

SECONDARY VISCOUS FLOW BETWEEN TWO ECCENTRIC SPHERES GENERATED BY THE HIGH-FREQUENCY SMALL-AMPLITUDE COAXIAL OSCILLATIONS OF THE INNER SPHERE

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Николина Ковачева, Христо Христов, Запрян Запрянв. Вторичные течения вязкой жидкости между двумя неконцентрическими сферами, порожденные высокочастотными осцилляциями внутренней сферы. Работа посвящена задаче об осесимметричном вязком течении, возникающем в зазоре между двумя неконцентрическими сферами. Внутренняя сфера осциллирует, а внешняя находится в покое. Получено аналитическое решение по методу малого параметра для случая малых амплитуд и высоких частот осцилляций. Структура вторичного стационарного течения показана графично для различных значений эксцентриситета и отношения радиусов сфер.

Nikolina Kovatcheva, Christo Christov, Zapryan Zapryanov. **Secondary viscous flow between two eccentric spheres generated by the high-frequency small-amplitude coaxial oscillations of the inner sphere.** The paper deals with the axisymmetrical problem of the oscillatory viscous incompressible flow in eccentric spherical annuli. The inner sphere performs torsional oscillations and the outer one is held at rest. The problem is studied analytically for the case of high frequency and low amplitude of oscillations by means of the method of matched asymptotic expansions. The structure of secondary flow is shown graphically for a variety of values of eccentricity and ratio of spheres radii.

INTRODUCTION

In recent years the oscillatory viscous flows attracted a considerable attention due to their importance in a number of applications, especially in chemical technology. When an oscillatory viscous flow interacts with a rigid boundary a secondary steady flow is generated and the latter enhances significantly the heat and mass transfer. On the other hand investigating the oscillatory flows offers an indispensable opportunity to reveal certain fundamental features of viscous flows which become conspicuous in transient situations with complicated interplay between inertial and viscous forces. For this reason it was yet Stokes [1] who studied the rotational oscillations of a disc of finite radius along with the translatory oscillations of small particles. Since then a great deal of work has been done both theoretically and experimentally in the field under consideration. The known papers can be divided into two major groups: translatory oscillations and rotational (torsional) oscillations. The present paper is chiefly concerned with the latter.

Helmholtz and Piotrowski [2] studied theoretically and experimentally the rotational oscillations of spheres and cylinders. Buchanan [3] investigated the torsional oscillations of a spheroid and Rosenblat [4] — torsional oscillations of a

plate. In all these works is assumed that the fluid trajectories are circuits whose centers belong to the axis of rotation. Henceforth we shall call this motion — primary flow. If, however, the inertial forces are acknowledged one sees that in the planes which consist of the line of rotation a secondary flow occurs. In case of a sphere the fluid is repulsed away of the sphere in the region of equator and attracted again to the sphere in the vicinity of poles. Carrier and Di Prima [5] and Di Prima and Liron [6] took into account both primary and secondary motions.

The general approach to oscillatory viscous flows with secondary streamings is essentially developed in works of Stuart and Riley (see e. g. [7, 8]) and since then is many times applied to a variety of different problems. The present paper is a descendant from this approach and can be viewed as its application.

The picture is significantly complicated if another rigid boundary is presented. The latter introduces a lot of difficulties on the way of theoretical treatment which are connected with the geometry of the region occupied by the flow. About problems of translatory oscillations of bodies in containers filled with viscous liquid we refer the reader to [9, 10] and literature cited there and focus our attention on the rotational motions of bodies in viscous liquid in presence of other rigid boundary. The most amenable to theoretical treatment appear to be regions with spherical boundaries at the same time when all the principal features of hydrodynamic interaction are not compromised. For this reason the predominant part of known works are aimed at the flow between or around two spheres.

The first to undergo investigation was the flow between two concentric spheres. Initially, the steady rotations of spheres were modelled asymptotically for low Reynolds numbers (cf. [11, 12]), numerically in [13, 14] for moderate to high Reynolds numbers and by asymptotic method of singular expansions [15] for very high Reynolds numbers. Among the experimental works are renowned the papers of Yaworskaya and co-workers (see [16] and literature cited there). Just recently the hydrodynamic interaction between two concentric spheres through the viscous liquid which fills the gap inbetween the spheres has been considered simultaneously in [17, 18] for low and high frequencies respectively. The gap in the theoretical results for this flow has been filled by the numerical solution [19] and now the solution to the problem could be thought of as virtually completed.

In the light of the capabilities of theoretical techniques the most natural development of the investigations on hydrodynamic interactions is the flow in an eccentric spherical annuli. Naturally, the first works on this topic are related to stationary relations. For a thin gap between the spheres and small eccentricity Snopov and Tchaikin [20] employed lubrication approximation. The general approach to the problem based on the use of bi-spherical coordinates was outlined much earlier in the works of Jeffery and co-workers [21, 22] and extended then to the case of a sphere rotating near a plane [23]. The last solution was still generalized by Majumdar [24] who solved the problem of slow motion in eccentric annuli when the inner sphere rotates with uniform angular velocity about an axis perpendicular to the line of centers and the outer sphere is fixed. In all these works [21—24] only the primary flow is considered. The secondary flow in steady rotating eccentric spherical annuli is modelled theoretically for small magnitudes of Reynolds number and eccentricity by Munson [25] and measured by Menguturk and Munson [26] for low Reynolds numbers and moderate magnitudes of eccentricity.

Following the line of thinking argued above we arrive to the conclusion that the next step is to consider the oscillatory flow in eccentric spherical annuli. In the present paper we focus our attention to the case of high frequency of oscillations which is the more important one to begin with, as in the case of low frequencies

one can get some qualitative feeling about patterns of secondary flow even on the base of results for stationary rotations. In order not to complicate the calculations we constrict ourselves to the case when only the inner sphere rotates while the outer one is held fixed. Obtaining results for other cases is straightforward. The method which we employ for solution is that of matched singular asymptotic expansions (see Van Dyke [27]).

1. GOVERNING EQUATIONS

We consider the region between two eccentric spheres of radii a and b ($a < b$) which is occupied by a viscous incompressible liquid with kinematic coefficient of viscosity ν . We consider also cylindrical coordinate system (φ', z', ρ') whose axis Oz' coincides with the line of centers (Fig. 1). It is convenient to introduce bi-spherical coordinates (ξ, η, φ) which are connected with the cylindrical ones through the following formulae:

$$(1.1) \quad \rho' = c \frac{\sin \eta}{\operatorname{ch} \xi - \cos \eta}, \quad z' = c \frac{\operatorname{sh} \xi}{\operatorname{ch} \xi - \cos \eta},$$

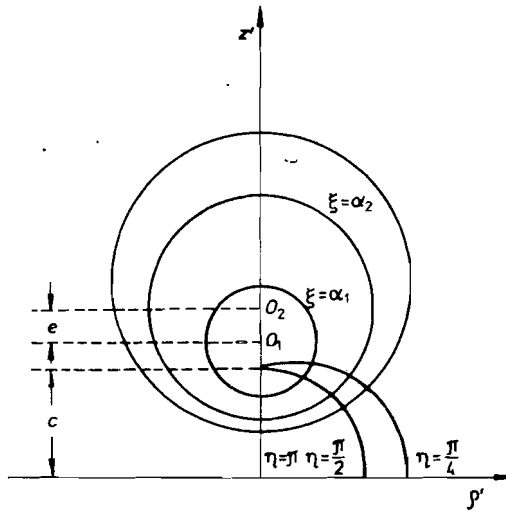


Fig. 1. Geometry of the problem and coordinate system.

where $0 \leq \eta \leq \pi$, $-\infty < \xi < \infty$ and c is the half-distance between the foci of a bi-spherical system (see for details Corn and Corn [28]). In the terms of bi-spherical coordinates the two spherical boundaries of flow are represented by $\xi = \alpha_1$, $\xi = \alpha_2$, where $\alpha_1 > \alpha_2$ if the sphere $\xi = \alpha_1$ is the smaller one. Connection between c , α_1 , α_2 from one side and a , b and eccentricity e —from the other is derived in section 4 of the present paper where the results are discussed.

