T_{10}(\mathbf{x})] \} = 0, \quad Q_{10} = \nabla T_{10}.$$

This is nothing else but the equation for the disturbance to the temperature field in a homogeneous unbounded medium introduced by a single spherical inclusion, when the temperature gradient has a constant prescribed value  $G$  at infinity. The solution to (8.3) is well known:

$$(8.4) \quad T_{10}(\mathbf{x}) = \begin{cases} -\beta G \cdot \mathbf{x} & \text{for } |\mathbf{x}| \leq a, \\ -\beta \frac{a^3}{|\mathbf{x}|^3} G \cdot \mathbf{x} & \text{for } |\mathbf{x}| > a, \end{cases} \quad \beta = \frac{[\kappa]}{\kappa_f + 2\kappa_m}.$$

This solution is obtained under the assumption that  $T_{10}(\mathbf{x})$  decays to zero at infinity and that the normal component of the heat flux along with the temperature itself are continuous functions at the inclusion surface  $|\mathbf{x}| = a$ . (The condition on the heat flux is in fact a corollary of (8.3) due to the discontinuous coefficient of the latter.) The gradient of (8.4) is then constant within the sphere  $V_a$

$$(8.5) \quad \mathbf{Q}_{10}(\mathbf{x}) = -\beta \mathbf{G}, \quad |\mathbf{x}| \leq a,$$

which, together with the linear version of (5.11), yields

$$(8.6) \quad \kappa^* = \kappa_m(1 + 3\beta c) + o(c).$$

The obtained formula (8.6) coincides with the linear with respect to the inclusion fraction  $c$  version of the Maxwell relation [16] for the overall conductivity of two-phase materials. In other words, the Maxwell relation reflects to order  $c$ , i.e. for dilute concentration, the overall properties for a perfectly disordered suspension of equi-sized spheres. (It is interesting to note here that that conclusion can be reached merely on the basis of the Poisson-Wiener expansion, since the function  $R(\mathbf{y}_i - \mathbf{y}_j)$  is not present anywhere in the above analysis.) Thus, the linear part of the Maxwell relation (8.6) is generated by the contribution of all the inclusions when they are supposed not to interact among themselves; the lack of interaction is seen here in the fact that the kernel  $T_{10}(\mathbf{x})$  is given by the solution (8.4) for a single inclusion embedded in the matrix material.

**9. Further development of concept of noninteracting inclusions: a heuristic approach.** Assume that the temperature field in the medium under consideration is given by the superposition of the equi-shaped effects of the inclusions

$$(9.1) \quad T(\mathbf{x}) = \mathbf{G} \cdot \mathbf{x} + \int T_1(\mathbf{x} - \mathbf{y}) C_\psi^{(1)}(\mathbf{y}) d^3\mathbf{y},$$

i.e. the higher order terms are a priori neglected in the PDS-Wiener expansion (5.4). As shown in the previous section, this assumption is consistent only if a linearization in  $\gamma$  is simultaneously performed in the equation (6.3) for the kernels  $T_1(\mathbf{x})$ ,  $T_2(\mathbf{x})$ . Such a linearization excludes  $T_2(\mathbf{x})$  and decisively simplifies (6.3), cf. (8.3). The approach we propose here is to neglect  $T_2(\mathbf{x})$  from the very beginning, while retaining terms of order  $o(\gamma)$  in (6.3), generated by  $T_1(\mathbf{x})$ . The obvious inconsistency of such an approach makes it a heuristic one. However, it has a spectacular consequence: it leads, within the frame of the so-called singular approximation [22], to the Maxwell formula for the overall conductivity of a two-phase material.

The equation (6.3) with the kernel  $T_2(\mathbf{x})$  neglected can be simplified, if the function  $S(\mathbf{x})$  introduced in (5.10) is summoned

$$(9.2) \quad \begin{aligned} & \kappa_m \Delta S + [\kappa] \nabla \cdot \left\{ h(\mathbf{x})[\mathbf{G} + \nabla S] - \gamma F(\mathbf{x})\mathbf{G} + \gamma[V_a - F(\mathbf{x})]\nabla S \right. \\ & \left. - \gamma \int \nabla S(\mathbf{x} - \mathbf{y}) h(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \right. \\ & \left. + \gamma^2 \iint h(\mathbf{x} - \mathbf{y}_1) \nabla T_1(\mathbf{x} - \mathbf{y}_2) R(\mathbf{y}_1 - \mathbf{y}_2) R(\mathbf{y}_2) [1 - R(\mathbf{y}_1)] d^3\mathbf{y}_1 d^3\mathbf{y}_2 \right\} = 0, \end{aligned}$$

where

$$(9.3) \quad F(\mathbf{x}) = \int h(\mathbf{x} - \mathbf{y}) R(\mathbf{y}) d^3\mathbf{y}.$$

