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THE METHOD OF VARIATIONAL IMBEDDING FOR PARABOLIC INCORRECT PROBLEMS OF COEFFICIENT IDENTIFICATION

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The problems of heat fields identification have attracted broad attention in the recent few decades (see [1]) due to their outstanding practical importance. The inverse problems for heat conduction may be roughly separated into three principle classes. First of them is the problem of identification of unknown heat conductivity coefficient from data concerning the temperature field; the second one is the identification of the thermal regimes on the outer boundary from the data concerning thermal quantities on the inner boundary or the problem of continuation; the third one is the reversed-time problem for identification of the initial distribution of the temperature from the data about the final-stage temperature distribution. There are a great number of mixed problems that belong to more than one of the listed classes, but it goes beyond the scope of the present short note to give a detailed classification of the inverse problems for the heat conduction equation. The first problem appears to be the most thoroughly studied both experimentally and theoretically, but still the solutions available are approximate ones in their very nature, not to speak of the truncation errors of the numerical implementation. Here we try an application of the method of variational imbedding proposed by the author for elliptic incorrect problems [2] and for identification of solutions [3]. The method proposed is displayed on the one-dimensional identification problem in which the sought coefficient of heat conductivity is a function of the spatial coordinate while the data concern the time evolution of the temperature and thermal fluxes at the boundaries. The method is readily generalized to multi-dimensional problems.

1. Posing the Problem. Consider the one-dimensional equation of heat conductivity

$$Au \equiv -\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left[\lambda(x)\frac{\partial u}{\partial x}\right] = 0, \quad 0<x<l, \quad 0<t<T, \quad \lambda>0, \tag{1}$$

with the initial condition

$$u\,|_{t=0} = u_0(x) \tag{2}$$

and boundary conditions

$$u(t,\,0)=f(t), \quad u(t,\,l)=g(t); \quad f(0)=u_0(0), \quad g(0)=u_0(l). \tag{3}$$

It is well known that the initial-boundary value problem (1)—(3) is correctly posed for evaluating the temperature $u(t,\,x)$ provided that the coefficient $\lambda(x)$ is a priori defined.

Let us now assume that $\lambda(x)$ is an unknown function and is to be estimated from the measurements of the heat flux at the boundaries

$$(4)\qquad \lambda(0)\frac{\partial u}{\partial x}\Big|_{x=0}=\psi(t),\quad \lambda(l)\frac{\partial u}{\partial x}\Big|_{x=l}=\varphi(t),$$

where ψ, φ are given functions and are sufficiently times differentiable.

The problem (1)—(4) is inversed incorrect problem for implicit evaluation of the coefficient $\lambda(x)$. The incorrectness of the problem stems from the fact that there is no explicit equation for the coefficient and the number of boundary conditions exceeds the required two conditions.

2. Variational Principle. We replace the original problem by the problem of minimization of the following functional

$$(5)\qquad I=\int_0^T\int_0^l\left[-\frac{\partial u}{\partial t}+\lambda\frac{\partial^2 u}{\partial x^2}+\frac{d\lambda}{dx}\,\frac{\partial u}{\partial x}\right]^2 dxdt=\min,$$

where u must satisfy the conditions (2)—(4). Functional I is a quadratic and homogeneous function of its argument Au and therefore attains its minimum only when $Au\equiv 0$ i. e. there is one-to-one correspondence between the original equation (1) and the minimization problem (5).