In the present paper we consider oscillation whose axis of rotation coincides with the line of centers. It means that the flow possesses axial symmetry and nothing depends on the polar angle φ . Then a stream function can be introduced according to the formulae

$$(1.2) \quad v'_\xi = \frac{(\operatorname{ch} \xi - \cos \eta)^2}{c^2 \sin \eta} \frac{\partial \psi'}{\partial \eta}, \quad v'_\eta = -\frac{(\operatorname{ch} \xi - \cos \eta)^2}{c^2 \sin \eta} \frac{\partial \psi'}{\partial \xi}$$

(see for details e. g. [22]). Then the Navier — Stokes equations yield the following equations for the dimensionless stream function $\psi = \psi' / \varepsilon c^3 \omega$ and angular velocity $\Omega = v_\varphi' / \varepsilon c^2 \omega$

$$(1.3) \quad \frac{\partial}{\partial t} (D^2 \Psi) + \varepsilon \frac{2\Omega (\operatorname{ch} \xi - \cos \eta)^4}{\sin^2 \eta} \frac{\partial \left(\Omega, \frac{\sin \eta}{\operatorname{ch} \xi - \cos \eta} \right)}{\partial (\xi, \eta)} - \varepsilon \frac{(\operatorname{ch} \xi - \cos \eta)^3}{\sin \eta} \frac{\partial (\Psi, D^2 \Psi)}{\partial (\xi, \eta)} + \varepsilon \frac{2D^2 \Psi (\operatorname{ch} \xi - \cos \eta)^4}{\sin^2 \eta} \frac{\partial \left(\Psi, \frac{\sin \eta}{\operatorname{ch} \xi - \cos \eta} \right)}{\partial (\xi, \eta)} = \frac{1}{M^2} D^4 \Psi,$$

$$(1.4) \quad \frac{\partial \Omega}{\partial t} - \varepsilon \frac{(\operatorname{ch} \xi - \cos \eta)^3}{\sin \eta} \frac{\partial (\Psi, \Omega)}{\partial (\xi, \eta)} = \frac{1}{M^2} D^2 \Omega,$$

where $t = t' \omega$ is dimensionless time. Respectively $M = \sqrt{\omega c^2 / \nu}$ and $\varepsilon = \bar{\Omega} / \omega$ are frequency and amplitude parameters. In the above notations $\bar{\Omega}$ is the amplitude of torsional oscillations and ω — their frequency. In eqs. (1.3), (1.4) is denoted

$$(1.5) \quad D^2 = \sin \eta (\operatorname{ch} \xi - \cos \eta) \times \left\{ \frac{\partial}{\partial \xi} \left[\frac{(\operatorname{ch} \xi - \cos \eta)}{\sin \eta} \frac{\partial}{\partial \xi} \right] + \frac{\partial}{\partial \eta} \left[\frac{(\operatorname{ch} \xi - \cos \eta)}{\sin \eta} \frac{\partial}{\partial \eta} \right] \right\}.$$

The boundary conditions for Ψ and Ω are derived from the non-slip boundary conditions for velocity components

$$(1.6) \quad v_\xi' = 0, \quad v_\eta' = 0, \quad v_\varphi' = \bar{\Omega} \cos \omega t' \quad \text{at } \xi = \alpha_1,$$

$$(1.7) \quad v_\xi' = 0, \quad v_\eta' = 0, \quad v_\varphi' = 0 \quad \text{at } \xi = \alpha_2$$

which give

$$(1.8) \quad \Psi = \frac{\partial \Psi}{\partial \xi} = 0 \quad \text{at } \xi = \alpha_1, \quad \xi = \alpha_2,$$

$$(1.9a) \quad \Omega = \frac{\sin^2 \eta}{(\operatorname{ch} \alpha_1 - \cos \eta)^2} \cos t \quad \text{at } \xi = \alpha_1,$$

$$(1.9b) \quad \Omega = 0 \quad \text{at } \xi = \alpha_2.$$

The boundary value problem is coupled by the symmetry relations at axis of symmetry

$$v'_\eta = \frac{\partial \Psi'}{\partial \xi} = 0, \quad v'_\phi = 0 \quad \text{at } \eta = 0, \pi,$$

which after integration and acknowledging (1.8) yield

$$(1.10) \quad \Psi = 0, \quad \Omega = 0 \quad \text{at } \eta = 0, \pi.$$

Solution of the problem under consideration depends on two dimensionless parameters M and ϵ . Here we try to obtain an analytical solution for small values of dimensionless amplitude ($\epsilon \ll 1$). Otherwise the non-linear terms can not be discarded for large values of M and we arrive unequivocally to nonlinear equations (see in this instance Lyne [29] for a similar situation for the oscillatory flow in curved tubes). So we develop Ψ and Ω into power series with respect to ϵ

$$(1.11) \quad \begin{aligned} \Psi &= \sum_{n=0}^{\infty} \Psi_n \left(\xi, \eta, t; \frac{1}{M} \right) \epsilon^n, \\ \Omega &= \sum_{n=0}^{\infty} \Omega_n \left(\xi, \eta, t; \frac{1}{M} \right) \epsilon^n. \end{aligned}$$

Here persists an implicit assumption that ϵ and M^{-1} are of different asymptotic orders. Fortunately, the latter affects the results only in higher-order approximations with respect to ϵ and since we are bound to obtain asymptotic solution of order $O(\epsilon)$ we are not to worry about the interaction of the higher-order approximations.

It is easily shown that $\Psi_0 \equiv 0$ and that equations for Ψ_1 and Ω_0 form a coupled system of equations

$$(1.12) \quad \frac{\partial \Omega_0}{\partial t} = \frac{1}{M^2} D^2 \Omega_0$$

$$(1.13) \quad \frac{\partial}{\partial t} (D^2 \Psi_1) + 2 \frac{\Omega_0 (\text{ch } \xi - \cos \eta)^4}{\sin^2 \eta} \frac{\partial \left(\Omega_0, \frac{\sin \eta}{\text{ch } \xi - \cos \eta} \right)}{\partial (\xi, \eta)} = \frac{1}{M^2} D^4 \Psi_1$$

with the boundary conditions (1.8)–(1.10). In order not to overload the notations we shall omit further the subscripts 'o' and '1' for Ω_0 and Ψ_1 without fear of confusion.

The last system is recast into more convenient in a sense form as follows:

$$(1.14) \quad \frac{\partial \Omega}{\partial t} = \frac{1}{M^2} D^2 \Omega,$$

$$(1.15) \quad \frac{\partial (D^2 \Psi_1)}{\partial t} - \frac{\text{Re} [\Omega] (\text{ch } \xi - \mu)^4}{\sqrt{1 - \mu^2}} \frac{\partial \left(\text{Re} [\Omega], \frac{\sqrt{1 - \mu^2}}{\text{ch} (\xi - \mu)} \right)}{\partial (\xi, \mu)} = \frac{1}{M^2} D^4 \Psi_1,$$

where

$$D^2 \equiv (\text{ch } \xi - \mu) \left\{ \frac{\partial}{\partial \xi} \left[(\text{ch } \xi - \mu) \frac{\partial}{\partial \xi} \right] + (1 - \mu^2) \frac{\partial}{\partial \mu} \left[(\text{ch } \xi - \mu) \frac{\partial}{\partial \mu} \right] \right\},$$

$$\mu = \cos \eta.$$

Respectively the boundary conditions read:

$$(1.16) \quad \Psi = \frac{\partial \Psi}{\partial \xi} = 0 \quad \text{at } \xi = \alpha_1, \xi = \alpha_2,$$

$$(1.17a) \quad \Omega = \frac{1 - \mu^2}{(\text{ch } \alpha_1 - \mu)^2} e^{i\mu} \quad \text{at } \xi = \alpha_1,$$

$$(1.17b) \quad \Omega = 0 \quad \text{at } \xi = \alpha_2,$$

$$(1.18) \quad \Psi = 0, \quad \Omega = 0 \quad \text{at } \mu = \pm 1.$$

In eq. (1.17a) it is introduced for convenience the complex function $e^{i\mu}$. Naturally, after solution is obtained only the real part of it is to be retained.

