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A FURTHER DEVELOPMENT OF THE CONCEPT OF RANDOM DENSITY FUNCTION WITH APPLICATION TO VOLTERRA-WIENER EXPANSIONS

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For treating the stochastic functions Wiener [1] proposed a method similar to that of Volterra functional expansions. He employed the Gaussian white noise as a basis function. The method was subsequently named 'Wiener-Hermite expansion' and proved to be a powerful tool for handling a number of nonlinear stochastic problems [2]. The very essense of this method — employment of Gaussian white noise — makes it unsuitable for a variety of physical problems where the fluctuations are not so frequent and each of them possesses a significant impact on the system investigated. Recently, the method of Wiener has been forwarded into a direction of more adequate choice of basis function considering the point random functions (for definition see [3,4]) and specifying the correlation properties of the generating set of random points [5-11].

The present paper is the next step in this line of thinking and deals with the problem of incorporation of still more information into the basis function. For this purpose the marked random point functions are considered for which a random variable called mark is associated with each random point of generating set [4]. Multipoint correlation functions of the random set [3] are employed, and statistics for a marked random point function are developed. Afterwards the way of constructiug the Volterra-Wiener series with such basis functions is outlined.

Consider the generalized random density function

$$\omega(\mathbf{x};\ \mathbf{u}) \equiv \omega(\mathbf{z}) = \sum_j{}' \delta(\mathbf{z}-\mathbf{z}_j) \tag{1}$$

where $\mathbf{z}=(x_1, x_2, x_3;\ u_1, \dots, u_m)$ is the outer product of the geometrical space $\mathbf{x}$ and m-dimensional mark vector space $\mathbf{u}$. There are no obstacles to include the time and it is not done only for the sake of brevity of notations. Respectively, $\delta(\cdot)$ are Dirac deltas of $(m+3)$-th order. It is well known [4] that every filtered random point function can be represented as a linear transformation of ω, namely

$$y(\mathbf{x};\ \mathbf{u}) = \int_{\mathbf{R}^3}\int_U K(\mathbf{x}-\boldsymbol{\xi};\ \mathbf{u})\,\omega(\boldsymbol{\xi};\ \mathbf{u})\,d^3\boldsymbol{\xi}\, d^m\mathbf{u} = \sum_j{}' K(\mathbf{x}-\mathbf{x}_j;\ \mathbf{u}), \tag{2}$$

where U is the mark space.

Let us now following [3] introduce the probability density functions $f_n(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{u}_1, \ldots, \mathbf{u}_n)$ of the system of random points $\{\mathbf{z}_j\}=\{\mathbf{x}_j; \mathbf{u}_j\}$ in the $(m+3)$-dimensional space, i. e.

$$dP=f_n(\mathbf{x}_1, \ldots, \mathbf{x}_n; \mathbf{u}_1, \ldots, \mathbf{u}_n)\, d^3\mathbf{x}_1, \ldots, d^3\mathbf{x}_n d^m\mathbf{u}_1 \ldots d^m\mathbf{u}_n \tag{3}$$

is the probability of simultaneous occurrence of n points $\mathbf{x}$, each of them with its mark $\mathbf{u}$ in the intervals

$$\mathbf{x}_i \leqq \mathbf{x} \leqq \mathbf{x}_i + d\,\mathbf{x}_i, \quad \mathbf{u}_i \leqq \mathbf{u} \leqq \mathbf{u}_i + d\,\mathbf{u}_i, \quad i=1, 2, \ldots, n.$$

In the case when the points are mutually independent in stochastic sense we arrive at the so-called compound Poisson random function [4] for which functions f_n can be decomposed into the following way

$$f_n=f_1(\mathbf{x}_1; \mathbf{u}_1) \ldots f_1(\mathbf{x}_n; \mathbf{u}_n)=\prod_i \lambda_i P(\mathbf{u}_i) \tag{4}$$

where λ_i is the intensity of the associated counting function and $P(\mathbf{u}_i)$ is the probability density of a mark.

If there exists a correlation between the points $\mathbf{x}$ and marks $\mathbf{u}$ it can be acknowledged by means of functions f_n. In this instance the following example is very instructive. Let $\mathbf{x}$ be the radius-vector of a point of the three-dimensional Eucledian space and the mark be simply a real positive number a. If this number is the radius of a rigid sphere situated at the respective random point and if spheres are not to intersect each other then for f_2 we have

$$f_2(\mathbf{x}_1, \mathbf{x}_2; a_1, a_2)=Q_{12} f_1(\mathbf{x}_1; a_1) f_1(\mathbf{x}_2; a_2) \tag{5}$$

where

$$Q_{12}=\begin{cases} 0 & \text{for } |\mathbf{x}_1-\mathbf{x}_2| \leqq a_1+a_2 \\ 1 & \text{otherwise.} \end{cases}$$

The higher-order probability densities f_n can be specified as

$$f_n(\mathbf{x}_1, \ldots, \mathbf{x}_n; a_1, \ldots, a_n)=f_1(\mathbf{x}_1; a_1) \ldots f_1(\mathbf{x}_n; a_n) Q_{12} \ldots Q_{1n} Q_{23} \ldots Q_{2n} \ldots Q_{n-1n}. \tag{6}$$

The random field created by this system of random points is a straight-forward generalization of the so-called PDS-field (Perfect Disorder of Spheres) introduced recently in [9] for the case of equi-sized non-overlapping spheres randomly dispersed throughout an unbounded three-dimensional space. The random function created by (5), (6) can be named PDSRR-field (Perfect Disorder of Spheres of Random Radii).

A number of other examples can be cited of random fields with various types of interconnection between random points and their marks, but it goes well beyond the frame of this short note and shall be done elsewhere.

