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STOCHASTIC REGIME FOR KURAMOTO–SIVASHINSKY EQUATION PART I — RANDOM POINT APPROXIMATION

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Христо И. Христов. СТОХАСТИЧЕСКИЙ РЕЖИМ ДЛЯ УРАВНЕНИЯ КУРАМОТО-СИВАШИНСКОГО. ЧАСТЬ I — СЛУЧАЙНОТОЧЕЧНАЯ АПРОКСИМАЦИЯ

Рассматривается метод аппроксимации точечными случайными функциями для моделирования стохастического поведения нелинейных систем в применении к эволюционному уравнению четвертого порядка по пространственной переменной, получаемому в слабонелинейном приближении для формы свободной поверхности стекающей пленки (называемому некоторыми авторами уравнением Курамото–Сивашинского). Точечная случайная функция, для которой предполагается, что лучше всего аппроксимирует решение, представляется в виде интеграла Винера от случайной функции плотности, и о ядре этого интеграла выводится замкнутое уравнение, содержащее двух и трех-точечные плотности вероятностей для системы точек, порождающей случайную функцию плотности. Результаты получены только для Пуассоновского частного случая, когда плотности распределения вероятностей точек равны единице и соответственно уравнение для ядра совпадает с исходным уравнением, но уже разыскивается решение, которое принадлежит пространству $L^2(-\infty, \infty)$, т. е. оно имеет форму уединенной волны (солитона). Солитонное решение для уравнения К–С найдено численно двумя разными способами в предыдущих работах автора, и на его основе в настоящей работе рассчитаны статистические характеристики Пуассоновской точечной случайной функции и сравнены качественно с существующими в литературе данными по прямому численному расчету. Поскольку эти данные недостаточны, во второй части настоящей работы проведен независимо численный эксперимент, и хорошее согласие с предсказаниями предлагаемого метода аппроксимации точечной случайной функцией является еще одним подтверждением его эффективности.

Christo I. Christov. STOCHASTIC REGIMÉ FOR KURAMOTO-SIVASHINSKY EQUATION. PART I — RANDOM POINT APPROXIMATION

The application of the random point approximation for modelling nonlinear stochastic systems is outlined for the case of fourth-order with respect to the spatial coordinate nonlinear equation of evolution presenting the weakly nonlinear approximation for the shape of the surface of falling thin viscous films (called by some authors Kuramoto–Sivashinsky equation). The expected to be a solution random point function is represented in the form of Wiener integral with random density function as a basis function and a closed equation for the kernel of the latter is derived containing the two- and three-point probability densities of the generating system of random points. Only the Poisson limiting case is considered when these probability densities are (trivially) equal to unity and therefore the equation for kernel reduces to the original one, but the solution is to be sought in the $L^2(-\infty, \infty)$ space, i. e. it is a solitary wave (soliton). The soliton solution of K–S equation is obtained, e. g. in the previous authors' works and by their means the statistical characteristics of the Poisson random point solution are calculated and compared qualitatively with the existing in the literature data from direct numerical simulation. As far as these data are unsufficient a thorough numerical simulation is performed in the part II of present paper and the good comparison speaks strongly in favour of the proposed here random point approximation.

The stochastic behaviour of deterministic nonlinear systems is one of the central problems in contemporary physics and mechanics. There exists a number of approximate approaches (see [1] for review) called hierarchy techniques, but their efficiency for qualitative prediction is rather modest and they serve well only as perturbation techniques in weakly nonlinear situations. Recently, a new approximate approach has been developed [2] which can be named properly enough "random point approximation". It is based on the notion that most of the physical random fields either are or can be well approximated by random point functions. Then the solutions sought are developed into functional series (called Volterra–Wiener series) with random point basis function and a closed hierarchical system for the kernels of functionals is derived. The method of functional series has proved very efficient for media with prescribed stochasticity (multiphase continia) [3, 4] for which the mathematical model consists of partial differential equations with random coefficients of random point nature. In this case the functional series turns out to be virial and an asymptotically rigorous procedure for truncation of the hierarchy for kernels can be devised.

