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ON THE RANDOM POINT STRUCTURE OF THE LORENZ ATTRACTOR: A NUMERICAL EXPERIMENT

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One of the simplest systems with random behaviour of solution was proposed by Lorenz [1] in connection with the natural convection in unbounded layers. Since then the Lorenz system has been thoroughly investigated from different points of view [2] and has become, perhaps, the principal example for deterministic dynamical system with stochastic solutions. Now it is frequently used when displaying novel approaches to randomization of dynamic systems (see e. g. [3]).

In author's works [4, 5] a new approach to stochasticity of nonlinear dynamic systems is proposed consisting of approximation of the sought solution with random point function. This approach has proved effective in modelling the turbulent Poiseuille flows [6, 7], giving for the turbulent characteristics predictions that compare well with the experimental data. Though the latter is an important verification of the said idea, it is still imperative to find a straightforward experimental evidence for this assumption. The real turbulent flows are too difficult to treat, and though the random point structure is virtually proved experimentally by the discovery of the so-called coherent structures, the respective results bear essentially qualitative meaning. In this instance it is important to investigate low-dimensional model systems in order to obtain quantitative results.

The present short note is devoted to investigation of the random point regimes of the Lorenz system by means of direct numerical experiment.

1. Stating the Problem. The Lorenz system [1] in terms of dimensionless variables has the form:

$$\dot{x}=-\sigma(x-y),\quad \dot{y}=-xz+rx-y,\quad \dot{z}=xy-bz, \tag{1}$$

where x is the first Fourier amplitude for the stream function; y, z — the first two for the temperature; b is combination of the wavelengths along the two spatial directions; σ is the Prandtl number and r is the Rayleigh number. Lorenz investigated (1) for $\sigma=10$ and $b=8/3$ and showed that for $r\geq 24.74$ all the three stationary points of the system became unstable. He conducted a numerical experiment for $r=28$ and confirmed the assumption that the trajectories can eventually go random for supercritical Rayleigh numbers.

The first quantitative numerical experiments concerned with the statistics of the random solution are those of Lücke [8] who obtained data for the correlation coefficients of the three functions x, y, z and cross-correlation of x and y. From those data, however, a conclusion cannot be reached whether the stochastic trajectories are random point functions. The latter can be done only through specially organized numerical experiments involving identification of structures.

2. Difference Scheme. In [8] is employed a difference approximation to (1) of fourth order, while in [1] it is of the second order. Here we choose the scheme of lower-order approximation in order to reduce the required computational time. It is important to note that the correlations obtained here with the second-order difference scheme coincide with the respective quantities calculated in [8] with fourth-order scheme within 0.1 %. This means that the stochastic behaviour is reproduced adequately by both schemes and allows us to use the scheme of lower-order approximation.

3. Preparing the Statistical Ensemble. Initially, a sufficient number of time steps are executed to bring the trajectory in the vicinity of the 'strange attractor' of the Lorenz system. Assuming a time increment $\tau=0.01$ a number of 10000 time steps proves to be more than enough to achieve that aim. The last 1001 time steps are recorded as the first realization of the process under study. Then by means of random number generator a random integer from the interval [0, 1000] is selected to define the number of time steps to be executed in order to obtain a new initial condition. After that the next 1001 steps are executed and the result recorded as the second realization of the random solution. This procedure is repeated for the third, fourth, etc. realizations.

The same correlation functions as in [8] are calculated by means of ensemble averaging. Let us denote by $\langle\cdot\rangle$ the ensemble averaging and by $\overline{}$ — the time averaging of [8] and the previous section. For ensemble of 10000 realizations the quantity $\langle z\rangle$ differs from $\bar{z}$ by 0.8 %; $\langle xy\rangle$ from $\overline{xy}$ — by 3.1 %; $\langle xx\rangle$ from $\overline{xx}$ — by 3 %; $\langle yy\rangle$ from $\overline{yy}$ — by 4 %; $\langle zz\rangle$ from $\overline{zz}$ — by 2.5 %. When the number of realizations is increased up to 120000 the differences are not discernible, which allows us to adopt the last number as the total size of the ensemble. This excellent agreement provides additional support to the assertion that the stochastic behaviour obtained is an intrinsic property of the differential system and is not affected by the specific difference scheme employed in computations.

4. Identifying the Structures. It is well known that for the considered Rayleigh number $r=28$ the solution for function z is positive. Hence we can define the centre of a structure to be the point in which z has a local maximum.

As mentioned in the previous section we have an ensemble of realizations which are numerically defined in 1001 points, regularly spaced in the interval [0, 10]. So we define all points in which the respective set function has a local maximum and then turn to the problem in which of the two adjacent intervals of length 0.01 the maximum is situated. Using second-order approximation it can be decided that the maximum is in that interval for which the value of the function at the other boundary point is higher. After the intervals containing local maxima have been identified, the value of 1 is assigned to them, 0 being assigned to the remaining intervals. Thus we arrive to a realization of the system of random points — centres of the structures. Let us denote the arrays of realizations by d_{ij}, where index i refers to the number of the realization and index j — to number of the interval. Then for the statistical average of the number of points per unit length we have

$$\gamma_j=\frac{1}{\tau N_{en}}\sum_{i=1}^{N_{en}} d_{ij}, \tag{2}$$

where N_{en} is the number of realizations (size of the ensemble). Accordingly, for the two-point probability density (stationarity of the process acknowledged) we obtain