The function  $F(\mathbf{x})$  possesses the following important properties

$$(9.4) \quad F(\mathbf{x}) = F(|\mathbf{x}|) = V_a \quad |\mathbf{x}| \leq a, \quad F'(|\mathbf{x}|) = 0, \quad |\mathbf{x}| = a.$$

The integro-differential equation (9.2) offers little hope for an exact solution to be readily obtained. If the evaluation of the overall conductivity of the medium is, however, the only goal instead of the full solution of (9.2), the values of  $\nabla S$  within the sphere  $|\mathbf{x}| \leq a$  are solely needed, cf. (5.11). The latter, at least approximately, can be found much easier.

Let us invoke the virial expansions of  $T_1(\mathbf{x})$  and  $S(\mathbf{x})$

$$(9.5) \quad T_1(\mathbf{x}) = \sum_{k=0}^{\infty} \gamma^k T_{1k}(\mathbf{x}), \quad S(\mathbf{x}) = \sum_{k=0}^{\infty} \gamma^k S_k(\mathbf{x}),$$

where, according to (5.10),

$$(9.6) \quad \begin{aligned} S_0(\mathbf{x}) &= T_{10}(\mathbf{x}), \\ S_n(\mathbf{x}) &= T_{1n}(\mathbf{x}) - \int T_{1n-1}(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y}, \quad n = 1, 2, \dots \end{aligned}$$

Inserting (9.5) into (9.2) and arguing as usual, we get

$$(9.7) \quad \kappa_m \Delta S_0 + [\kappa] \nabla \cdot \{h(\mathbf{x})[\mathbf{G} + \nabla S_0]\} = 0,$$

$$(9.8) \quad \begin{aligned} \kappa_m \Delta S_1 + [\kappa] \nabla \cdot \left\{ h(\mathbf{x}) \nabla S_1 - F(\mathbf{x})[\mathbf{G} + \nabla S_0] \right. \\ \left. + V_a \nabla S_0 - \int \nabla S_0(\mathbf{x}-\mathbf{y}) h(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} \right\} = 0, \end{aligned}$$

$$(9.9) \quad \begin{aligned} \kappa_m \Delta S_n + [\kappa] \nabla \cdot \left\{ h(\mathbf{x}) \nabla S_n + [V_a - F(\mathbf{x})] \nabla S_{n-1} \right. \\ \left. - \int \nabla S_{n-1}(\mathbf{x}-\mathbf{y}) h(\mathbf{x}-\mathbf{y}) R(\mathbf{y}) d^3\mathbf{y} + \mathbf{J}_{n-2}(\mathbf{x}) \right\} = 0, \end{aligned}$$

$n = 2, 3, \dots,$

where

$$(9.10) \quad \mathbf{J}_{n-2}(\mathbf{x}) = \iint h(\mathbf{x}-\mathbf{y}_1) \nabla T_{1n-2}(\mathbf{x}-\mathbf{y}_2) R(\mathbf{y}_1-\mathbf{y}_2) R(\mathbf{y}_2) [1 - R(\mathbf{y}_1)] d^3\mathbf{y}_1 d\mathbf{y}_2.$$

The zeroth order equation (9.7) coincides with the equation (8.3) for the function  $T_{10}(\mathbf{x})$ , which is no wonder due to (9.6). Thus  $S_0(\mathbf{x})$  is given by (8.4) and, in particular, its gradient is constant within the sphere  $V_a$ , cf. (8.5).

Using the so obtained solution for  $S_0(\mathbf{x})$ , we rewrite (9.8)

$$(9.11) \quad \kappa_m \Delta S_1 + [\kappa] \nabla \cdot \{h(\mathbf{x}) \nabla S_1 - (1 - \beta) F(\mathbf{x}) \mathbf{G} + [V_a - F(\mathbf{x})] \nabla S_0\} = 0$$

which implies, in virtue of (9.4), that

$$(9.12) \quad \begin{aligned} \Delta S_1 &= 0 \quad \text{for } |\mathbf{x}| < a, \\ \kappa_m \Delta S_1 + [\kappa] \nabla \cdot \{(\beta - 1) F(\mathbf{x}) \mathbf{G} + [V_a - F(\mathbf{x})] \nabla S_0\} &= 0 \quad \text{for } |\mathbf{x}| > a. \end{aligned}$$