The situation is much more complicated for the case of nonlinear systems with deterministic coefficients which go random in the wake of instability. For such systems the solution still can be developed into functional series with random point basis function, but the question of whether the first or two functionals provide a sufficient approximation remains open. Reducing here the series to the lower-order functionals is only a heuristic procedure. Nevertheless it seems important to check whether the mere approximation with random point function (only first-order functional retained) can provide an acceptable approximation for the random solution. Even this is a formidable task since it requires solving bifurcation problems in unbounded regions which problems need a special treatment [5, 6]. All the more, some encouraging results have already been obtained for systems modelled by: ordinary differential equations (Lorenz system [7]); by partial differential equations (Burgers equation [8]); by PDE of two spatial dimensions (stochastic Poiseuille problem [9]). In all these cases for the first kernels it has been obtained the respective solutions and on their base the statistical characteristics of the stochastic regimés have been calculated. It turns out that the predictions for the average characteristics

and for the correlation functions with moderate distances between the points are in very good quantitative or qualitative agreement with the respective nature or numerical experiments. In this connection it is important to enlarge the number of systems treated by the technique of random-point approximation in order to verify its features that are concerned with the intrinsic behaviour of the nonlinear unstable systems. In the present paper one more nonlinear system described by a PDE is considered. That is the equation governing in the frame of the weakly nonlinear approximation the shape of the free surface of thin viscous capillary film falling down a vertical plane. In the first part of the paper a random point solution is obtained and compared in the second part [10] with the results from a direct numerical simulation conducted there. The new numerical experiment is necessary since the output of the only known one from the literature [11] is insufficient for decisive quantitative comparisons.

1. KURAMOTO–SIVASHINSKY EQUATION

Consider a thin viscous capillary film falling down a vertical plane (Fig. 1). The first comprehensive experimental investigation of the flow regimés was conducted by Kapitza [12] who discovered that the plane shape of the surface exists only for very small Reynolds numbers Re. For small Re on the surface occur harmonic waves [13] and for $\mathrm{Re} \sim O(1)$ the periodic regimé is confined only to the inlet region while downflow the free surface adopts a random shape. This means that the falling film presents a system of distributed parameters that exhibits a random behaviour for certain values of governing parameters and can be employed as featuring example for different stochastic models.

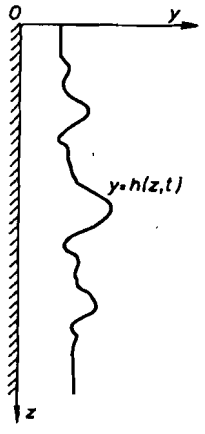

Fig. 1. Flow geometry

Assume that the flow is two dimensional and also that nothing depends on the spanwise coordinate x. The equations of motion and continuity can be integrated to obtain the following system (see, e.g. [14] for details) for longitudinal component v and shape function h

$$\frac{\partial}{\partial t}\int_0^{h(z,t)} v(y,z,t)\,dy + \frac{\partial}{\partial z}\int_0^{h(z,t)} v^2(y,z,t)\,dy = \frac{\sigma h}{\rho}\frac{\partial^3 h}{\partial z^3} - \nu\frac{\partial v}{\partial y}\Big|_{y=0} + gh, \tag{1.1}$$

$$\frac{\partial h}{\partial t} = -\frac{\partial}{\partial z}\int_0^{h(z,t)} v(y,z,t)\,dy. \tag{1.2}$$

Here has already been implicitly assumed that the waves are not steep (long-wave approximation) and the term representing the curvature (the third derivative of h in (1.1)) has tacitly been linearized. Under the provisions of long-wave approximation the Kapitza assumption for the velocity profile in the bulk phase is fully adequate

$$v(y,z,t) = 3\langle v\rangle\left(\frac{y}{h} - \frac{1}{2}\frac{y^2}{h^2}\right), \tag{1.3}$$

where $\langle v \rangle$ is the mean-profile velocity. Substituting (1.3) into (1.1) and (1.2) and applying the simplifications of the so-called weakly nonlinear approximation (see [15], [16], [17]) one arrives to the following equation for the relative thickness $H = (h - h_0)/h_0$ (h_0 — the equilibrium thickness):

$$\frac{\partial H}{\partial t} + 3\frac{\partial H}{\partial z} + 6H\frac{\partial H}{\partial z} + \frac{6}{5}\mathrm{Re}\frac{\partial^2 H}{\partial z^2} + \mathrm{We}\frac{\partial^4 H}{\partial z^4} = 0. \tag{1.4}$$

Here $\mathrm{Re} = u_0 h_0/\nu$, $\mathrm{We} = \sigma\left(\rho g h_0^2\right)^{-1}$, g — gravity acceleration and u_0 is the mean-profile velocity for a laminar film. It is convenient to introduce the following scaled variables (see, among others, [18]):

$$H = a\varphi,\ z = x/b,\ t = \tau/ab,\ a = \mathrm{We}b^3,\ b = \sqrt{6\mathrm{Re}/5\mathrm{We}}, \tag{1.5}$$

and then Eq. (1.4) adopts the form

$$\frac{\partial \varphi}{\partial \tau} + \frac{3}{a}\frac{\partial \varphi}{\partial x} + 6\varphi\frac{\partial \varphi}{\partial x} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^4 \varphi}{\partial x^4} = 0 \tag{1.6}$$

with a single dimensionless parameter a. The last equation is not quite properly called by some authors [11, 21, 22] Kuramoto–Sivashinsky equation in the wake of the works of Kuramoto [19] and Sivashinsky [20]. In the former an equivalent to (1.6) equation is proposed for modelling the propagating flame fronts. In the latter there exist no new fundamental ideas in comparisons with Roskes' [16] and Nepomnyashchii's [17] works.