2. METHOD OF MATCHED ASYMPTOTIC EXPANSIONS

As promised in the Introduction we shall try to solve equations (1.14), (1.15) under the assumption that $M^{-1} \ll 1$. The latter allows us to expand Ψ_1 and Ω_0 (henceforth named simply Ψ and Ω) into asymptotic series with respect to small parameter M^{-1} . As seen the parameter M^{-1} multiplies the term with the higher-order spatial derivatives which means that the respective series shall be singular one. Then one is to expect occurrence of two boundary layers at sphere walls and a core flow inbetween the two spheres. So there are three different situations in these regions. Let us denote them by $\Psi^{(i)}$, $\Omega^{(i)}$, where $i=0$ refers to core flow, $i=1$ — to the boundary layer at the inner sphere ($\xi = \alpha_1$) and $i=2$ — to the outer sphere ($\xi = \alpha_2$) (Fig. 1). Obviously, it is assumed that the gap between the spheres is wide enough in order not to overlap the two boundary layers. Otherwise one is to consider the entire annuli as a boundary-layer region and to employ the lubrication approximation (as for instance it is done in [20]). The latter goes beyond the scope of the present paper.

The asymptotic consequences for the two functions Ψ and Ω are in general not known priori, so we write

$$(2.1) \quad \Omega^{(i)} = \sum_{n=0}^{\infty} \Omega_n^{(i)}(\xi, \eta, t) \alpha_n^{(i)}(M^{-1}),$$

$$(2.2) \quad \Psi^{(i)} = \sum_{n=0}^{\infty} \Psi_n^{(i)}(\xi, \eta, t) \beta_n^{(i)}(M^{-1}),$$

where

$$\lim_{M^{-1} \rightarrow 0} \frac{\alpha_{n+1}^{(i)}(M^{-1})}{\alpha_n^{(i)}(M^{-1})} = 0 \quad \text{and} \quad \lim_{M^{-1} \rightarrow 0} \frac{\beta_{n+1}^{(i)}(M^{-1})}{\beta_n^{(i)}(M^{-1})} = 0.$$

Introducing now (2.1) into (1.14) one obtains

$$(2.3) \quad \frac{\partial \Omega_0^{(i)}}{\partial t} = 0, \quad \text{i. e.} \quad \Omega_0^{(i)} = \Omega_0^{(i)}(\xi, \mu).$$

This solution can not satisfy the two boundary conditions (1.17) simultaneously. The latter can be achieved by the so-called 'inner' expansions in the boundary layers at the spheres. Having in mind that the coordinate ξ increases toward inner sphere, the scaled coordinate in the first boundary layer is the following

$$(2.4a) \quad \zeta = (\alpha_1 - \xi) \frac{M}{\sqrt{2}}.$$

Respectively in the second boundary layer (when the latter exists) we have

$$(2.4b) \quad \bar{\zeta} = (\xi - \alpha_2) \frac{M}{\sqrt{2}},$$

where the factor $\sqrt{2}$ is introduced for the sake of convenience. Now, on the basis of (2.1) and (2.4) the following equation for $\Omega_0^{(1)}$ is derived

$$(2.5) \quad \frac{\partial \Omega_0^{(1)}}{\partial t} = \frac{(\text{ch } \alpha_1 - \mu)^2}{2} \frac{\partial^2 \Omega_0^{(1)}}{\partial \zeta^2}.$$

The solution of (2.5) which satisfies the boundary condition (1.17a) is

$$\Omega_0^{(1)} = \left[C_1 \exp \left(\frac{i+1}{\text{ch } \alpha_1 - \mu} \zeta \right) + (1 - C_1) \exp \left(- \frac{i+1}{\text{ch } \alpha_1 - \mu} \zeta \right) \right] \frac{1 - \mu^2}{(\text{ch } \alpha_1 - \mu)^2} e^{it}.$$

Obviously, the first term should vanish because it is unbounded at $\zeta \rightarrow \infty$ and, therefore, does not match anything in the core. The latter is achieved by taking $C_1 = 0$. Then, finally, we have

$$(2.6a) \quad \Omega_0^{(1)} = \frac{1 - \mu^2}{(\text{ch } \alpha_1 - \mu)^2} \exp \left(it - \frac{i+1}{\text{ch } \alpha_1 - \mu} \zeta \right).$$

In the same manner from (1.17b) is obtained that

$$(2.6b) \quad \Omega_0^{(2)} \equiv 0.$$

Obviously (2.3) and (2.6a,b) match if only

$$(2.7) \quad \Omega_0^{(0)} \equiv 0.$$

Then (2.6a,b) and (2.7) present the zeroth-order solution of eq. (1.14) with respect to small parameter M^{-1} which we call further first approximation. It is instructive to note that the boundary condition (1.17a) defines also the first term $\alpha_0^{(1)}(M^{-1}) \equiv 1$ of the asymptotic sequence.

The second approximation Ω_1 with respect to M^{-1} for Ω is sought in the same way. Once again we have

$$(2.8) \quad \frac{\partial \Omega_1^{(0)}}{\partial t} = 0, \text{ i. e. } \Omega_1^{(0)} = \Omega_1^{(0)}(\xi, \eta)$$

and

$$(2.9) \quad \Omega_1^{(2)} = 0,$$

due to the boundary condition (1.17 b). In the boundary layer at the inner sphere we obtain

$$\begin{aligned} \alpha_1^{(1)}(M^{-1}) \frac{\partial \Omega_1^{(1)}}{\partial t} &= \alpha_1^{(1)}(M^{-1}) \frac{(\text{ch } \alpha_1 - \mu)^2}{2} \frac{\partial^2 \Omega_1^{(1)}}{\partial \xi^2} \\ &- \alpha_0^{(1)}(M^{-1}) M^{-1} \sqrt{2} \xi \text{sh } \alpha_1 (\text{ch } \alpha_1 - \mu) \frac{\partial^2 \Omega_0^{(1)}}{\partial \xi^2} \\ &- \alpha_0^{(1)}(M^{-1}) M^{-1} \frac{(\text{ch } \alpha_1 - \mu) \text{sh } \alpha_1}{\sqrt{2}} \frac{\partial \Omega_0^{(1)}}{\partial \xi} + O[M^{-2} \alpha_0^{(1)}(M^{-1})]. \end{aligned}$$

Substituting here $\Omega_0^{(1)}$ from (2.6a) the last equation is rendered into the form

$$\begin{aligned} (2.10) \quad & \frac{\partial \Omega_1^{(1)}}{\partial t} - \frac{(\text{ch } \alpha_1 - \mu)^2}{2} \frac{\partial^2 \Omega_1^{(1)}}{\partial \xi^2} \\ &= \frac{M^{-1}}{\alpha_1^{(1)}(M^{-1})} \left[- \frac{2\sqrt{2}i(1-\mu^2)}{(\text{ch } \alpha_1 - \mu)^3} \xi \text{sh } \alpha_1 \exp \left(it - \frac{i+1}{\text{ch } \alpha_1 - \mu} \xi \right) \right. \\ & \quad \left. + \frac{(i+1)\text{sh } \alpha_1 (1-\mu^2)}{\sqrt{2}(\text{ch } \alpha_1 - \mu)^2} \exp \left(it - \frac{i+1}{\text{ch } \alpha_1 - \mu} \xi \right) \right] \end{aligned}$$

with the trivial boundary condition

$$(2.11) \quad \Omega_1^{(1)} = 0,$$

because (1.17b) is satisfied for the first approximation $\Omega_0^{(1)}$. The only way to give a non-trivial solution to (2.10) is to have

$$(2.12) \quad \alpha_1^{(1)}(M^{-1}) = M^{-1},$$

i. e. the second member of the asymptotic sequence $\alpha_1^{(1)}$ is already defined. Respectively for the second approximation in the inner boundary layer is obtained

$$(2.13) \quad \Omega_1^{(1)} = - \frac{(i+1)(1-\mu^2)\text{sh } \alpha_1}{\sqrt{2}(\text{ch } \alpha_1 - \mu)^4} \xi^2 \exp \left(it - \frac{i+1}{\text{ch } \alpha_1 - \mu} \xi \right).$$

This procedure can be extended further but with increasing technical difficulties. Higher-order approximations are not necessary for the purposes of the present paper since we attempt a first approximation for the stream function.