The solution to (9.12) has the form

$$(9.13) \quad S_1(\mathbf{x}) = S_{10}(\mathbf{x}) + \chi_1(\mathbf{x}) + \chi_2(\mathbf{x}).$$

Here  $S_{10}(\mathbf{x})$  solves the homogeneous equation (9.12); guided by (8.4), we choose it as

$$(9.14) \quad S_{10}(\mathbf{x}) = \begin{cases} p_{10} \mathbf{G} \cdot \mathbf{x} & \text{for } |\mathbf{x}| < a, \\ q \frac{a^3}{|\mathbf{x}|^3} \mathbf{G} \cdot \mathbf{x} & \text{for } |\mathbf{x}| > a. \end{cases}$$

The functions  $\chi_1$  and  $\chi_2$ , in turn, are solutions to the equations

$$(9.15) \quad \Delta \chi_i = 0 \quad \text{for } |\mathbf{x}| < a, \quad i = 1, 2,$$

$$(9.16) \quad \kappa_m \Delta \chi_1 + [\kappa](\beta - 1) \nabla \cdot \{F(\mathbf{x}) \mathbf{G}\} = 0, \quad \text{for } |\mathbf{x}| > a,$$

$$(9.17) \quad \kappa_m \Delta \chi_2 + [\kappa] \nabla \cdot \{(V_a - F(\mathbf{x})) \nabla S_0\} = 0,$$

Because of (9.4), the equations (9.16), (9.17), when considered at  $|\mathbf{x}| < a$ , coincide with (9.15). That is why we can consider them over the whole  $\mathbb{R}^3$ . A bounded and continuously differentiable solution to (9.15), (9.16) is given then by the integral

$$(9.18) \quad \chi_1(\mathbf{x}) = \frac{(\beta - 1)[\kappa]}{4\pi\kappa_m} \int \frac{\nabla \cdot \{F(\mathbf{y}) \mathbf{G}\}}{|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y}.$$

This solution can obviously be represented as  $\mathbf{G} \cdot \nabla \varphi_1(|\mathbf{x}|)$ . Due to (9.15) we have for  $\varphi_1$  then equation  $\Delta \varphi_1 = p_{11}$  at  $|\mathbf{x}| < a$ , with a certain constant  $p_{11}$ , whose only bounded solution is  $\varphi_1 = \frac{1}{2} p_{11} |\mathbf{x}|^2$ ; thus

$$(9.19) \quad \chi_1(\mathbf{x}) = p_{11} \mathbf{G} \cdot \mathbf{x} \quad \text{for } |\mathbf{x}| < a.$$

For the constant  $p_{11}$  we have from (9.18)

$$(9.20) \quad p_{11} \mathbf{G} = \nabla \chi_1(0) = \frac{(\beta - 1)[\kappa]}{\kappa_m} \mathbf{G} \cdot \int F(\mathbf{y}) \nabla \nabla \frac{1}{4\pi|\mathbf{y}|} d^3\mathbf{y}.$$

As known, the second gradient of the Green function  $1/4\pi|\mathbf{y}|$  for the Laplace equation is a sum of a singular part, proportional to the Dirac delta function, and a regular part, which is an ordinary function of the spatial coordinates

$$(9.21) \quad \nabla \nabla \frac{1}{4\pi|\mathbf{y}|} = -\frac{1}{3} \delta(\mathbf{y}) \mathbf{I} + \frac{1}{|\mathbf{y}|^3} (3\mathbf{e}_y \mathbf{e}_y - \mathbf{I}), \quad \mathbf{e}_y = \frac{\mathbf{y}}{|\mathbf{y}|},$$

cf., [22, p. 100]. As  $F(\mathbf{y}) = F(|\mathbf{y}|)$ , the contribution to the right-hand side of (9.20) of the regular part vanishes, so that

$$(9.22) \quad p_{11} = \frac{(1 - \beta)[\kappa]}{3\kappa_m} V_a.$$

Repeating the same arguments for the equation (9.15), (9.17), we find consecutively

$$(9.23) \quad \chi_2(\mathbf{x}) = \frac{[\kappa]}{4\pi\kappa_m} \int \frac{\nabla \cdot \{(V_a - F(\mathbf{y})) \nabla S_0(\mathbf{y})\}}{|\mathbf{x} - \mathbf{y}|} d^3\mathbf{y},$$

$$(9.24) \quad \chi_2(\mathbf{x}) = p_{12} \mathbf{G} \cdot \mathbf{x} \quad \text{for } |\mathbf{x}| < a,$$

$$(9.25) \quad p_{12} \mathbf{G} = \nabla \chi_2(0) = \frac{[\kappa]}{\kappa_m} \int (V_a - F(\mathbf{y})) \nabla S_0(\mathbf{y}) \nabla \nabla \frac{1}{4\pi|\mathbf{y}|} d^3\mathbf{y}.$$