As it is elucidated in [23, 24] the Eq. (1.6) is not capable of accurate enough quantitative prediction of the known experimental correlations of type "amplitude-celerity" of a wave. Still it is of great fundamental interest just in the same way as its famous counterparts Korteweg-de Vries and Burgers equations are. The K–S equation (1.6) is a nonlinear equation of evolution that contains nonlinearity, fourth-order dissipative term ($\partial^4\varphi/\partial x^4$) and second-order sourse term ("antidissipation" $\partial^2\varphi/\partial x^2$). Hence, the occurrence and the quantitative characteristics of the stochastic regimés for (1.6) are of primary interest for the theory of stochastic behaviour of deterministic nonlinear systems.

RANDOM-POINT APPROXIMATION

Consider the homogeneous random flux (random point function)

$$\varphi = \varphi_0 + \int_{-\infty}^{\infty}\left[\int_{V_1}^{V_2} K(x - \xi; V)\omega(\xi; V)\, dV\right] d\xi, \tag{2.1}$$

where φ_0 is not a function of the spatial variable x and ω is the random density function

$$\omega(x; V) = \sum_{\alpha}\delta(x - x_\alpha)\delta(V - V_\alpha), \tag{2.2}$$

generated by a set of marked random points $\{x_\alpha; V_\alpha\}$. The index α can formally be thought of as not necessarily denumerable. The most important trait of the

random density function ω is that the random points from the generating set can translate spatially by celerities

$$\frac{dx_\alpha}{dt} = V_\alpha, \tag{2.3}$$

i. e. (2.1) is a random train of solitons each of them being a stationary wave in a moving with velocity V coordinate frame. Interaction among the solitons is not taken into account. In [2] is shown that double, tripple, etc. interactions among the structures can be rigorously incorporated in the model by means of Volterra–Wiener functional series with ω in the role of basis function. When no higher-order Wiener functionals are considered and only the first-order one (namely (2.1)) is retained we shall speak of random point approximation.

Let us denote

$$\gamma = \langle \omega(\xi;\, V) \rangle, \tag{2.4}$$

which is in fact the mean number of points falling per unit length. Then (2.1) can be rewritten in the form

$$\varphi = \langle \varphi \rangle + \int\int K(x - \xi;\, V) C_\omega^{(1)}(\xi;\, V)\, dV d\xi, \tag{2.5}$$

where $\langle \varphi \rangle = \varphi_1 + \varphi_0$ and

$$\varphi_1 \equiv \gamma \int\int K(x - \xi;\, V) P(V)\, dV d\xi. \tag{2.6}$$

Respectively,

$$C_\omega^{(1)}(\xi;\, V) \equiv \omega(\xi;\, V) - \gamma P(V), \quad \langle C_\omega^{(1)} \rangle = 0 \tag{2.7}$$

is the generalized Charlier polynomial [2], [25] of first order which is a centred random variable. Here and henceforth the bounds for the integrals are omitted assuming tacitly that $\xi \in (-\infty, \infty)$ and $V \in [V_1, V_2]$, where V_1 and V_2 are certain minimal and maximal values for the celerity V. Respectively $P(V)$ is the probability density of the celerity. We assume also that

$$P_n(V_1, \ldots, V_n) = P(V_1) \ldots P(V_n), \quad n = 1, 2, \ldots, \tag{2.8}$$

which means that the celerities of two different structures (solitons) are statistically independent. The latter is the first condition for the presence of the so-called "perfect disorder" [2, 25]. The relationship between the multipoint probability densities of the system of generating points and the cumulants of the random density function is derived in [26, 27] and generalized by the author to the case of marked random points [2, 25]. For purpose of the present paper the higher-order Charlier polynomials are not needed and hence we cite here only the formulae for the second and third moments of $C_\omega^{(1)}$:

