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THE METHOD OF VARIATIONAL IMBEDDING FOR ELLIPTIC INCORRECT PROBLEMS

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One of the most spoken of incorrect problems for partial differential equations is the Cauchy initial value problem for the Laplace equation. This example was brought up by Hadamard [1] as a typical one for the so-called inverse problems (see also [2]). As shown by Hadamard, the solution to an initial value problem for elliptic equations turns out to be unstable and hence cannot be obtained by means of approximate, e. g. numerical, methods and is solved only in the extremely rare cases when analytical solution can be found out. At the time the sought solution exists and is unique under the casual requirements for the smoothness of boundary and the functions that play the role of the initial condition. So the problem is how to devise a method for identification of the solution from the whole set of solutions to different Dirichlet problems. For the existing methods of approximate evaluation of solutions of incorrect problems see, e. g. [2].

A completely new method of solving incorrect problems is proposed in the present paper. Guided by the notion that the initial value problem is too 'stiff' to be succesfully treated we replace it with the minimization of a functional that is just the integral over the region under consideration of the square of the Laplace equation. A necessary condition for minimization of this functional is the Euler-Lagrange equation which in our case appears to be the biharmonic equation. A specific kind of boundary value problem is posed for the biharmonic equation which appears to possess a unique solution and this solution coincides with the solution of the original incorrect problem. In a sense we imbed the incorrect problem into a higher-order but correct boundary value problem derived from the minimization of a certain functional. So we call the new technique, method of variational imbedding' and it is a generalization to the case of partial differential equations of the method for identification of homoclinics and solutions proposed by the author [3]. In a sence the new method is akin to the method of quasi-reversibility [4] as far as a higher-order boundary value problem is considered but is free of the shortcomings of that method connected with the small parameter before the highest derivatives.

1. **Posing the problem.** Consider a region D in the two-dimensional Euclidean space with boundary Γ that is a closed curve. There are no principle difficulties to consider multidimensional problems but it is not done only for the sake of simplicity. Let also Γ_1 be a portion of Γ with non-zero lenght and Γ_2 be the reminder of Γ. Then for the Laplace equation

$$\Delta u=0 \tag{1}$$

we consider the following initial value problem discussed by Hadamard

(2) $$u=\varphi(x, y),\ \frac{\partial u}{\partial n}\Big|_{\Gamma}=\psi(x, y)\quad \text{for}\quad (x, y)\in\Gamma_1,$$

where n is the outward normal vector to Γ_1.

2. **Variational principle.** Informally speaking, the problem (1), (2) is too stiff to be solved directly, and we prefer to consider the more flexible minimization problem

(3) $$I=\iint_D(\Delta u)^2dxdy=\min,$$

where $u\in C^\infty(D)$ is to satisfy (2). The functional I (if it exists) is a quadratic and homogeneous function of its argument Δu and as so attains its minimal value only when $\Delta u=0$, i. e. there is one-to-one correspondence between the original equation (1) and the minimization problem (3).

A necessary condition for minimization of I is the Euler-Lagrange equation (see for details of derivation, e. g., [5]), which in our case appears to be the bi-harmonic equation

(4) $$\Delta\Delta u=0.$$

The last equation is of fourth order and generally requires two boundary conditions at each point of the boundary Γ. At the portion Γ_1 these two conditions are just eqs (2). On the reminder Γ_2 of Γ we couple the boundary conditions by the so-called natural conditions for a functional, namely

(5) $$\Delta u=0,\quad \frac{\partial}{\partial n}\Delta u=0,\quad (x, y)\in\Gamma_2.$$

Thus (4), (2), (5) form a coupled fourth-order boundary value problem for evaluating the function u which we hereafter call 'imbedding problem'. On the basis of considerations presented below one can easily show that it is an elliptic boundary value problem.

3. **Existing and uniqueness of solution.** Let us consider now the Hilbert space $H(D)$ comprised by the functions that satisfy the following boundary conditions

(6) $$a=\frac{\partial a}{\partial n}=0,\quad (x, y)\in\Gamma_1;\quad \Delta a=\frac{\partial}{\partial n}\Delta a=0,\quad (x, y)\in\Gamma_2.$$

We do not seek here the most general posing of the problem but rather concentrate on the main idea and hence we expect that the functions under consideration are as many time differentiable as it is required for correctness of the method.

The following scalar product is introduced in

(7) $$[a, b]=\iint_D\Delta a\Delta b\,dxdy$$

which is a scalar product indeed since the Hadamard problem $\Delta a=0$ with the first two of the boundary conditions (6) possesses only a trivial solution (see, e. g. [4]), i. e. $[a, a]=0$ is true only when $a=0$.

Let us introduce the sufficiently times differentiable function $\chi(x, y)$ satisfying the boundary conditions (2), i. e. it is a continuation of function $\varphi(x, y)$ in the interior of D with prescribed normal derivative $\psi(x, y)$ at the boundary. Then, a generalized solution of (4), (2), (5) is called any function u for which holds

(8) $$[u, \Phi]=\iint_D\Delta u\Delta\Phi\,dxdy=0,$$

where $\Phi\in H(D)$ and $u-\chi\in H(D)$. It is easily seen that the classical solution of (4) (2), (5) is also a generalized solution since

(9) $$0=\iint_D\Phi\Delta\Delta u\,dxdy=\iint\nabla\cdot(\Phi\nabla u)dxdy-\iint\nabla\Phi\cdot\nabla\Delta u\,dxdy$$

$$=\int_{\Gamma}\Phi(\nabla\Delta u).n ds-\iint\nabla.[\Delta u\nabla\Phi]\,dxdy+\iint\Delta u\Delta\Phi dxdy$$

$$=-\int_{\Gamma}(\Delta u)\frac{\partial\Phi}{\partial n}ds+\iint\Delta u\Delta\Phi dxdy=\iint\Delta u\Delta\Phi dxdy,$$

where the second pair (5) of boundary conditions for u and the conditions (6) that must satisfy Φ as a member of H are acknowledged.