Having functions f_n one can obtain the statistics (moments, cumulants, etc.) of the random density function ω. This is possible due to results of Stratonovich [3] which apply to our case

$$\begin{aligned} &\langle \omega(\mathbf{x}; \mathbf{u}) \rangle = f_1(\mathbf{x}; \mathbf{u}) \\ &\langle \omega(\mathbf{x}_1; \mathbf{u}_1)\, \omega(\mathbf{x}_2; \mathbf{u}_2) \rangle = f_1(\mathbf{x}_1; \mathbf{u}_1)\, \delta(1,2) + f_2(\mathbf{x}_1, \mathbf{x}_2; \mathbf{u}_1, \mathbf{u}_2) \\ &\langle \omega(\mathbf{x}_1; \mathbf{u}_1)\, \omega(\mathbf{x}_2; \mathbf{u}_2)\, \omega(\mathbf{x}_3; \mathbf{u}_3) \rangle = f_1(\mathbf{x}_1; \mathbf{u}_1)\, \delta(1,2) \delta(1,3) \\ &\quad + 3\{\delta(1,2) f_2(\mathbf{x}_1, \mathbf{x}_3; \mathbf{u}_1, \mathbf{u}_3)\}_s + f_3(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3; \mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) \end{aligned} \tag{7}$$

where $\delta(i, j)=\delta(\mathbf{z}_i-\mathbf{z}_j)$ and $\{\cdot\}_s$ is the symmetrization operation.

The very outlook of (2) hints how to construct the Volterra functional series with the generalized random density function as a basis function, namely

$$(8) \qquad F(\mathbf{x}) = \sum_{n=0}^{\infty} V_{\omega}^{(n)}[F]$$

where $V_{\omega}^{(n)}$ are the respective Volterra functionals

$$V_{\omega}^{(n)}[F] = \int_{\mathbf{R}^3} \dots \int_{\mathbf{R}^3} \int_{U} \dots \int_{U} K_F^{(n)}(\mathbf{x}-\boldsymbol{\xi}_1, \dots, \mathbf{x}-\boldsymbol{\xi}_n; \mathbf{u}_1, \dots, \mathbf{u}_n)$$

$$\omega(\boldsymbol{\xi}_1; \mathbf{u}_1)\,\omega(\boldsymbol{\xi}_2; \mathbf{u}_2) \dots \omega(\boldsymbol{\xi}_n; \mathbf{u}_n)\, d^3\boldsymbol{\xi}_1 \dots d^3\boldsymbol{\xi}_n d^m \mathbf{u}_1 \dots d^m \mathbf{u}_n.$$

It is clearly seen that the asymmetry between the spatial coordinates and marks from (2) is retained in (8). The latter is crucial for the physical application and appears to be novel.

Functionals (8) have no direct physical meaning, though it is clear that the n-th order functional is related to the n-tuple interaction between the inhomogenities. In addition, these functionals are neither centered stochastic functions nor orthogonal in a stochastic sense. Part of these shortcomings are removed if instead of simple n-th order products of ω the following quantities are employed in functionals:

$$(9) \quad \Delta_{\omega}^{(n)} = \omega(\boldsymbol{\xi}_1, \mathbf{u}_1)\,[\omega(\boldsymbol{\xi}_2, \mathbf{u}_2) - \delta(1, 2)] \dots [\omega(\boldsymbol{\xi}_n, \mathbf{u}_n) - \delta(2, n) - \dots - \delta(n-1, n)].$$

On the basis of (7) after some tedious manipulations it is proved that

$$(10) \qquad \langle \Delta_{\omega}^{(n)} \rangle = f_n(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{n}_1, \dots, \mathbf{u}_n)$$

Now we have at our disposal the centered functionals

$$(11) \qquad G_{\omega}^{(n)}[F] = \int_{\mathbf{R}^3} \dots \int_{\mathbf{R}^3} \int_{U} \dots \int_{K} K_F^{(n)}(\mathbf{x}-\boldsymbol{\xi}_1, \dots, \mathbf{x}-\boldsymbol{\xi}_n, \mathbf{u}_1, \dots, \mathbf{u}_n)$$

$$[\Delta_{\varphi}^{(n)} - f_n(\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_n; \mathbf{u}_1, \dots, \mathbf{u}_n)]\, d^3\boldsymbol{\xi}_1 \dots d^3\boldsymbol{\xi}_n d^m \mathbf{u}_1 \dots d^m \mathbf{u}_n,$$

which are still non-orthogonal in stochastic sense. Orthogonal functionals can be constructed only for the limiting cases of compound Poisson function or functions of the type of perfect disorder through obvious generalization of the techniques of [5,8] and [9], respectively. The physical meaning of the n-th order functional (11) is the quantity which has to be added to the field in order to incorporate the n-tuple interaction if all the lower-order interactions are already acknowledged.

It is interesting to note that under certain not very restrictive conditions on functions f_n, the Volterra-Wiener expansion exhibits virial property. Consider a homogeneous generalized random function with intensity λ, the latter being the number of random points falling per unit volume. In the case of compound Poisson random density function a relation of type (4) holds. In the general case one has:

$$(12) \qquad f_n = \lambda^n P(\mathbf{u}_1)\, P(\mathbf{u}_2) \dots P(\mathbf{u}_n)\, \varphi(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{u}_1, \dots, \mathbf{u}_n)$$

If we now require for function φ to be of order 0(1), then we obtain that the contribution of each functional (11) of n-th order to the average characteristics is of order of λ_n. In other words, φ does not have to include delta functions in order to secure viriality. An example of functions φ, which are delta functions, is given in [10]. The virial property of Volterra-Wiener expansion is of outstanding importance in obtaining asymptotically correct results for various nonlinear stochastic systems.

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