In the end it is to be mentioned that the matching procedure turned out to be somewhat oversimplified due to the specific shape of solution for angular velocity

Ω . This is not the case, however, with the stream function of secondary flow where the full scale matching technique is required.

Turning to eq. (1.15) we can notice that the solution for ψ should be comprised of steady and unsteady part, namely

$$(2.14) \quad \Psi = \bar{\Psi}(\xi, \mu) + e^{2\mu} \bar{\bar{\Psi}}(\xi, \mu),$$

because of the specific expressions (see (2.6a) and (2.13)) for angular velocity. Indeed (1.15) contains terms proportional to

$$\operatorname{Re}[e^{\mu}] \operatorname{Re}[e^{\mu}] = \frac{1}{2} \operatorname{Re}(1 + e^{2\mu}),$$

which is the explanation of the form (2.14).

Substituting the asymptotic series (2.2) for Ψ into (1.15) and acknowledging (2.7) we obtain that the core flow is governed by the following homogeneous equation

$$(2.15) \quad \frac{\partial}{\partial t} (D^2 \Psi^{(0)}) = M^{-2} D^4 \Psi^{(0)}.$$

Introducing here the asymptotic expansion (2.2) we derive:

$$(2.16) \quad \frac{\partial}{\partial t} D^2 \Psi_0^{(0)} = 0, \quad \frac{\partial}{\partial t} D^2 \Psi_1^{(0)} = 0,$$

$$\frac{\partial}{\partial t} D^2 \Psi_2^{(0)} = D^4 \Psi_0^{(0)}, \quad \frac{\partial}{\partial t} D^2 \Psi_3^{(0)} = D^4 \Psi_1^{(0)}, \dots$$

The first of these equations has the following general solution

$$(2.17) \quad \Psi_0^{(0)} = \bar{\Psi}_0^{(0)}(\xi, \eta) + e^{2\eta} \bar{\bar{\Psi}}_0^{(0)}(\xi, \eta),$$

where

$$(2.18) \quad D^2 \bar{\bar{\Psi}}_0^{(0)}(\xi, \eta) = 0$$

and the specific shape of the time dependence $e^{2\eta}$ is taken into account. Obviously

$$D^2 \Psi_0^{(0)} = D^2 \bar{\Psi}_0^{(0)}(\xi, \eta)$$

and, therefore,

$$D^4 \Psi_0^{(0)} \equiv D^4 \bar{\Psi}_0^{(0)}(\xi, \eta).$$

Introducing the latter into the third equation (2.16) we obtain

$$\frac{\partial}{\partial t} D^2 \Psi_2^{(0)} = D^4 \bar{\Psi}_0^{(0)}(\xi, \eta),$$

i. e.

$$D^2\Psi_2^{(0)} = tD^4\bar{\Psi}_0^{(0)}(\xi, \eta) + \Phi(\xi, \eta),$$

which implies that $D^2\Psi_2^{(0)}$ increases to infinity with time unless

$$(2.19) \quad D^4\bar{\Psi}_0^{(0)} = 0.$$

So that (2.18) and (2.19) are the governing equations for the unsteady and steady part of secondary flow respectively. The higherorder approximations are treated in absolutely the same manner if necessary.

We postpone solving eq. (2.19) to next Section and focus here our attention on boundary layers at the rigid boundaries. For the inner sphere the governing equation (1.15) in terms of scaled variable (2.4a) yields the following equation for the first term in the asymptotic series (2.2)

$$(2.20) \quad \beta_0^{(1)}(M^{-1}) \left[\frac{(\text{ch } \alpha_1 - \mu)^2}{2} \frac{\partial^4 \Psi_0^{(1)}}{\partial \zeta^4} - \frac{\partial^3 \Psi_0^{(1)}}{\partial t \partial \zeta^2} \right] \\ = -\sqrt{2} M^{-1} \frac{2(\mu \text{ch } \alpha_1 - 1)}{1 - \mu^2} \text{Re}[\Omega_0^{(1)}] \text{Re} \left[\frac{\partial \Omega_0^{(1)}}{\partial \zeta} \right].$$

If now we take $\beta_0^{(1)}(M^{-1})M \rightarrow \infty$ for $M \rightarrow 0$, then we obtain a homogeneous equation and after satisfying the boundary conditions (1.16) and matching with the solution of homogeneous eq. (2.19) we unequivocally arrive to the trivial solution for stream function $\Psi_0^{(1)}$. The only reasonable limit for $\beta_0^{(1)}$ is $\beta_0^{(1)}(M^{-1}) = \sqrt{2} M^{-1}$. Then separating the solution of (2.20) into steady and unsteady part according to (2.14) we obtain the following two equations:

$$(2.21a) \quad \frac{\partial^4 \bar{\Psi}_0^{(1)}}{\partial \zeta^4} = \frac{2(\mu \text{ch } \alpha_1 - 1)}{(\text{ch } \alpha_1 - \mu)^7} \exp \left(-\frac{2\zeta}{\text{ch } \alpha_1 - \mu} \right),$$

$$(2.21b) \quad \frac{\partial^4 \bar{\Psi}_0^{(1)}}{\partial \zeta^4} - \frac{4i}{(\text{ch } \alpha_1 - \mu)^2} \frac{\partial^2 \bar{\Psi}_0^{(1)}}{\partial \zeta^2} \\ = \frac{2(i+1)(\mu \text{ch } \alpha_1 - 1)(1 - \mu^2)}{(\text{ch } \alpha_1 - \mu)^7} \exp \left[-2(1+i) \frac{\zeta}{\text{ch } \alpha_1 - \mu} \right].$$

Being reminded of the boundary conditions (1.16) the solutions of (2.21) of interest are

$$(2.22a) \quad \bar{\Psi}_0^{(1)} = \frac{(\mu \text{ch } \alpha_1 - 1)(1 - \mu^2)}{(\text{ch } \alpha_1 - \mu)^3} \left[\frac{1}{8} \left(e^{-2\zeta/(\text{ch } \alpha_1 - \mu)} - 1 \right) + \frac{\zeta}{4(\text{ch } \alpha_1 - \mu)} \right] \\ + E_2^{(1)} \zeta^2 + E_3^{(1)} \zeta^3$$

$$(2.22b) \quad \bar{\Psi}_0^{(1)} = \frac{(\mu \text{ch} \alpha_1 - 1)(1 - \mu^2)}{(\text{ch} \alpha_1 - \mu)^3} \left[\frac{(i+1)}{16} \left(1 - e^{-2(1+i)\zeta/(\text{ch} \alpha_1 - \mu)} \right) - \frac{i\zeta}{4(\text{ch} \alpha_1 - \mu)} \right] \\ + C^{(1)} \left[e^{(i+1)\sqrt{2}\zeta/(\text{ch} \alpha_1 - \mu)} + \frac{(i+1)\sqrt{2}\zeta}{\text{ch} \alpha_1 - \mu} - 1 \right].$$

Obtaining the last expression the terms which increase infinitely with $\zeta \rightarrow \infty$ are discarded. Constants $E_2^{(1)}$, $E_3^{(1)}$ and $C^{(1)}$ are to be estimated after matching with the 'outer' solution $\bar{\Psi}_0^{(0)}$.