$$\begin{aligned} M^{(2)}(x_1, x_2; V_1, V_2) &\equiv \langle C_\omega^{(1)}(x_1;\, V_1) C_\omega^{(2)}(x_2;\, V_2) \rangle \\ &= \gamma P(V_1) \delta(x_1 - x_2) \delta(V_1 - V_2) + \gamma^2 P(V_1) P(V_2) [Q_{12}^{(2)} - 1] \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} M^{(3)}(x_1, x_2, x_3; V_1, V_2, V_3) &\equiv \langle C_\omega^{(1)}(x_1;\, V_1) C_\omega^{(1)}(x_2;\, V_2) C_\omega^{(1)}(x_3;\, V_3) \rangle \\ &= \gamma P(V_1) \delta(x_1 - x_2) \delta(x_1 - x_3) \delta(V_1 - V_2) \delta(V_1 - V_3) \\ &+ 3\gamma^2 \left\{ P(V_1) P(V_3) \delta(x_1 - x_2) \delta(V_1 - V_2) [Q_{13}^{(2)} - 1] \right\}_{\text{sym}} \\ &+ \gamma^3 P(V_1) P(V_2) P(V_3) [Q_{123}^{(3)} - Q_{12}^{(2)} - Q_{23}^{(2)} - Q_{31}^{(2)} + 2], \end{aligned} \tag{2.10}$$

where $\{\cdot\}_{\text{sym}}$ is the symmetrization operation and $Q_{12}^{(2)}$ and $Q_{123}^{(3)}$ are connected with the two- and tree-point probability densities of the system of random points. Namely,

$$dP = Q_{12}^{(2)}(x_1, x_2; V_1, V_2)P(V_1)P(V_2)\,dx_1dx_2dV_1dV_2 \tag{2.11}$$

is the probability to find two members of the system $\{x_\alpha; V_\alpha\}$ in the vicinities of the two spatial positions x_1 and x_2 while the markers are in the vicinities of the values V_1 and V_2, respectively. Further:

$$dP = Q_{123}^{(3)}(x_1, x_2, x_3; V_1, V_2, V_3)P(V_1)P(V_2)P(V_3)\,dx_1dx_2dx_3dV_1dV_2dV_3 \tag{2.12}$$

is the respective probability for a triplet. Here is to be mentioned that for the case of Poisson random function when the members are statistically independent one has $Q^{(2)} \equiv 1$ and $Q^{(3)} \equiv 1$ and $P(V) = \delta(V - V_0)$. When the latter does not hold, one arrives to the so-called compound Poisson function [28] which is to be discussed below.

3. BOUNDARY VALUE PROBLEM FOR THE KERNEL

The governing equation for kernel $K(x; V)$ is derived from (1.6) upon introducing (2.5). It is easily seen that for the spatial derivatives one has

$$\frac{\partial^k \varphi}{\partial x^k} = \int\int \frac{\partial^k}{\partial x^k} K(x - \xi; V) C_\omega^{(1)}(\xi; V)\,dV d\xi, \quad k = 1, 2, 3, 4. \tag{3.1}$$

Some special care requires the time derivative of φ because the time dependence enters (2.1) only througt the random density function ω. Let us begin with the equivalent representation

$$\varphi = \sum_\alpha K(x - x_\alpha; V_\alpha) + \varphi_0, \tag{3.2}$$

that follows straightforward from (2.1) upon introducing (2.2). Acknowledging also (2.3) we have

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi_0}{\partial t} - \sum_\alpha \frac{\partial}{\partial x} K(x - x_\alpha; V_\alpha)\frac{dx_\alpha}{dt} \equiv \frac{\partial \varphi_0}{\partial t} - \sum_\alpha V_\alpha \frac{\partial}{\partial x} K(x - x_\alpha; V_\alpha).$$

Returning to the original notations we have

$$\frac{\partial \varphi}{\partial t} = \frac{\partial \varphi_0}{\partial t} - \int\int V \frac{\partial K(x - \xi; V)}{\partial x} C_\omega^{(1)}(\xi; V)\,dV d\xi. \tag{3.3}$$

Here is already taken into account the following property of the random flux (2.1)

$$\frac{\partial \varphi_1}{\partial t} \equiv -\gamma \int\int V \frac{\partial}{\partial x} K(x - \xi; V)P(V)\,d\xi dV = -\gamma \int V\,dV K\Big|_{-\infty}^{\infty} = 0, \tag{3.4}$$

i. e.

$$\frac{\partial \varphi_0}{\partial t} = \frac{\partial (\varphi)}{\partial t}. \tag{3.5}$$

