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ON THE METHOD OF VECTOR AND SCALAR POTENTIALS FOR FLOWS OF INHOMOGENEOUS INVISCID LIQUIDS

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One of the principle difficulties on the way of numerical treatment of the flows in which the density is not a function of the pressure appears to be the implicit estimation of the pressure in a way satisfying the continuity equation. The most popular in two dimensions is, perhaps, the vorticity/stream function method. Unfortunately, generalizing this method for three-dimensional flows of incompressible liquids increases the number of dynamic equations to six and this apparently is one of the reasons why the method of vector potential is not enjoying much attention. The second reason is that there is a problem with coupling the boundary conditions for vector potential, especially in the case of inviscid liquids.

The present paper deals with the problem of boundary conditions for the method of vector and scalar potentials for flows of inviscid inhomogeneous fluids. A curvilinear coordinate system is considered whose coordinate lines are normal to the boundaries of the region of the flow, but are generally non-orthogonal in the interior. Two-dimensional boundary value problems for the components of the vector potential are separated at the boundary in terms of these coordinates and are shown to be correct. The solutions of these two-dimensional problems provide the boundary conditions for the three-dimensional problem under consideration. Two kinds of boundary conditions for the scalar potential are discussed: of Dirichlet and Neumann type. It is argued that the first one is preferable in numerical implementation. In author's opinion the simplicity and numerical amenability of the boundary value problem proposed fully compensate for the increase of governing equations to seven in comparison with the original four ones.

1. Governing Equations. Consider an inviscid inhomogeneous liquid whose motion is governed by the Euler equations

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla(\rho\mathbf{v}) = -\nabla p + \mathbf{F}, \tag{1}$$

where $\mathbf{v}$ is velocity vector, p — pressure, ρ — density, $\mathbf{F}$ — applied body forces. Among the latter can be indicated the gravity force, the Coriolis force, etc.

Eqs (1) are coupled by the continuity equation

$$\nabla\cdot(\rho\mathbf{v}) = \Phi, \tag{2}$$

where Φ stands for the total source at given point. It takes into account not only the source due to external reasons, but also the time derivative of the density, the latter being thought of as a known function of the spatial coordinates and time. Such a situation is present in all kinds of natural convection problems where the density is a function of the temperature governed by separate equation.

Let us denote by D the region occupied by the flow and by Γ — the boundary of D. Eqs (1), (2) are valid in the interior D, while at the boundary the well known (see e. g. [1]) conditions for ideal flows, namely

(3) $$\mathbf{v}\cdot\mathbf{n}|_{x\in\Gamma}=v_n(\mathbf{x},\ t),$$

have to be satisfied where $\mathbf{n}$ is the outward normal vector to Γ at specific point $\mathbf{x}$. Respectively, v_n is a given function of the point and time t. The above stated problem is correct iff the following relation holds:

(4) $$\iiint_D \Phi dxdydz=\oiint_\Gamma \mathbf{v}_n ds.$$

The difficulties connected with the boundary value problem (1)-(3) are now well known; they are based on the fact that the pressure p is implicitly defined through the continuity equation (2).

2. Vector and Scalar Potentials. One of the ways to exclude the pressure from the equations is to introduce a vector potential and vorticity function (see [2] and the literature cited there). The main difference between the inhomogeneous liquid and the homogeneous incompressible one is the presence of the source term Φ in the continuity equation. This means that it is no longer enough to introduce a vector potential, but rather both vector and scalar potentials should be employed according to the scheme

(5) $$\rho\mathbf{v}=\nabla\times\mathbf{A}+\nabla\varphi,$$

where $\mathbf{A}$ is called 'vector potential' and ρ — 'scalar potential'.

It is obvious that there exists certain a degree of freedom in substitution (5) since the three quantities $\rho\mathbf{v}$ are replaced by the four quantities $\mathbf{A}$, φ. Then, one is free to postulate one more relation between $\mathbf{A}$, φ in order to get one-to-one transformation. As usual, we choose the following relation

(6) $$\nabla\cdot\mathbf{A}=0.$$

Introducing further the 'dynamic' vorticity function $\zeta=\Delta\times(\rho\mathbf{v})$ we arrive to the expression

(7) $$\zeta=\nabla\times(\nabla\times\mathbf{A})=-\Delta\mathbf{A}+\nabla(\nabla\cdot\mathbf{A})=-\Delta\mathbf{A}.$$

Respectively, the equations for the components of the dynamic vorticity function are obtained after applying the operator rot to eq. (1)

(8) $$\frac{\partial\zeta}{\partial t}+(\mathbf{v}\cdot\nabla)\zeta+\zeta(\nabla\cdot\mathbf{v})-(\zeta\cdot\nabla)\mathbf{v}+\nabla\frac{v^2}{2}\times\nabla\rho+\nabla\times[(\mathbf{v}\times\nabla\rho)\times\mathbf{v}]=\nabla\times\mathbf{F}.$$

For the scalar potential a Poisson equation is easily derived

(9) $$\Delta\varphi=\Phi,$$

and hence (5), (7), (8) and (9) form a coupled set of equations for estimating the unknowns $\mathbf{v}$, $\mathbf{A}$, φ. In fact, the only unknowns are $\mathbf{A}$, φ, since velocity $\mathbf{v}$ is related to these through the explicit formula (5).