Considering the boundary layer at the resting outer sphere we scale the variable ξ according to formula (2.4b). In that region the equations adopt the following simple form

$$(2.23) \quad \beta_0^{(2)}(M^{-1}) \left[\frac{(\text{ch} \alpha_2 - \mu)^2}{2} \frac{\partial^4 \bar{\Psi}_0^{(2)}}{\partial \zeta^4} - \frac{\partial^3 \bar{\Psi}_0^{(2)}}{\partial t \partial \zeta^2} \right] = 0$$

because of the fact that $\Omega_0^{(2)} = 0$ (see (2.6b)). Once again the solution is divided into steady $\bar{\Psi}_0^{(2)}$ and unsteady $\bar{\Psi}_0^{(2)}$ parts and the respective solutions which satisfy boundary conditions (1.16) are:

$$(2.24a) \quad \bar{\Psi}_0^{(2)} = E_2^{(2)} \bar{\zeta}^2 + E_3^{(2)} \bar{\zeta}^3,$$

$$(2.24b) \quad \bar{\Psi}_0^{(2)} = C^{(2)} \left[\exp \left(- \frac{(i+1)\sqrt{2}\bar{\zeta}}{\text{ch} \alpha_2 - \mu} \right) + \frac{\sqrt{2}(i+1)\bar{\zeta}}{\text{ch} \alpha_2 - \mu} - 1 \right].$$

As we are primarily concerned with the steady part of secondary streaming we shall outline here only the matching procedure for that case. Matching the unsteady parts is similar and in a sense simpler for the problem under consideration. Starting with the boundary layer at the inner sphere we recast function $\bar{\Psi}_0^{(0)}$ into the 'inner' variable

$$(2.25) \quad \hat{\Psi}_0^{(0)} \equiv \bar{\Psi}_0^{(0)} \left(\alpha_1 - \frac{\sqrt{2}}{M} \zeta, \eta \right)$$

and expand $\hat{\Psi}_0^{(0)}$ into McLoren series with respect to variable ζ , namely

$$(2.26) \quad \hat{\Psi}_0^{(0)} = \hat{\Psi}_0^{(0)}(0, \eta) + \frac{\partial \hat{\Psi}_0^{(0)}}{\partial \zeta} \Big|_{\zeta=0} \zeta + \frac{\partial^2 \hat{\Psi}_0^{(0)}}{\partial \zeta^2} \Big|_{\zeta=0} \frac{\zeta^2}{2!} + \dots \\ = \bar{\Psi}_0^{(0)}(\alpha_1, \eta) - \frac{\partial \bar{\Psi}_0^{(0)}}{\partial \xi} \Big|_{\xi=\alpha_1} \frac{\sqrt{2}}{M} \zeta + \frac{\partial^2 \bar{\Psi}_0^{(0)}}{\partial \xi^2} \Big|_{\xi=\alpha_1} \frac{1}{M^2} \zeta^2 + \dots$$

where (2.25) is acknowledged. Matching this solution to order M^{-1} (see Van Dyke [27] for details of matching procedure) to the function $\beta_0^{(1)}(M^{-1})\bar{\Psi}_0^{(1)}$, where $\bar{\Psi}_0^{(1)}$ is from (2.22a) and being reminded that $\beta_0^{(1)}(M^{-1}) = \sqrt{2}M^{-1}$ we obtain for the first approximation

$$(2.27) \quad \bar{\Psi}_0^{(0)}(\alpha_1, \mu) = 0,$$

respectively

$$-\frac{\partial \bar{\Psi}_0^{(0)}}{\partial \xi} \Big|_{\xi=\alpha_1} \zeta = \frac{(\mu \operatorname{ch} \alpha_1 - 1)(1 - \mu^2)}{4(\operatorname{ch} \alpha_1 - \mu)^4} \zeta$$

and, therefore,

$$(2.28) \quad \frac{\partial \bar{\Psi}_0^{(0)}}{\partial \xi} \Big|_{\xi=\alpha_1} = -\frac{(\mu \operatorname{ch} \alpha_1 - 1)(1 - \mu^2)}{4(\operatorname{ch} \alpha_1 - \mu)^4},$$

which are the boundary conditions for $\bar{\Psi}_0^{(0)}(\xi, \eta)$ at $\xi = \alpha_1$. The rest of the boundary conditions are obtained after matching with the boundary layer solution $\bar{\Psi}_0^{(2)}$ from (2.24a). Then

$$(2.29) \quad \bar{\Psi}_0^{(0)}(\alpha_2, \mu) = 0, \quad \frac{\partial \bar{\Psi}_0^{(0)}}{\partial \xi} \Big|_{\xi=\alpha_2} = 0$$

are the boundary conditions at the outer sphere.

So far we have a coupled boundary value problem (2.19), (2.27), (2.28), (2.29) for estimating the function $\bar{\Psi}_0^{(0)}$.

Here should be mentioned that the matching also provides the values of $E_2^{(1)}$, $E_3^{(1)}$ which are easily shown to be equal to zero

$$(2.30) \quad E_2^{(1)} = 0, \quad E_3^{(1)} = 0.$$

3. SOLUTION FOR THE FIRST APPROXIMATION OF THE STEADY PART OF SECONDARY MOTION

The general solution of (2.19) is due to Stimson and Jeffery [22] and has the form

$$(3.1) \quad \bar{\Psi}_0^{(0)} = (\operatorname{ch} \xi - \mu)^{-3/2} \sum_{n=1}^{\infty} \left[A_n \operatorname{ch} \left(n - \frac{1}{2} \right) \xi + B_n \operatorname{sh} \left(n - \frac{1}{2} \right) \xi + C_n \operatorname{ch} \left(n + \frac{3}{2} \right) \xi + D_n \operatorname{sh} \left(n + \frac{3}{2} \right) \xi \right] C_{n+1}^{-1/2}(\mu),$$

where $C_{n+1}^{-1/2}(\mu)$ are the Gegenbauer polynomials of order $n + 1$ and degree $-1/2$. Constants A_n , B_n , C_n and D_n are to be estimated after boundary conditions. We start considerations with the only nontrivial boundary condition (2.28). From (3.1) one has

$$\frac{\partial \bar{\Psi}_0^{(0)}}{\partial \xi} \Big|_{\xi=\alpha_1} = -\frac{3}{2} \operatorname{sh} \alpha_1 (\operatorname{ch} \alpha_1 - \mu)^{-5/2} \sum_{n=1}^{\infty} \left[A_n \operatorname{ch} \left(n - \frac{1}{2} \right) \alpha_1 + B_n \operatorname{sh} \left(n - \frac{1}{2} \right) \alpha_1 \right]$$

$$\begin{aligned}
& + C_n \operatorname{ch}\left(n + \frac{3}{2}\right) \alpha_1 + D_n \operatorname{sh}\left(n + \frac{3}{2}\right) \alpha_1 \left] C_{n+1}^{-1/2}(\mu) + (\operatorname{ch} \alpha_1 - \mu)^{-3/2} \sum_{n=1}^{\infty} \left[A_n \left(n - \frac{1}{2}\right) \right. \right. \\
& \times \operatorname{sh}\left(n - \frac{1}{2}\right) \alpha_1 + B_n \left(n - \frac{1}{2}\right) \operatorname{ch}\left(n - \frac{1}{2}\right) \alpha_1 + C_n \left(n + \frac{3}{2}\right) \operatorname{sh}\left(n + \frac{3}{2}\right) \alpha_1 \\
& \left. \left. + D_n \left(n + \frac{3}{2}\right) \operatorname{ch}\left(n + \frac{3}{2}\right) \alpha_1 \right] C_{n+1}^{-1/2}(\mu) = -\frac{1}{4} \frac{(\mu \operatorname{ch} \alpha_1 - 1)(1 - \mu^2)}{(\operatorname{ch} \alpha_1 - \mu)^4}.
\end{aligned}$$