Taking now the average of the Eq. (1.6) it is obtained

$$0 = \frac{\partial\langle\varphi\rangle}{\partial t} + \frac{\partial}{\partial x}\left[\frac{3}{a}\langle\varphi\rangle + 3\langle\varphi^2\rangle + \frac{\partial}{\partial x}\langle\varphi\rangle + \frac{\partial^3}{\partial x^3}\langle\varphi\rangle\right] \equiv \frac{\partial\langle\varphi\rangle}{\partial t}, \tag{3.6}$$

because of the spatial homegeneity. Hence

$$\langle\varphi\rangle = \text{const}, \tag{3.7}$$

and due to (3.5)

$$\varphi_0 = \text{const}. \tag{3.8}$$

In fact, (3.6) is the equation for "zeroth-order" kernel $\langle\varphi\rangle$. For the first-order kernel $K(x; V)$, the equation is obtained by multiplying (1.6) by $C_\omega^{(1)}(0; V)$ after the representation (2.5) is introduced there and the average is taken (see [2, 3, 8, 29]). It is instructive to display the technique for each term entering (1.6). For the time derivative ((2.9) acknowledged) one has

$$\begin{aligned}
\langle C_\omega^{(1)}(0;V)\frac{\partial\varphi}{\partial t}\rangle &\equiv \langle C_\omega^{(1)}(0;V)\frac{\partial\varphi_0}{\partial t}\rangle \\
&- \int\int V_1\frac{\partial K}{\partial x}(x-\xi;V_1)\langle C_\omega^{(1)}(\xi;V_1)C_\omega^{(1)}(0;V)\rangle\,dV_1\,d\xi \\
&= -\int\int V_1\frac{\partial K}{\partial x}(x-\xi;V_1)\Big\{\gamma P(V_1)\delta(\xi)\delta(V-V_1) \\
&\quad + \gamma^2 P(V_1)P(V)[Q^{(2)}(0,\xi;V,V_1)-1]\Big\}\,dV_1\,d\xi \\
&= -\gamma V\frac{\partial}{\partial x}K(x;V) - \gamma^2 P(V)\int\int\frac{\partial}{\partial x}K(x-\xi;V_1)P(V_1)V_1 \\
&\times\ [Q^{(2)}(0,\xi;V,V_1)-1]\,dV_1\,d\xi.
\end{aligned} \tag{3.9}$$

In the same manner are treated the spatial derivatives (note that $\int P(V)\,dV \equiv 1$)

$$\begin{aligned}
\langle C_\omega^{(1)}(0;V)\frac{\partial^k\varphi}{\partial x^k}\rangle &= \gamma\frac{\partial^k K(x;V)}{\partial x^k} \\
&+ \gamma^2 P(V)\int\int\frac{\partial^k}{\partial x^k}K(x-\xi;V_1)P(V_1)[Q^{(2)}(0,\xi;V,V_1)-1]\,dV_1\,d\xi.
\end{aligned} \tag{3.10}$$

The most important feature of random point approximation is that it yields an equation for kernel that is nonlinearly coupled (see, e. g. [29])

$$\begin{aligned}
\langle C_\omega^{(1)}(0;V)\varphi(x,t)\frac{\partial\varphi(x,t)}{\partial x}\rangle &\equiv \int\int\int\int K(x-\xi_1;V_1)\frac{\partial}{\partial x}K(x-\xi_2;V_2) \\
&\times \langle C_\omega^{(1)}(0;V)C_\omega^{(1)}(\xi_1;V_1)C_\omega^{(1)}(\xi_2;V_2)\rangle\,dV_1dV_2d\xi_1d\xi_2 \\
&= \gamma K(x;V)\frac{\partial K(x,V)}{\partial x} + \gamma^2 P(V)\int\int\Big[K(x;V)\frac{\partial}{\partial x}K(x-\xi_1;V_1) \\
&+ \frac{\partial K(x;V)}{\partial x}K(x-\xi_1;V_1) + K(x-\xi_1;V)\frac{\partial}{\partial x}K(x-\xi_1;V_1)\Big]
\end{aligned} \tag{3.11}$$

$$\times P(V_1)[Q^{(2)}(0,\xi_1;\,V,V_1)-1]\,dV_1d\xi_1$$
$$+\gamma^3\int\int\int\int K(x-\xi_1;\,V_1)\frac{\partial}{\partial x}K(x-\xi_2;\,V_2)P(V_1)P(V_2)$$
$$\times\left[Q^{(3)}(0,\xi_1,\xi_2;\,V,V_1,V_2)-Q^{(2)}(0,\xi_1;\,V,V_1)\right.$$
$$\left.-Q^{(2)}(0,\xi_2;\,V,V_2)-Q^{(2)}(\xi_1,\xi_2;\,V_1,V_2)+2\right]dV_1dV_2d\xi_1d\xi_2.$$