To calculate the integral (9.25) approximately, we shall recall the so-called singular approximation, cf. [22, Chap. V.8]. It consists in retaining in (9.21) only the singular part, while neglecting the regular part. Let us point out that the singular approximation

has been successfully used in theory of heterogeneous materials; it has allowed, e.g., the reproduction of the well-known Hashin-Shtrikman bounds for two-phase materials and for polycrystals, cf. [22, Chap. V.9].

As it follows from (9.4), within the frame of the singular approximation, we have  $p_{12} = 0$ . Making use of (9.13), (9.14), (9.20) and (9.24), we find then the solution to (9.12) to be

$$(9.26) \quad S_1(\mathbf{x}) = \begin{cases} p_1 \mathbf{G} \cdot \mathbf{x} & \text{for } |\mathbf{x}| < a, \\ q_1 \frac{a^3}{|\mathbf{x}|^3} \mathbf{G} \cdot \mathbf{x} + \chi_1(\mathbf{x}) + \chi_2(\mathbf{x}) & \text{for } |\mathbf{x}| > a, \end{cases}$$

where  $p_1 = p_{10} + p_{11}$ . To specify the unknown constants  $p_1, q_1$  in this solution, we employ the above-mentioned conditions of continuity for the functions  $S_1(\mathbf{x})$  and for its normal heat flux at  $|\mathbf{x}| = a$ , which follow from the very equation (9.11). These conditions yield

$$(9.27) \quad p_1 = q_1 + p_{11}, \quad \kappa_f p_1 = \kappa_m (-2q_1 + p_{11}),$$

where it is acknowledged that both functions  $\chi_1, \chi_2$  are continuously differentiable over the whole  $\mathbb{R}^3$  and  $p_{12} = 0$ . From (9.27) and (9.22) we find eventually

$$(9.28) \quad p_1 = \beta(1 - \beta)V_a.$$

A fully similar analysis of the equations (9.9), employing the fact that the integrals  $\mathbf{J}_{n-2}(\mathbf{x})$ , cf. (9.10), vanishes at  $|\mathbf{x}| < a$ , shows that within the frame of the above-mentioned singular approximation

$$(9.29) \quad S_n(\mathbf{x}) = p_n \mathbf{G} \cdot \mathbf{x} \quad \text{for } |\mathbf{x}| < a, \quad p_n = (1 - \beta)\beta^n V_a^n, \quad n = 1, 2, \dots$$

Making use of (5.11), (8.4), (9.26), (9.28) and (9.29), we find the overall conductivity of the medium to be

$$\begin{aligned} \kappa^* \mathbf{G} &= \langle \kappa \rangle \mathbf{G} + \gamma[\kappa] \sum_{n=0}^{\infty} \gamma^n \int_{V_a} \nabla S_n(\mathbf{x}) d^3 \mathbf{x} \\ &= \left\{ \kappa_m + c[\kappa](1 - \beta) \sum_{n=0}^{\infty} \beta^n c^n \right\} \mathbf{G}. \end{aligned}$$

As  $|\beta| \leq 1$ , cf. (8.4), the series is absolutely convergent at  $0 \leq c < 1$  which always holds since the concentration  $c$  cannot exceed 0.74—the value corresponding to the maximum packing of spheres. The summation of the series is obvious—

$$(9.30) \quad \frac{\kappa^*}{\kappa_m} = \frac{1 + 2\beta c}{1 - \beta c}, \quad \left( \beta = \frac{\kappa_f - \kappa_m}{\kappa_f + 2\kappa_m} \right)$$

which is the full-scale Maxwell relation for the overall conductivity for a two-phase material.