It is obvious that the first sum in the right-hand side of the last equation equals zero because of the boundary condition (2.27) which reads

$$\begin{aligned}
(3.2) \quad & (\operatorname{ch} \alpha_1 - \mu)^{-3/2} \sum_{n=1}^{\infty} \left[A_n \operatorname{ch}\left(n - \frac{1}{2}\right) \alpha_1 + B_n \operatorname{sh}\left(n - \frac{1}{2}\right) \alpha_1 \right. \\
& \left. + C_n \operatorname{ch}\left(n + \frac{3}{2}\right) \alpha_1 + D_n \operatorname{sh}\left(n + \frac{3}{2}\right) \alpha_1 \right] C_{n+1}^{-1/2}(\mu) = 0.
\end{aligned}$$

The second sum of that equation, however, does not vanish trivially and the only way to satisfy analytically the boundary condition is to develop the following function

$$(3.3) \quad f(\mu) = -\frac{(\mu \operatorname{ch} \alpha_1 - 1)(1 - \mu^2)}{4(\operatorname{ch} \alpha_1 - \mu)^{5/2}}$$

into series with respect to Gegenbauer polynomials. This is, in fact, the crucial point of the solution. The easiest way to develop the function $f(\mu)$ is in our opinion to make use of the properties of so-called generation function of Gegenbauer polynomials (see Bateman and Erdelyi [30]). For the particular case under consideration for which the degree of polynomials is $-1/2$ we have

$$(3.4) \quad g(s; \mu) = (1 - 2s\mu + s^2)^{1/2} = \sum_{n=0}^{\infty} C_n^{-1/2}(\mu) s^n, \quad |s| < 1.$$

It is seen that function $f(\mu)$ from (3.3) is akin to generating function (3.4) since one can set

$$\operatorname{ch} \alpha_1 = \frac{1 + s^2}{2s} \quad \text{or} \quad s = e^{-\alpha_1} \quad (\text{since } |s| < 1),$$

and so that to obtain

$$(\operatorname{ch} \alpha_1 - \mu) = \frac{1}{2s} (1 - 2s\mu + s^2).$$

Then

$$f(\mu) = - (2s)^{3/2} \frac{[\mu(1 + s^2) - 2s] (1 - \mu^2)}{4(1 - 2s\mu + s^2)^{5/2}}.$$

The last function is a linear combination of generating function $g(s; \mu)$ and its first three derivatives with respect to parameter s , namely

$$\begin{aligned} G'_s &= \frac{s - \mu}{(1 - 2s\mu + s^2)^{1/2}} = \sum_{n=0}^{\infty} C_{n+1}^{-1/2}(\mu) (n + 1) s^n, \\ (3.5) \quad G''_s &= \frac{1 - \mu^2}{(1 - 2s\mu + s^2)^{3/2}} = \sum_{n=0}^{\infty} C_{n+2}^{-1/2}(\mu) (n + 1) (n + 2) s^n, \\ G'''_s &= -3 \frac{(1 - \mu^2)(s - \mu)}{(1 - 2s\mu + s^2)^{5/2}} = \sum_{n=0}^{\infty} C_{n+3}^{-1/2}(\mu) (n + 1) (n + 2) (n + 3) s^n. \end{aligned}$$

After some trivial algebra one obtains

$$(3.6) \quad f(\mu) = \frac{(2s)^{3/2}}{4} \sum_{n=1}^{\infty} \left[(n + 1) n s^n + \frac{s^2 - 1}{3} (n + 1) n (n - 1) s^{n-2} \right] C_{n+1}^{-1/2}(\mu).$$

Then the non-trivial boundary condition yields the following algebraic relation for the coefficients A_n, B_n, C_n, D_n being reminded that it has to be satisfied for each value of argument μ

$$\begin{aligned} (3.7a) \quad & (2n - 1) \left[A_n \operatorname{sh} \left(n - \frac{1}{2} \right) \alpha_1 + B_n \operatorname{ch} \left(n - \frac{1}{2} \right) \alpha_1 \right] \\ & + (2n + 3) \left[C_n \operatorname{sh} \left(n + \frac{3}{2} \right) \alpha_1 + D_n \operatorname{ch} \left(n + \frac{3}{2} \right) \alpha_1 \right] = a_n(\alpha_1), \end{aligned}$$

where

$$a_n(\alpha_1) = \frac{(2e^{-\alpha_1})^{3/2}}{2} \left[n(n + 1) e^{-\alpha_1 n} + \frac{e^{-2\alpha_1} - 1}{3} (n - 1) n (n + 1) e^{-\alpha_1(n-2)} \right]$$

In the same manner from (3.2) is derived that

$$(3.7b) \quad A_n \operatorname{ch}\left(n - \frac{1}{2}\right) \alpha_1 + B_n \operatorname{sh}\left(n - \frac{1}{2}\right) \alpha_1 \\ + C_n \operatorname{ch}\left(n + \frac{3}{2}\right) \alpha_1 + D_n \operatorname{sh}\left(n + \frac{3}{2}\right) \alpha_1 = 0.$$

Respectively, boundary conditions (2.29) yield:

$$(3.7c) \quad A_n \operatorname{ch}\left(n - \frac{1}{2}\right) \alpha_2 + B_n \operatorname{sh}\left(n - \frac{1}{2}\right) \alpha_2 \\ + C_n \operatorname{ch}\left(n + \frac{3}{2}\right) \alpha_2 + D_n \operatorname{sh}\left(n + \frac{3}{2}\right) \alpha_2 = 0,$$

$$(3.7d) \quad (2n - 1) \left[A_n \operatorname{sh}\left(n - \frac{1}{2}\right) \alpha_2 + B_n \operatorname{ch}\left(n - \frac{1}{2}\right) \alpha_2 \right] \\ + (2n + 3) \left[C_n \operatorname{sh}\left(n + \frac{3}{2}\right) \alpha_2 + D_n \operatorname{ch}\left(n + \frac{3}{2}\right) \alpha_2 \right] = 0.$$

Equations (3.7a–d) form a coupled system for unknowns A_n, B_n, C_n, D_n for each n . This system is easily solved provided some trivial manipulations are performed in order to ascertain good convergence at high α_1, α_2 .

So, having the solution of system (3.7a–d) and introducing it into the general formula (3.1) one obtains the sought solution for the steady part of secondary streaming in eccentric spherical annuli.

4. NUMERICAL RESULTS AND DISCUSSION

It is instructive now to return to the original (dimensional) variables since in the present paper the role of length scale is played by the focal semi-distance c which is convenient in computations but is not useful in comparison with other works, and more specifically with the works where a single sphere or a concentric spherical annuli are considered. We shall do this only for stream function of the secondary flow as the latter is the prime objective of our work. Being reminded that Ψ is, in fact, the second approximation with respect to ε , we find that the dimensional stream function Ψ' is given by

$$(4.1) \quad \Psi' = \varepsilon^2 c^3 \omega \Psi,$$

where Ψ is the solution of (3.1). It is convenient to introduce a different dimensionless stream function $\bar{\Psi}$ with the radius of the inner sphere a as a length scale, i. e.