Multiplying (1.6) by $C_\omega^{(1)}(0;\,V)$, taking the average and acknowledging (3.9)–(3.11), one obtains the governing equation for kernel $K(x;\,V)$ which turns out to be an integro-differential equation whose main (differential) part follows directly (1.6), namely

$$\gamma\left[\left(\frac{3}{a}-V\right)\frac{dK}{dx}+6K\frac{dK}{dx}+\frac{d^2K}{dx^2}+\frac{d^4K}{dx^4}\right]+\gamma^2[\cdots]+\gamma^3[\cdots]=0. \tag{3.12}$$

The chief difference between the problems for φ and K is included in the fact that $K\in L^2(-\infty,\infty)$ in order to have a finite "energy" ($\langle\varphi^2\rangle<+\infty$) for the stochastic function φ from (2.1), or which is the same — from (2.5). So, the Eq. (3.12) is coupled by the following requirement:

$$\int_{-\infty}^{\infty}\int_{V_1}^{V_2}K(x;\,V)\,dV\,dx<+\infty, \tag{3.13}$$

which form the boundary value problem for K.

4. THE POISSON LIMITING CASE

The boundary value problem from previous section is not coupled because the two- and three-point probability densities $Q^{(2)}$ and $Q^{(3)}$ of the system of random points are not known. The most consistent way to couple the problem is to derive a system of ordinary differential equations governing the motion of the different members x_α of the system of random points, namely

$$x_\alpha(t)=F_\alpha(x_1,\ldots,x_\alpha,\ldots;\,K) \tag{4.1}$$

and to use the approach of Bogolyubov [30] to derive equations of evolution for the multipoint probability densities. Unfortunately, on the way of doing that there exist a number of difficulties. The law of motion (4.1) is not priory known and depends as a functional on the kernel K. The approach of [30] is effective only for quite particular functions $F_\alpha(x_\alpha)$. All that means that at the beginning one has to resort to certain semiempirical hypotheses concerning the multipoint probability densities. The simplest and most natural assumption is that points (centres of solitons) are statistically independent, i. e. $Q^{(k)}\equiv 1$ for $k=2,3,\ldots$ For two-phase media $Q^{(k)}$ can be generalized in a way to prevent the particles of the particulate phase to overlap [3], but for the nonlinear systems with deterministic coefficients there exist no specific hints how to model the functions $Q^{(k)}$. When certain a-priori information is known it can be incorporated into model as it is done in [8], where $Q^{(2)}$ from [31] is employed and the achieved prediction is decisively improved.

For the present case such information is not yet available and, hence, we confine ourselves to Poisson approximation, and more specificly to the compound Poisson random function. Then the Eq. (3.12) reduces to

$$-cK' + 6KK' + K'' + K^{IV} = 0, \tag{4.2}$$

where $c = V - (3/a)$ and the solution is to satisfy

$$K \longrightarrow 0 \quad \text{for} \quad x \longrightarrow \pm\infty. \tag{4.3}$$

It is easily proved that due to the Eq. (4.2) the condition (4.3) yields also (3.13).

Equation (4.2) can be integrated once and acknowledging (4.3) it is obtained

$$-cK + 3K^2 + K' + K''' = 0. \tag{4.4}$$

Thus (4.3) and (4.4) form the boundary value problem for evaluating the kernel K.

5. SKETCH OF THE METHOD OF SOLUTION

For solving the soliton problems similar to that of the previous section, which are in fact inverse problems, two different methods were developed by the author and coworkers. The first of them, named "method of variational imbedding", is proposed in [5] in connection with calculation of the homoclinic solution of Lorenz system and consists in replacing the original problem by an correct boundary problem for the Euler-Lagrange equations for minimization of a quadratic functional of the original set of equations. To the case of K–S equation the method was applied in [32]. In order to make the present paper self-contained we cite briefly the main features of the technique.

Let us recast (4.4) in the form of a system

$$K'(x) = Y(x), \quad Y' = Z(x), \quad Z' = -Y - 3K^2 + cK, \tag{5.1}$$

for which the following boundary conditions are imposed:

$$K = Y = Z = 0 \quad \text{for} \quad x \longrightarrow \pm\infty. \tag{5.2}$$

Consider the functional

$$J = \int_{-\infty}^{\infty} \left[(K' - Y)^2 + (Y' - Z)^2 + (Z' + Y + 3K^2 - cK)^2\right] dx. \tag{5.3}$$

When the sought solution exists, the functional J adopts the trivial value and as far as it is a quadratic one the value $J = 0$ is a local minimum. So we seek for the minimum of J with respect to its functional arguments K, Y, Z. A necessary condition for a local minimum is given by the Euler–Lagrange equations which for the functional under consideration take the form

$$\left|\begin{array}{l} K'' - (18K^2 + c^2)K = Y' - 9cK^2 + (6K - c)(Z' + Y), \\ Y'' - 2Y = 2Z' - K' + 3K^2 - cK, \\ Z'' - Z = -2Y' - (6x - c)X'. \end{array}\right. \tag{5.4}$$

This system is solved by means of iteration procedure in [32]. In Fig. 2 it is shown the calculated shape of the soliton (represented by the kernel K) for different c.