A necessary condition for minimization of I is the Euler-Lagrange equation (see, e. g. [4]) which reads

$$(6)\qquad -\frac{\partial^2 u}{\partial t^2}+\frac{\partial}{\partial x}\lambda\frac{\partial^2}{\partial x^2}\lambda\frac{\partial u}{\partial x}=0.$$

This equation is of second order with respect to time and requires along with (2) a boundary condition for $t=T$. We impose at this temporal boundary the natural condition that appears to be simply the original equation (1)

$$(7)\qquad -\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\lambda\frac{\partial u}{dx}=0\quad \text{at } t=T.$$

In order to couple the problem one needs to derive also the Euler-Lagrange equation for $\lambda(x)$, namely

$$(8)\qquad -\frac{\partial}{\partial x}\left[\lambda\int_0^T u_x u_{xx}dt+\lambda_x\int_0^T u_x^2 dt-\int_0^T u_t u_x dt\right]$$

$$+\lambda\int_0^T u_{xx}^2\,dt+\lambda_x\int_0^T u_x u_{xx}\,dt-\int_0^T u_t u_{xx}dt=0,$$

coupled with the respective natural boundary conditions

$$(9)\qquad \lambda\int_0^T u_x u_{xx}dt+\lambda_x\int_0^T u_x^2\,dt=\int_0^T u_t u_x dt,\quad x=0,\,l.$$

Though the form (8) of the equation for $\lambda(x)$ is convenient for the general considerations it is still helpful to have a more concise form, especially for the numerical implementation:

$$(10)\qquad \frac{d}{dx}F(x)\frac{d\lambda}{dx}-\lambda\int_0^T u_x u_{xxx}dt=\int_0^T u_{tx}u_x dt,\quad F(x)=\int_0^T u_x^2\,dt.$$

3. Existence and Uniqueness of the Solution. Let us consider now the Hilbert space $H(D)$ of functions defined in the region D: $\{(0\leqq x\leqq l)\times(0\leqq t\leqq T)\}$ and satisfying the boundary conditions

$$(11)\qquad a(0,\,x)=0,\quad \left[-\frac{\partial a}{\partial t}+\frac{\partial}{\partial x}\lambda\frac{\partial a}{\partial x}\right]_{t=T}=0,$$

$$(12)\qquad a(t,\,0)=a(t,\,l)=\lambda(0)\frac{\partial a}{\partial x}\Big|_{x=0}=\lambda(l)\frac{\partial a}{\partial x}\Big|_{x=l}=0,$$

where function $\lambda(x)$ is sufficiently times differentiable in the entire region. We do not seek for the most general posing of the problem but rather we are bound to display the method and hence we assume all of the required properties of the functions from H.

The following scalar product is introduced in H:

$$[a, b]=\int_0^T \int_0^l \left[\frac{\partial a}{\partial t}\frac{\partial b}{\partial t}+\left(\frac{\partial}{\partial x}\lambda\frac{\partial a}{\partial x}\right)\left(\frac{\partial}{\partial x}\lambda\frac{\partial b}{\partial x}\right)\right] dxdt+\int_0^l \lambda\left(\frac{\partial a}{\partial x}\frac{\partial b}{\partial x}\right)_{t=T} dx \tag{13}$$

which is a scalar product since $[a, a]=0$ is possible only when $a\equiv 0$. The last assertion is easily proved since $[a, a]$ is a quadratic functional that adopts the trivial value only when each term in it is equal to zero. The latter, together with the boundary conditions yield the said property.

Consider now the sufficiently times differentiable function $\chi(t, x)$ that satisfies the boundary conditions (2), (3), (4), (7), (8), i. e. it is a continuation of functions $f(t)$ and $g(t)$ in the interior of D with prescribed values of the normal derivatives according to fluxes $\psi(t)$ and $\varphi(t)$. Then a generalized solution to (2)—(4), (6), (7) is called any function for which holds

$$[u, \Phi]=0 \tag{14}$$

with $\Phi\in H(D)$ and $u-\chi\in H(D)$. It is easily seen that the classical solution to the problem (if it exists) is also a generalized one since

$$0=\int_0^T\int_0^l \Phi\left(-\frac{\partial^2 u}{\partial t^2}+\frac{\partial}{\partial x}\lambda\frac{\partial^2}{\partial x^2}\lambda\frac{\partial u}{\partial x}\right)dtdx$$

$$=\int_0^T\int_0^l\left[\frac{\partial u}{\partial t}\frac{\partial \Phi}{\partial t}+\left(\frac{\partial}{\partial x}\lambda\frac{\partial u}{\partial x}\right)\left(\frac{\partial}{\partial x}\lambda\frac{\partial \Phi}{\partial x}\right)\right]dxdt+\int_0^l\lambda\left(\frac{\partial u}{\partial x}\frac{\partial \Phi}{\partial x}\right)_{t=T}dx=[u, \Phi].$$