$$(4.2) \quad \Psi' = \varepsilon a^3 \omega \bar{\Psi}.$$

Obviously the last dimensionless quantity is connected with Ψ as follows

$$(4.3) \quad \bar{\Psi} = \left(\frac{c}{a}\right)^3 \varepsilon \Psi.$$

As Ψ is not function of ε , one sees that for small ε $\bar{\Psi}$ is a linear function of ε . So that we shall present results for the ratio $\bar{\Psi}/\varepsilon$ in order to reduce the number of plots. Further we notice that the solution Ψ of the the previous section does not depend on frequence parameter M . On the latter depends only the thickness of the boundary layer. This thickness is very small, because we are concerned with $M^{-1} \ll 1$, and it cannot be discerned if plotted in the same scale as the core flow (3.1). So we decide to plot results for the core flow regardless of the specific value of frequence parameter M . Finally the only governing parameters of our study of steady secondary motion remain the geometric characteristics: radii a and b of inner and outer sphere and eccentricity e . These three characteristics completely define the above three α_1 , α_2 and c through the following relations

$$(4.4) \quad c = \frac{\sqrt{a^4 + b^4 + e^4 - 2a^2b^2 - 2b^2e^2 - 2a^2e^2}}{2e}$$

$$(4.5) \quad \alpha_1 = \operatorname{arcsch} \frac{c}{a}, \quad \alpha_2 = \operatorname{arcsch} \frac{c}{b}.$$

For our method it is more convenient to use $\lambda = b/a$ and $\sigma = c/a$ as independent variables and to relate the dimensionless eccentricity to them as follows:

$$(4.6) \quad \frac{e}{a} = \frac{1}{a} \left(\sqrt{\lambda^2 a^2 + c^2} - \sqrt{a^2 + c^2} \right) = \sqrt{\lambda^2 + \sigma^2} - \sqrt{1 + \sigma^2}.$$

The numerical implementation of the solution of section 3 is achieved through calculating the values of Ψ on the base of (3.1) with expressions for A_n , B_n , C_n , D_n acknowledged on a given grid. Then the obtained two-dimensional array is treated by means of standard numerical procedure for approximate tracing of the equilines of a function presented in discrete way. In order to have good accuracy for large values of eccentricity e (or which is the same-small c for given a and b) we employ a non-uniform mesh in η -direction according to the rule

$$(4.7) \quad \eta = 2 \operatorname{arctg} \left[\frac{\operatorname{sh} \alpha_2}{\operatorname{ch} \alpha_2 + 1} \operatorname{tg} \frac{s \cdot \operatorname{sh} \alpha_2}{2c} \right],$$

$$s = (j - 1) \frac{c\pi}{(\operatorname{sh} \alpha_2)(N - 1)}.$$

The latter secures that the reference points are uniformly spaced along a meridian of outer sphere. Parameters of the mesh are $i = 1, \dots, K$ and $j = 1, \dots, N$, for which

$$(4.8) \quad \xi_i = \alpha_2 + (i - 1)h_\xi, \quad \eta_j = 0 + (j - 1)h_\eta,$$

where

$$h_\xi = (\alpha_1 - \alpha_2)/(K - 1) \text{ and } h_\eta = \pi/(N - 1).$$

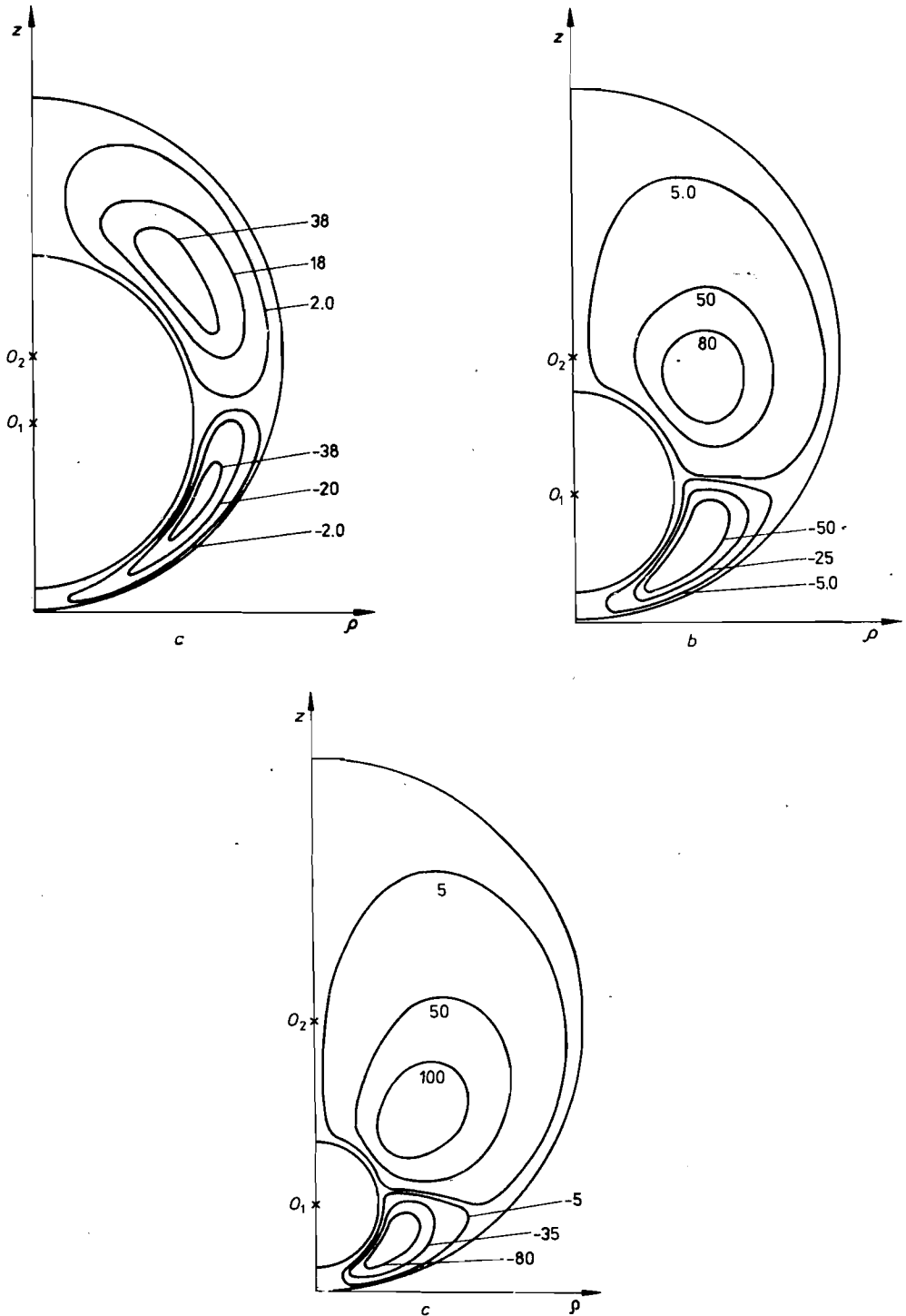


Fig. 2. Steady part of secondary flow $\Psi/\epsilon^2 a^3 \omega \cdot 10^4$ for intermediate value of eccentricity $\sigma=1$ and three different values of radii ratios $\lambda=1.5$ (a), $\lambda=2.5$ (b), $\lambda=4$ (c)