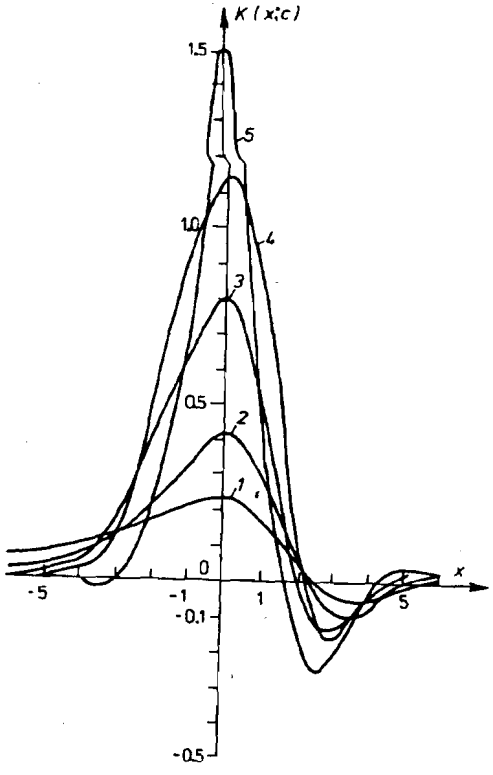

Fig. 2. The calculated shape of the kernel $K(x;c)$ for different values of c:
1 — $c = 0.55$; *2* — $c = 1.00$; *3* — $c = 2.00$; *4* — $c = 3.00$; *5* — $c = 4.00$

The interesting finding in [32] is that soliton solution does exist for quite wide range of c and this interval was further widened in [2] to reach the size $0.12 \leqq c \leqq 10$ and the way in which the results develop strongly suggest that the solution exists even for $0 < c < +\infty$. At the same time in [18] it is reported only one value $c = 1.216$ for which the solitary wave exists. This is in contrast not only with our results, but also with the earlier work of the same author [33], where results are obtained also for $c = 1$. This discrepancy forced us to develop another completely different numerical technique in order to verify the results for the spectrum of the eigen-value parameter c. It is important in order to find whether the random point solution is a marked random point function (compound Poisson function).

As it is mentioned in the introduction, a new complete orthonormal (CON) system of functions is proposed in [6]. The main advantage of the new system is that it possesses a formula, representing a product of two its members in series with respect to the system. The latter is not known for the existing systems in infinite regions and allows one to solve nonlinear problems (see [34] for further details). To the specific problem under consideration the new CON system is applied in [35], where the soliton solution is obtained not only for $c \in [0.12, 10]$ but also up to $c = 30$ and down to $c = 0.01$ and the shape is in very good quantitative agreement with that calculated by means of the method of variational imbedding. The latter justifies the results presented in Fig. 2.

6. STATISTICAL CHARACTERISTICS

In the previous section the method for evaluating the kernel is outlined and is shown that a continuous spectra takes place for the nonlinear eigenvalue parameter c. Unfortunately, the absence of knowledge on this stage about the probability distribution $P(V)$ of the marker $V = c + 3/a$ hinders the full scale application of the techniques involving marked random point functions. What can be done under these circumstances is to calculate the statistical characteristics for the "monosoliton" random point regimés with different kinds of basic structures (different solitons

with different c). Numerically this is expressed by the formula

$$P(V) = \delta(V - V_0), \quad V_0 = 3/a + c_0. \tag{6.1}$$

In the frame of the Poisson limiting case with (6.1) acknowledged, one has for the spatial correlation function

$$Q_2(x = x_2 - x_1) \equiv \langle \varphi(x_1, t)\varphi(x_2, t)\rangle - \langle \varphi\rangle^2 = \gamma \int_{-\infty}^{\infty} K(\xi; V_0)K(\xi + x; V_0)\, d\xi. \tag{6.2}$$

For the simplest case (6.1) the temporal correlation function is easily expressed by means of the spatial one

$$\overline{Q}_2(t = t_2 - t_1) \equiv \langle \varphi(x, t_1)\varphi(x, t_2)\rangle - \langle \varphi\rangle^2 = \gamma \int_{-\infty}^{\infty} K(\xi; V_0)K(\xi + V_0\tau; V_0)\, d\xi = Q_2(V_0\tau). \tag{6.3}$$

The latter allows us to consider only one of these two correlation functions. For definiteness we chose the spatial one, moreover that it is easily obtained in numerical simulations. In Fig. 3 is shown the correlation function predicted for two values of c on the base of the respective kernel shown in Fig. 2.