$$f_2(\tau_j)=\frac{1}{\tau^2 N_{en}}\sum_{i=1}^{N_{en}} d_{i1}d_{ij}. \tag{3}$$

It is convenient to use the normalized function

$$Q(\tau)=\gamma^{-2}f_2(\tau_j). \tag{4}$$

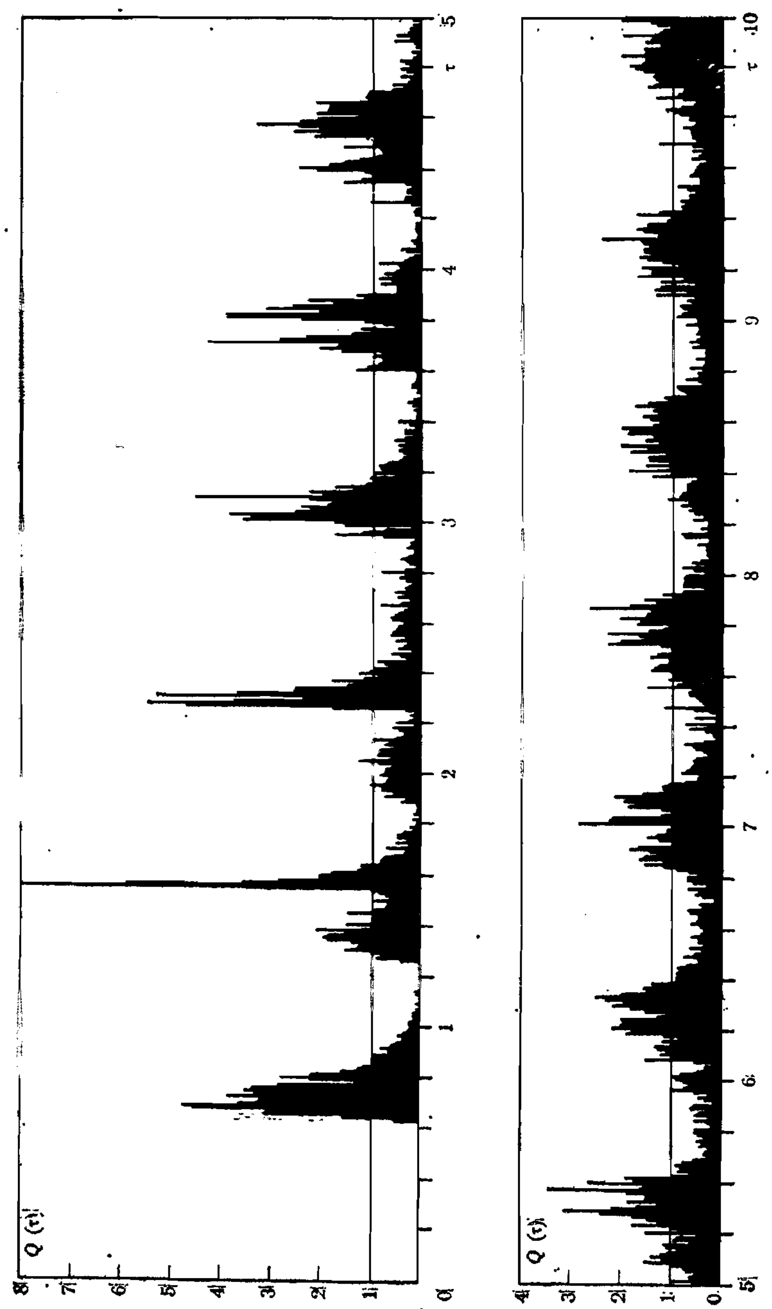

Fig.

5. Results and Discussion. As is mentioned above, the number of computed realizations of the random solution of Lorenz system (1) is 120000. It is interesting to note that even for this quite a formidable an ensemble the mean number γ of random points per unit length (the intensity of the random point process) vary significantly (about 8 %) from point to point in the interval [0, 10]. The time average over the entire interval under consideration gives $\gamma = 1.327$ which we adopt as the result for the said quantity. The slow conversion with the increase of the size of ensemble can be attributed to the method of estimation of the interval in which a structure is centered. Sometimes the location of maximum can appear in the interval with the lower value of function at the other boundary point than the point of maximum of set function. This is especially probable when in the two adjacent intervals to the point of maximum the function z adopts at the other boundary points approximately the same magnitudes. This is no surprise, because it is well known that slight variations in the functional value can result in significant error in estimating the roots or extremal points when the gradients of the function are small.

The Fugure presents the normalized two-point probability density function $Q(\tau)$ of the system of random points generating the random point solution to the Lorenz system. It is obvious that for $\tau < 0.63$ the probability is exactly equal to zero which means that two structures cannot be situated closer than that distance. In other words, they cannot significantly overlap each other which is very similar to the situation present in the particulate two-phase media with random structure [8].

For large separation distances between the structures they are statistically independent and $Q(\tau)$ eventually approaches unity for $\tau \to \infty$. This process, however, is slow and for moderate values of the argument (up to approximately 10—15) the function $Q(\tau)$ exhibits a number of peaks whose positions are related to the length scale of the structure. It is so because the most probable moment for occurrence of a structure is when the previous one is fairly decayed.

The results obtained in the present work prove that the stochastic solution to the Lorenz system is a random point function for $r = 28$ when the trajectory belongs to the strange attractor. This is an important conclusion which opens a new avenue for modelling the stochastic behaviour of the solutions of nonlinear dynamic systems, consisting in approximating with random point functions. Application of this approach for predicting the statistics of the Lorenz system and comparison with the numerical experiments is due in a following paper.

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