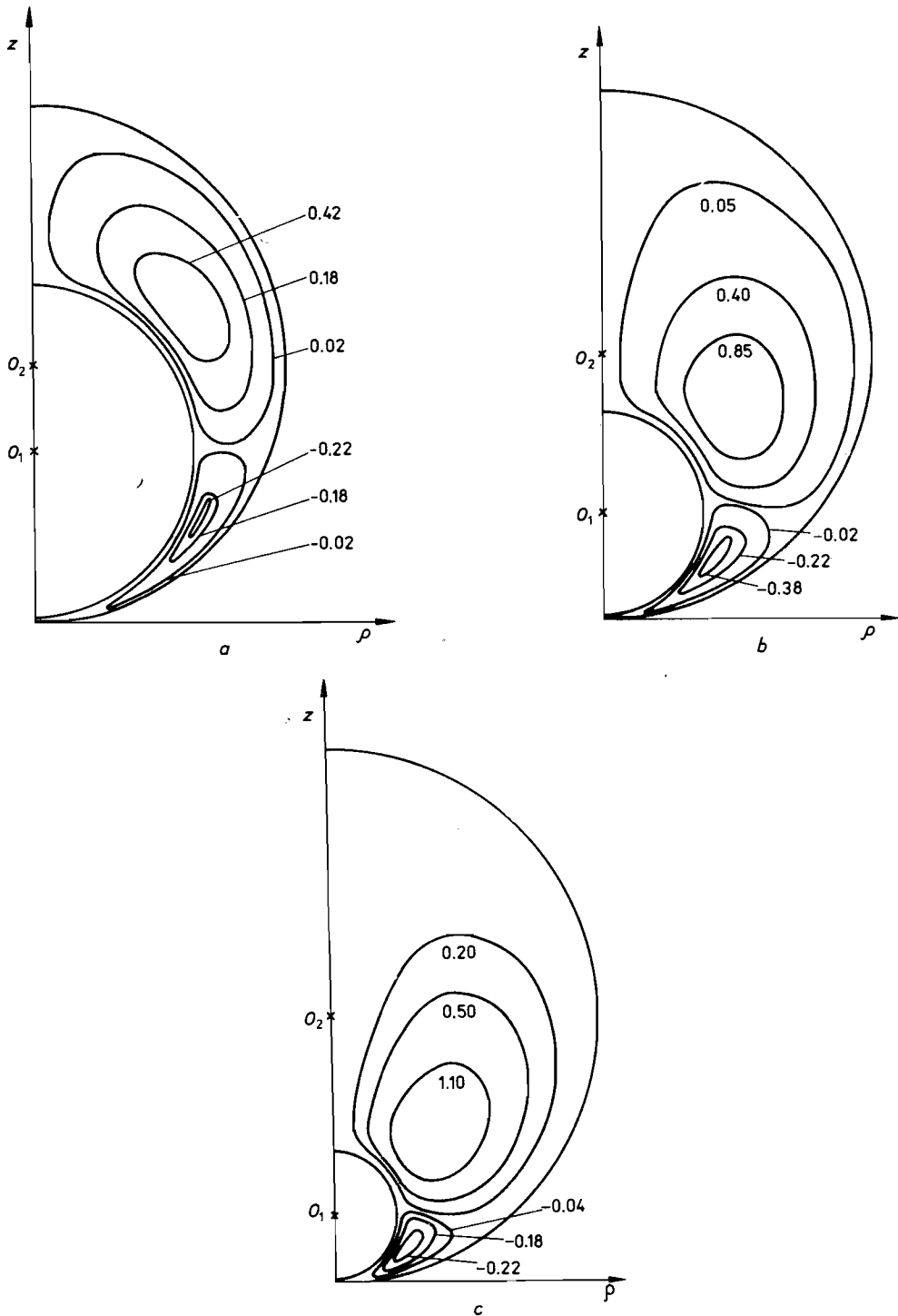


Fig. 3. Steady part of secondary flow $\Psi/\varepsilon^2 a^3 \omega$ for large value of eccentricity $\sigma=0.2$ and three different values of radii ratios $\lambda=1.5$ (a), $\lambda=2.5$ (b), $\lambda=4$ (c)

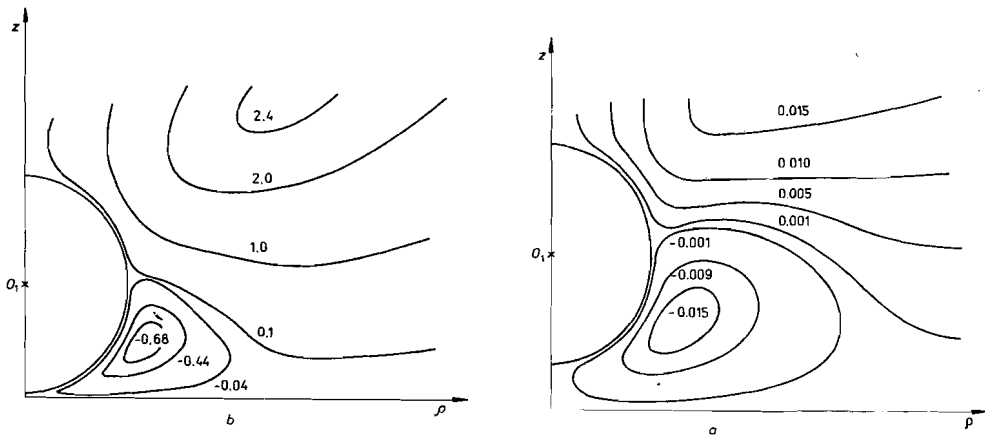


Fig. 4. Steady part of secondary flow $\Psi/\varepsilon^2 a^3 \omega$ for large value of radii ratio $\lambda=100$ and two different values of eccentricity: $\tau=1.0$ (a) and $\tau=0.2$ (b)

Computations are run with a couple of different pairs of K , N in order to check the accuracy of procedure which is responsible for tracing the equilines (in this case — stream lines). It turned out that $N = 41$ and $K = 51$ is the optimal mesh size, because the employment of more fine meshes does not improve significantly the results and only raises the magnitude of computational time.

So far the technical details of the numerical algorithm are described.

The first result to be obtained is for the case of large focal distance σ and fixed radius ratio λ , which according to (4.6) corresponds to small eccentricity. The calculated here $\max[\bar{\Psi}]$ is compared with the respective results of Tabakova and Zapryanov [18] for $\lambda = 1.5$ and $\lambda = 2.5$. For $c = 100$ our results compare with their computations up to the fourth digid. We have reasonable good comparison within 2% accuracy even for $\sigma = 10$. Having in the mind that in the previous section we mentioned the case of large c (or σ), as the hardest to be treated by the present method, one can see the agreement with the concentric case as a certificate for the good performance of the method described here. So that the results for moderate and small σ can be trusted.

In Fig. 2 are shown the dimensionless streamlines $\bar{\Psi}/\varepsilon a^3 \omega$ for moderate eccentricity $\sigma = 1$. It is seen that the two-vortex structure is preserved but the lower vortex decreases in size and increases its intensity. This tendency is apparent with further increase of eccentricity (Fig. 3) and virtually disappears for small σ and λ . It is not reasonable, however, to decrease more σ , because the gap between the spheres becomes so small that there is no place for the two boundary layers unless extremely high values of frequency parameter M are considered.

Another interesting limiting case is $\lambda \gg 1$ and $e \sim O(1)$ which approximately corresponds to the case of sphere oscillating near a plane. It is shown graphically in Fig. 4.

In [31] is observed that the steady streaming changes its direction of rotation in the boundary layer at the wall of oscillating spherical cell containing a fluid

drop. This effect is not observed in present work which we attribute to the fact that in our case the oscillating particle is rigid while in [31] the inner sphere is with tangentially mobile surface.

Concluding remarks

The oscillatory viscous flow which arises in the gap between two eccentric spheres when the inner one executes torsional oscillations is investigated by means of the method of matched asymptotic expansions. The problem depends on two parameters — dimensionless amplitude ϵ and frequency parameter M . The case of small amplitudes $\epsilon \ll 1$ and high frequencies $M \gg 1$ is considered when the Reynolds number of stationary part $Re_s = \epsilon^2 M^2$ is much lesser in comparison with unity. Approximate solution is obtained asymptotically correct within an order of approximation of $o(\epsilon)$, $o(M^{-1})$ for main rotational flow and for the secondary transversal motion. It turns out that in the core region the secondary flow is comprised exclusively of steady part within the adopted asymptotic order of approximation.

Formulae presenting the steady secondary motion are implemented numerically and results are obtained for various values of eccentricity and ratio of spheres radii. For small values of eccentricity the results are shown to compare very well quantitatively with the known solution for oscillatory flow between two concentric spheres.

Typical flow patterns are shown graphically for different eccentricities and radii ratios.

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