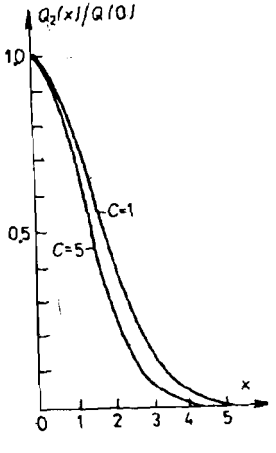

Fig. 3. The spatial correlation function *1* — $c = 1.00$, *2* — $c = 5.00$

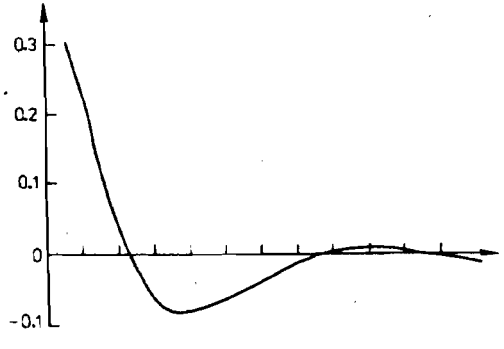

Fig. 4. Flux correlation from the numerical experiment of Pomeau and Zaleski [11] (arbitrary scale)

The only known numerical experiment with K–S equation is reported in [11], but no data concerning the correlation function is presented there. Instead, a special kind of correlation called by the authors of [11] "flux-correlation" is considered (see Fig. 4). Roughly, the correlation of the spatial derivative of φ

$$R(x) \equiv \langle \varphi_x(x_1,t)\varphi_x(x_2,t)\rangle = \gamma \int_{-\infty}^{\infty} K_x(\xi; V_0)K_x(\xi + x; V_0)\, d\xi \tag{6.4}$$

corresponds to that quantity. In Fig. 5 is shown the correlation coefficient for $R(x)$. Qualitatively it does correspond to the flux-correlation of [11].

More quantitative comparisons are due in the second part of the present paper [10].

Having the correlation function $Q_2(x)$, one can calculate the spectrum

$$E(k) = \int_{-\infty}^{\infty} Q_2(x)\cos(2\pi k x)\, dx, \tag{6.5}$$

where k is the wavenumber. In Fig. 6 is shown the calculated on the base of random point approximation spectrum. Similarly to the Burgers equation (see the numerical simulation in [36] and random point prediction [8]), the spectrum does not vanish at $k = 0$ as it is well-known to be the case for real turbulence [37]. This can be attributed to the fact that Burgers and K–S models are one-dimentional while the real turbulent flows are essentially three-dimensional and the velocity fields satisfy the continuity equation.

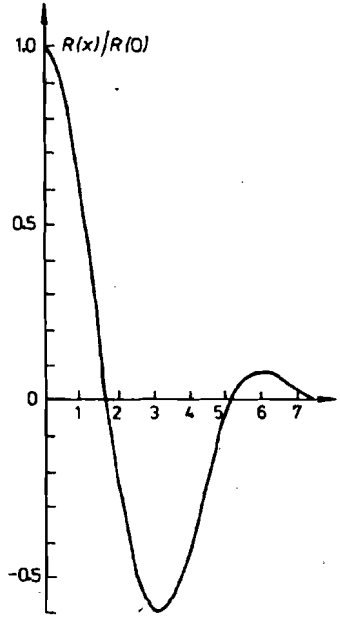

Fig. 5. The correlation $R(x)$ of the spatial derivative of shape function φ for random point solution with $c = 1$

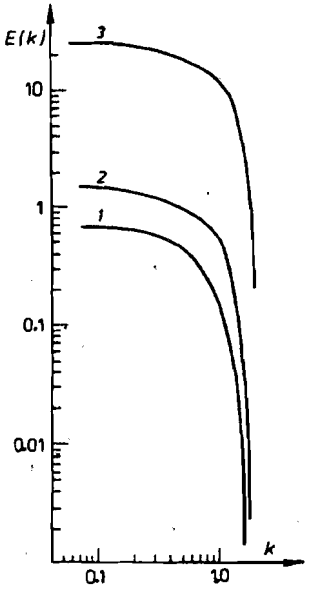

Fig. 6. The spectrum of the random point solution with different c:
1 — $c = 0.55$; *2* — $c = 1.00$; *3* — $c = 5.00$

CONCLUDING REMARKS

In the present paper it is shown that the Kuramoto–Sivashinsky equation admits as a solution a (generally marked) random point function. For the Poisson limiting case the random solution is obtained in closed form and various statistical characteristics are calculated and shown graphically.

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