It is to be noted that neglecting in the expansion (9.4) the  $n$ -tuple terms,  $n \geq 2$ , cf. (9.1), we neglect in a sense binary, triple, etc., interactions among inclusions. However, retaining the higher order degrees in  $\gamma$  generated by  $T_1$  means that to a certain extent the inclusions' interaction is accounted for in the proposed heuristic approach. (This is also seen in the fact that the kernel for the case of noninteracting inclusions, cf. (8.4), differs from the kernel  $T_1$  obtained here, the latter being influenced by the presence of all the inclusions.) As far as the overall conductivity is only concerned, the heuristic approach in the singular approximation is equivalent to the so-called effective field method for which the interaction between particles in a composite material makes appearance through influencing the average temperature gradient

around each particle due to the presence of the rest of them (cf. [18], and, for a more general situation, [14]). The reason is that the effective field method for a suspension of spheres, as shown in [18], also leads to the Maxwell formula (9.30).

**10. Second-order approximation.** If one is bound to find a consistent second-order approximation to the solution of the basic system (6.3), (6.4), etc., one is to retain in (6.3) the terms proportional to  $\gamma^2$ , including those with  $\mathbf{Q}_2$ . This automatically brings into view the equation (6.4) in which all terms of order  $\gamma^3$  and  $\gamma^4$  are disregarded. As a result, the complexity of the problem raises essentially. If, however, only the overall conductivity is to be predicted, so that full stochastic information for the temperature and heat flux fields up to the order  $c^2$  is not required, a certain simplification of the problem can be attained—again due to the fact that to specify  $\kappa^*$  only the kernel  $T_1$  is needed, cf. (5.11). Thus, the explicit expression for  $T_2$  is not needed. Moreover, only the values of  $S(\mathbf{x})$  inside the sphere  $|\mathbf{x}| < a$  are required to calculate  $\kappa^*$ . The latter appears of critical importance, because these values can be obtained at least approximately—by means of the singular approximation—without solving the respective full-scale equations for the second approximation. (The same has happened in § 9 when dealing with equation (9.2).)

Let us retain in (6.3) only the terms of order  $\gamma$  and  $\gamma^2$ . The result can be recast to the form

$$(10.1) \quad \kappa_m \Delta S + [\kappa] \nabla \cdot \left\{ h(\mathbf{x})[\mathbf{G} + \nabla S] - \gamma F(\mathbf{x})\mathbf{G} + \gamma[V_a - F(\mathbf{x})]\nabla S - \gamma \int \nabla S(\mathbf{x}-\mathbf{y})R(\mathbf{y})h(\mathbf{x}-\mathbf{y}) d^3\mathbf{y} + 2\gamma I_2(\mathbf{x}) \right\} = 0,$$

where

$$(10.2) \quad I_2(\mathbf{x}) = \int h(\mathbf{x}-\mathbf{y})\mathbf{Q}_2(\mathbf{x}-\mathbf{y}, \mathbf{x}) d^3\mathbf{y}.$$

The equation (10.1) differs, to the order  $\gamma^2$ , from (9.2) only by the term containing  $I_2(\mathbf{x})$ . The function  $I_2(\mathbf{x})$ , as readily seen from its definition (10.2), possesses the properties

$$(10.3) \quad I_2(\mathbf{x}) = 0 \quad \text{at } |\mathbf{x}| < a, \quad \frac{dI_2(\mathbf{x})}{dn} = 0 \quad \text{at } |\mathbf{x}| = a.$$

That is why, within the frame of the singular approximation, the term with  $I_2(\mathbf{x})$  in (10.1) will not contribute anything to the solution within the sphere  $|\mathbf{x}| < a$ . It means that to the order  $c^2$  the Maxwell formula (9.30) gives the overall conductivity for a perfectly disordered suspension of spheres in the singular approximation.

In order to have full stochastic description of the temperature field to the order  $c^2$ , let us consider finally the needed terms in the virial expansion (7.1) for the second-order approximation

$$(10.4) \quad T_1(\mathbf{x}) = T_{10}(\mathbf{x}) + \gamma T_{11}(\mathbf{x}), \quad T_2(\mathbf{x}) = T_{20}(\mathbf{x}),$$

since everywhere  $T_1$  and  $T_2$  appear multiplied at least by  $\gamma$  and  $\gamma^2$  respectively. The equation (6.4), when truncated to the order  $\gamma^2$ , then reads

$$(10.5) \quad \nabla \cdot \{ 2(\kappa_m + [\kappa])[h(\mathbf{x}) + h(\mathbf{x}-\mathbf{z})]\nabla T_{20}(\mathbf{x}, \mathbf{x}-\mathbf{z}) + [\kappa][h(\mathbf{x})\mathbf{Q}_{10}(\mathbf{x}-\mathbf{z}) + h(\mathbf{x}-\mathbf{z})\mathbf{Q}_{10}(\mathbf{x})] \} = 0.$$