The existence of a generalized solution follows directly from the Riesz Theorem because, as has been shown above, (7) defines a scalar product and therefore a functional. In order to prove the uniqueness we consider the difference $v=u_1-u_2$ between two supposed solutions u_1 and u_2. It is obvious that $v\in H(D)$. On the other hand eq. (8) holds also for v. Then taking simply $\Phi\equiv v$ we have

(10) $$[v, v]=0 \text{ and then } v\equiv 0.$$

4. **The essence of imbedding.** It has so far been shown that the Euler-Lagrange equation possesses a unique solution under the boundary conditions (2), (5). It remains only to check whether this solution is also a solution to the original problem (1), (2).

Obviously, the problem (4), (5) is once again a Hadamard problem with trivial initial conditions at Γ_2 but for the unknown $w=\Delta u$. That problem possesses a unique trivial solution and hence in the interior of D as well as on the Γ_2 holds (1). The imbedding solution satisfies also (2). As a result the solution of the problem (1), (2) is obtained which is the gist of the method proposed.

5. **Numerical implementation.** Boundary value problem (4), (2), (5) is inconvenient for numerical treatment due to the non-local character of the second pair of boundary conditions (5). The difficulties, however are only apparent and still a correct difference scheme can be constructed by means of the splitting type fractional-step method provided a time derivative with respect to certain fictitious time t is added in (4). Then, generalizing the scheme of Douglas (called recently 'stabilyzing correction' [6]), we have

(11a) $$\frac{\tilde{u}-u^n}{\tau}=-\Lambda_{11}u^n-2\Lambda_{12}u^n-\Lambda_{22}\tilde{u};$$

(11b) $$\frac{u^{n+1}-\tilde{u}}{\tau}=-\Lambda_{11}(u^{n+1}-u^n),$$

where τ is the fictitious time increment, and Λ_{11}, Λ_{12}, Λ_{22} are difference approximations of the partial derivatives $\partial^4/\partial x^4$, $\partial^4/\partial x^2\partial y^2$, $\partial^4/\partial y^4$, respectively.

The first pair of boundary conditions (2) remain as they are, but the problem with the second pair (5) is more intricate. The first of them is split in a standard manner

(12a) $$\frac{\tilde{n}-u^n}{\tau}=\Lambda_1u^n+\Lambda_2\tilde{u}, \quad (x, y)\in\Gamma_2;$$

(12b) $$\frac{n^{n+1}-\tilde{u}}{\tau}=\Lambda_1(u^{n+1}-u^n), \quad (x, y)\in\Gamma_2,$$

where Λ_1 and Λ_2 stand for the difference approximations of $\partial^2/\partial x^2$, $\partial^2/\partial y^2$. The second of conditions (5) can be split in time in different ways. The most convenient is, perhaps, the incorporation of the mixed derivative $\partial^2 u/\partial t\partial x$ instead of the simple time derivative as it is done in (12). Then

(13a) $$\bar{\delta}_n\left(\frac{\tilde{u}-u^n}{\tau}\right)=\bar{\delta}_n(\Lambda_2\tilde{u}+\Lambda_1u^n), \quad (x, y)\in\Gamma_2;$$

(13b) $$\bar{\delta}_n\left(\frac{u^{n+1}-u^n}{\tau}\right)=\bar{\delta}_n[\Lambda_1(u^{n+1}-u^n)], \quad (x, y)\in\Gamma_2,$$

where $\overline{\delta}_n$ denotes a finite-difference operator that approximates the normal deriva tive $\partial/\partial n$.

At each time stage the eqs (11a) with the boundary conditions (2), (12a), (13a are satisfied and then — (11b) with (2), (12b), (13b). Each of these systems is we posed as a difference approximation of the respective one-dimensional problem. F nally, it is easily shown that excluding the half-time step yields a scheme with fu approximation which is stable and correct because the operators — Λ_{11}, $-\Lambda_{22}$, Λ_1, Λ are negative ones. Computations are conducted until convergence is attained, i. until the norm of the difference between two consecutive time steps becomes smalle than certain a priori described small value.

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REFERENCES

[1] Hadamard, J. Le probleme de Cauchy et les equations aux derivees partielles lineares hype boliques. Hermann, Paris, 1932. [2] Тихонов, А. Н., В. Я. Арсенин. Методы решения некорректны задач. Наука, Москва, 1974. [3] Christov, C. I. Proc. 14 th Conf. Union of Mathematicans in Bulgari Sunny Beach, 6-9. IV. 1985, 571-577. [4] Lattés, R., J.-L. Lions. Méthode de quasi-reversibilité applications. Dunod, Paris, 1967. [5] Цлафф, Л. Я. Вариационное исчисление и интегральные ура нения, Справочник. Наука, Москва, 1966. [6] Яненко, Н. Н. Метод дробных шагов решения матем тической физики. Наука, Новосибирск, 1967.