The last equality is derived after acknowledging the conditions (10), (11) that are to be satisfied by Φ as a member of $H(D)$.

The existence of the generalized solution follows directly from Riesz theorem because (12) defines a scalar product and therefore a functional. In order to prove the uniqueness we consider the difference $v=u_1-u_2$ between two supposed solutions u_1 and u_2. It is obvious that $v\in H(D)$. On the other hand (13) holds for v too. Then simply taking $\Phi\equiv v$ we arrive at $[v, v]=0$ and therefore $v\equiv 0$. It should be reminded here that the results are valid for arbitrary sufficiently smooth function $\lambda(x)$. Turning to the problem of evaluating $\lambda(x)$ we consider the Hilbert space $\bar{H}[0, l]$ of functions $\alpha(x)$ that satisfy the conditions

$$\alpha\int_0^T u_x u_{xx}\,dt+\frac{d\alpha}{dx}\int_0^T u_x^2\,dt=0, \tag{15}$$

with scalar product defined as

$$[\alpha, \beta]=\int_0^T\int_0^l(\alpha_x u_x+\alpha u_{xx})(\beta_x u_x+\beta u_{xx})dtdx. \tag{16}$$

It is seen that $[\alpha, \alpha]=0$ yields $\alpha\equiv 0$, since then we have $\alpha_x u_x+\alpha u_{xx}=0$ where u is a function also of time t and the equality must be satisfied for each value of t.

Generalized solution of (8) is called the function λ that satisfies

$$[\lambda, \bar{\Phi}]=\int_0^l\left[\bar{\Phi}_x\int_0^T u_t u_x\,dt+\bar{\Phi}\int_0^T u_t u_{xx}\,dt\right]dx=0, \tag{17}$$

where $\bar{\Phi}\in\bar{H}$ and $\lambda-\mu\in\bar{H}$. Here μ is a function which is a continuation of both boundary conditions (9). Repeating the reasoning concerning the existence and uniqueness of the solution for u we obtain the existence and the uniqueness of the solution for λ.

4. The Essence of Imbedding. Up to this point we have shown that the two Euler-Lagrange equations (6) and (8) for u and λ possess unique solutions provided that in each of them the other function is thought of as known. This yields that there exists a unique solution to the system as a whole. The latter means that the functional I possesses only one stationary point since the Euler-Lagrange equations are necessary conditions for existence of a stationary point of a functional. On the other hand the functional I adopts its minimal value because it is a convex function. Hence, the only stationary point of the functional is its minimum. It remains only to show that this minimum is equal to zero. Unlike the incorrect problems with prescribed coefficients (see [2]) the functional of the present work is a square of a bi-linear function rather than of a linear and hence it is not obvious that its only minimum is the trivial value. The latter, however, is easily shown by multiplying (6) by u, (8) by λ and integrating over the respective domains provided that the boundary conditions for the said functions are taken into account. Hence the solution to the imbedding problem coincides with the solution of the original inverse problem which is the gist of the method of variational imbedding.

5. Numerical Implementation. The boundary value problem (2)—(4), (6), (7) is elliptic and can be solved by means of convergence method similar to that described in [2] provided that a fictitious time is incorporated. The boundary conditions do not pose any difficulties and only (7) needs some special care to be splitted too. The ordinary differential equation for λ if solved by a standard method for two-points boundary value problems.

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