By means of (8.3) this equation can be rewritten as

$$(10.6) \quad \nabla \cdot \{(\kappa_m + [\kappa][h(\mathbf{x}) + h(\mathbf{x} - \mathbf{z})])[\mathbf{G} + \nabla T^{(2)}(\mathbf{x}; \mathbf{z})]\} = 0,$$

where

$$T^{(2)}(\mathbf{x}; \mathbf{z}) = 2T_{20}(\mathbf{x}, \mathbf{x} - \mathbf{z}) + T_{10}(\mathbf{x}) + T_{10}(\mathbf{x} - \mathbf{z}).$$

This is nothing but the equation for the disturbance  $T^{(2)}(\mathbf{x}; \mathbf{z})$  to the temperature field in a homogeneous unbounded matrix, introduced by a pair of identical spherical inclusions whose centers are at the origin and at the point  $\mathbf{z}$ , when the temperature gradient at infinity is constant. Each one of these inclusions, if it were alone, would disturb the temperature field in the homogeneous matrix by  $T_{10}(\mathbf{x})$  and  $T_{10}(\mathbf{x} - \mathbf{z})$ , respectively. Thus the kernel  $T_{20}(\mathbf{x}, \mathbf{x} - \mathbf{z})$  is the field which should be added to the single-inclusion disturbances  $T_{10}(\mathbf{x})$ ,  $T_{10}(\mathbf{x} - \mathbf{z})$  in order to obtain the double-inclusion disturbance  $T^{(2)}(\mathbf{x}; \mathbf{z})$ . Note that the kernels  $T_{n0}$  will be similarly connected with the disturbance to the temperature field in a homogeneous matrix, introduced by  $n$  spherical inclusions.

To get the equation for  $T_{11}(\mathbf{x})$ , (10.4) is to be inserted into (6.3) and the coefficient of  $\gamma^2$  is to be taken zero

$$(10.7) \quad \kappa_m \Delta S_1 + [\kappa] \nabla \cdot \{h(\mathbf{x}) \nabla S_1(\mathbf{x}) + \mathbf{Q}_{10}(\mathbf{x})[V_a - F(\mathbf{x})] + (\beta - 1)F(\mathbf{x})\mathbf{G} + 2\mathbf{I}_2(\mathbf{x})\} = 0,$$

where

$$S_1(\mathbf{x}) = T_{11}(\mathbf{x}) - \int T_{10}(\mathbf{x} - \mathbf{y})R(\mathbf{y})d^3\mathbf{y}$$

is the first-order term in the virial expansion (9.5) for the function  $S(\mathbf{x})$ , and  $\mathbf{I}_2(\mathbf{x})$  is given in (10.2). Thus, having obtained the solution for the two-inclusion problem,  $T^{(2)}(\mathbf{x}; \mathbf{z})$ , one encounters a problem for an infinite matrix with a single spherical inclusion undergoing a known field of body forces.

**Conclusions.** In the present paper an application to random heterogeneous media of the Volterra–Wiener functional expansions is treated. It is argued first of all that the basis function of such expansions has to be most suited to the physical problem under consideration and may well differ from the Gaussian white noise employed in the original Wiener works. As far as random suspensions of spheres are concerned, it is shown that the most suited basis function is the introduced perfect disorder of spheres—a system of random points forbidden to appear closer than the diameter  $2a$  of a single sphere and statistically independent when the distance between them exceeds  $2a$ . The orthogonal Wiener functionals for the PDS-field are explicitly constructed by means of the multivariate Charlier polynomials. To display the performance of the Wiener expansion with respect to the PDS-field the problem of determining the temperature field in a random medium containing a perfectly disordered system of spherical inclusions is considered, when the averaged temperature gradient is constant. The method leads to an infinite hierarchy of equations for the unknown kernels of the expansion, for which no closure procedure is to be devised due to the proven virial property of the said expansion. As a result, the hierarchy can be solved to any prescribed order of accuracy  $c^n$ , where  $c$  is the volume fraction of the inclusions. Once solved, the respective equations give full statistical information up to the order  $c^n$  for the random fields under study, and not only the overall conductivity. Moreover, unlike a number of known approaches, cf. [17] for details, no conditionally convergent integrals appear when calculating the overall properties by means of the Volterra–Wiener series.

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