3. Coordinate System. Let us now assume that in the region D curvilinear coordinates (q^1, q^2, q^3) are introduced and that the transformed region D' appears to be the unit cube. For the sake of simplicity of the sought boundary value problem we assume that the coordinate lines which cross the walls of the unite cube are orthogonal to the latter. This does not mean at all that the coordinate system is orthogonal in the interior or that the lines which coincide with the cube walls are orthogonal two-dimensional coordinate systems. It is obvious that our assumption is not very restrictive and it is easy to construct the required coordinate system, e. g. by means of the method proposed of Coons (see [3]) and named later 'transfinite interpolation'. Then, the differential operators adopt the form (see [4]):

(10)
$$\nabla\cdot\mathbf{A}=\frac{1}{\sqrt{|g|}}\frac{\partial}{\partial q^i}(\sqrt{|g|}A^i),$$

and

(11)
$$\nabla\times\mathbf{A}=\frac{1}{|g|}\frac{DA_j}{\partial q^i}e^{ijk}=\frac{1}{|g|}\frac{DA^j}{\partial q^i}e^{ik}_{j},$$

where g^{ij} are the contravariant components of fundamental tensor, g — its determinant and e^{ijk}, e^{ik}_{j} are the respective Levi-Chivita symbols.

4. Boundary Conditions for the Vector Potential. Let us temporarily leave apart the boundary value problem for scalar potential and consider the latter as a prescribed quantity. Then the condition (3) in conjunction with (5) yield

(13)
$$\mathbf{n}\cdot(\nabla\times\mathbf{A})|_\Gamma=\rho v_n-\frac{\partial\varphi}{\partial n}|_\Gamma.$$

This is the first condition on the vector potential. The second one stems from (6). The latter must be satisfied at the boundary in order to be valid in the entire region D'.

It is obvious that one more condition is needed in order to couple the boundary value problem for $\mathbf{A}$. This condition is fully at our disposal. Let us consider now for definiteness the boundaries of the unit cube D', at which $q^1=\text{const}$. At these two boundaries we choose the additional condition to be

(14)
$$\frac{\partial}{\partial q^1}(\sqrt{|g|}A^1)=0.$$

Then eq. (6) reduces to

(15)
$$\frac{\partial}{\partial q^2}(\sqrt{|g|}A^2)+\frac{\partial}{\partial q^3}(\sqrt{|g|}A^3)=0.$$

As the q^1-coordinate lines are normal to the considered boundaries, the norma vector to them can be represented in the local basis as follows

(16)
$$\mathbf{n}=(\pm 1,\ 0,\ 0).$$

where the sign indicates whether the direction of the q^1-line coinsides with outward or inward direction. Then

(17)
$$\mathbf{n}\cdot(\nabla\times\mathbf{A})|_\Gamma=\frac{\pm g_{11}}{2\sqrt{|g|}}\left(\frac{\partial A^3}{\partial q^2}-\frac{\partial A^2}{\partial q^3}\right).$$

Now we introduce a "stream function of the vector potential" (potential of the potential) which allows us to satisfy (15), namely

(18)
$$\sqrt{|g|}A^2=\frac{\partial\psi^1}{\partial q^3},\quad \sqrt{|g|}A^3=-\frac{\partial\psi^1}{\partial q^2},$$

and then (17) yields

(19)
$$\frac{\partial}{\partial q^2}\frac{1}{\sqrt{|g|}}\frac{\partial\psi^1}{\partial q^2}+\frac{\partial}{\partial q^3}\frac{1}{\sqrt{|g|}}\frac{\partial\psi^1}{\partial q^3}=\mp\frac{2\sqrt{|g|}}{g_{11}}\left(\rho v_n-\frac{\partial\varphi}{\partial q^1}\right)_{q^1=\text{const}},$$

which is an elliptic equation for function $\psi^1(q^2, q^3)$. The boundary conditions for this function can be selected arbitrarily and we choose them to be the homogeneous one

(20)
$$\psi^1=0 \text{ for } q^2=\text{const},\quad q^3=\text{const},$$

and therefore we obtain a well posed Dirichlet problems for ψ^1. After (19) and (20) ar solved, the components A^2 and A^3 are extracted from (18) and together with (14) ar used as boundary conditions for the vector potential at the boundary $q^1=\text{const}$. Th calculations are repeated in the same manner for the other boundaries where $q^2=\text{cons}$ or $q^3=\text{const}$.

5. Boundary Value Problem for the Scalar Potential. As far as the eq. (13) holds, on can state a Neumann-type boundary value problem for the scalar potential

$$\left.\frac{\partial\varphi}{\partial n}\right|_{\Gamma}=\rho v_n. \tag{21}$$

In this case the respective two-dimensional problems for ψ^i are homogeneous ones an possess only trivial solutions. Therefore the boundary conditions for the vector poter tial adopt the simplest form

$$\frac{\partial}{\partial q^1}(\sqrt{|g|}\,A^1)=0, \quad A^2=A^3=0 \text{ at } q^1=\text{const}. \tag{22}$$

At the rest of the unit cube walls, the conditions are obtained by means of permut tion of indeces in (22).

Though apparently simple, the boundary value problem (21), (22) happens to b inconvenient for numerical treatment because of the well acknowledged difficulties con nected with the algorithms for difference implementation of the condition (4). For thi reason we prefer a different way that makes use of a Dirichlet problem for the scal potential

$$\varphi=0 \quad \text{for} \quad \mathbf{x}\in\Gamma, \tag{23}$$

and the partial derivative of φ in (19) is thought of as a known magnitude. The la approach enables one to avoid the capricious Neumann problem and appears to b novel. In addition, the proposed boundary value problem is more natural than the pr vious one since for homogeneous liquids one has that $\Phi\equiv 0$ and then (23) yields a tr vial solution for scalar the potential $\varphi\equiv 0$ and the number of unknowns is reduce only to the three components of vector potential.

In the end it should be noted that (8) is solved by means of a quasi-Lagrangia method and boundary conditions are needed for ζ only at those portions of the bou dary at which the normal component of velocity is pointed inward.

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