Доклады Болгарской академии наук
Comptes rendus de l'Académie bulgare des Sciences
Tome 40, N° 6, 1987

MATHEMATIQUES
Equations différentielles

THE METHOD OF VARIATIONAL IMBEDDING FOR TIME REVERSAL PARABOLIC INCORRECT PROBLEMS

C. I. Christov

(*Submitted by Academician Bl. Sendov on December 15, 1986*)

One of the most impressive examples of instability is represented by the parabolic equation with reversed direction of time, i. e. when the initial condition is defined at certain $t=T$ and the solution is sought for $t<T$. It is easily shown that such a problem is incorrect in the sense of Hadamard (see [1]). At the time it is of great practical importance (see [2]) and as such an abundant literature is available displaying a variety of mathematical approaches. The most successful, perhaps, turns out to be the method of quasi-reversibility [2], [3] in which the original heat conduction equation is replaced by a fourth-order problem with respect to the spatial coordinates that is correct. The shortcomings of that method are founded in the presence of a small parameter before the highest-order derivatives. In the present paper we employ the method of variational imbedding outlined in the previous author's works [4], [5]. Similar to quasi-reversibility the present method employs higher-order equations in the place of the original problem but it is considerably more convenient and does not use an artificial small parameter. Another distinctive feature of the method proposed is that the solution of the higher-order elliptic boundary value problem boils down to a strictly identical one with the solution of the initial incorrect problem.

1. Posing the Problem. Consider the heat conduction equation

$$Au \equiv -\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}=0, \tag{1}$$

with homogeneous boundary conditions

$$u(t,\ 0)=u(t,\ l)=0. \tag{2}$$

There are no difficulties for the consideration of non-homogeneous boundary conditions and it is not done only for the sake of simplicity. The 'initial' condition is

$$u(T,\ x)=\varphi(x), \quad \varphi(0)=\varphi(l)=\varphi''(0)=\varphi''(l)=0. \tag{3}$$

Requirements on function $\varphi(x)$ stem from the fact that it is to satisfy the equation (1) and boundary conditions (2).

2. Variational Principle. The gist of the method of variational imbedding is that the original problem is replaced by a minimization problem

$$I=\int_0^T \int_0^l \left(-\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}\right)^2 dx\,dt=\min, \tag{4}$$

where u must satisfy (2), (3). Functional I is a quadratic function of its argument Au, i. e. there is a one-to-one correspondence between the original equation (1) and the minimization problem (4). A necessary condition for minimization of I is the Euler-Lagrange equation (see, e. g. [6]):

(5) $$-\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}=0.$$

In order to secure that the solution of (5) is a solution also of (1) it is necessary to impose (1) as a boundary condition. Being reminded of (2) this gives:

(6) $$\frac{\partial^2 u}{\partial x^2}=0, \quad x=0,\ l.$$

Eq. (5) is of second order with respect to time and along with (3) requires still another condition. It is natural to try to impose (1) as a condition for (5) at $t=0$, but for such a problem neither the existence nor the uniqueness of the solution can be proved and there are reasons to believe that those are not present at all since the operator of the resulted boundary value problem is not positive. Setting the temporal condition at the final time stage we have

(7) $$\left(-\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x^2}\right)_{t=T}=0 \ \therefore\ \left.\frac{\partial u}{\partial t}\right|_{t=T}=\varphi''(x),$$

and hence arrive to an initial value problem for the elliptic equation (5) which is much similar to the classical Hadamard example of incorrect problems.

To prove directly the existence and uniqueness of the problem (5), (2), (3), (6), (7) is not a trivial task and goes beyond the frame of present short note. Due to the closest similarity with the Hadamard problem, however, there is strong belief in favour of this supposition. For the considerations in what follows it is enough the heuristic assumption that the solution does exists and is unique.

3. The Repeated Imbedding. The boundary-initial value problem which we have arrived at in the previous section appears to be much more convenient than the original problem because of its resemblance to the Hadamard problem. The latter is treated successfully in [4] by means of method of variational imbedding. This prompts us to repeat the procedure of variational imbedding for (5), i. e. to consider the following minimization problem

(8) $$I_r=\int_0^T\int_0^l\left(-\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}\right)^2 dxdt=\min.$$

The respective Euler-Lagrange equation reads

(9) $$\frac{\partial^4 u}{\partial t^4}-2\frac{\partial^6 u}{\partial t^2\partial x^4}+\frac{\partial^8 u}{\partial x^8}=0.$$

Here we repeat the reasoning concerning the derivation of (6) and impose eq (5) as a boundary condition at $x=0,\ l$. It is easy to select the second condition at these boundaries to be the second spatial derivative of (5) which leads to a well posed problem. Taking the first spatial derivative of (5), along with (2) and (4) we arrive to an ill-posed problem.

So,

(10) $$\frac{\partial^4 u}{\partial x^4}=\frac{\partial^6 u}{\partial x^6}=0, \quad x=0,\ l.$$

Proceeding with the boundary conditions at the initial time stage following [4] we require eq. (5) to be satisfied as well as its normal to that boundary derivative, namely

(11) $$-\frac{\partial^2 T}{\partial t^2}+\frac{\partial^4 T}{\partial x^4}=0, \quad -\frac{\partial^3 T}{\partial t^3}+\frac{\partial^5 T}{\partial t\partial x^5}=0, \quad t=0,$$

which couples the boundary conditions for eight-order elliptic equation (8).

4. Existence and Uniqueness of Solution. Let us consider now the Hilbert space $H(D)$ of functions that are defined in D: $(0 \leq x \leq l) \times (0 \leq t \leq T)$ which satisfy the following boundary conditions:

(12a)
$$-\frac{\partial^2 a}{\partial t^2}+\frac{\partial^4 a}{\partial x^4}=0, \quad \frac{\partial^3 a}{\partial t^3}+\frac{\partial^5 a}{\partial t \partial x^4}=0, \quad t=0,$$

(12b)
$$a=0, \quad \frac{\partial a}{\partial t}=0, \quad t=T,$$

(13)
$$a=\frac{\partial^2 a}{\partial x^2}=\frac{\partial^4 a}{\partial x^4}=\frac{\partial^6 a}{\partial x^6}=0, \quad x=0, l.$$

The following scalar product is introduced in H:

(14)
$$[a, b]=\int\int\left(-\frac{\partial^2 a}{\partial t^2}+\frac{\partial^4 a}{\partial x^4}\right)\left(-\frac{\partial^2 b}{\partial t^2}+\frac{\partial^4 b}{\partial x^4}\right) dxdt.$$

The last equality defines a scalar product since we have already assumed that the Hadamard-like problem possesses a unique solution, i. e. $[a, a]$ is true only when $a \equiv 0$.

Consider now the sufficiently times differentiable function $\chi(t, x)$ satisfying at the boundaries the whole set of boundary conditions, i. e. it is a continuation of $\varphi(x)$ in D. Then a generalized solution of (9), (2), (3), (6), (7), (10), (11) is called any function u for which the following holds

(15)
$$[u, \Phi]=0,$$

where $\Phi \in H(D)$ and $u-\chi \in H(D)$. The classical solution is also a generalized solution since

$$B_1=\int\int \Phi\left(\frac{\partial^4 u}{\partial t^4}-\frac{\partial^6 u}{\partial t^2 \partial x^4}\right) dxdt=\int\left(\Phi \frac{\partial^3 u}{\partial t}-\Phi\frac{d^5 u}{\partial t \partial x^4}\right)_0^T dx$$
$$+\int\int \frac{\partial^2 \Phi}{\partial t^2}\left(\frac{\partial^2 u}{\partial t}-\frac{\partial^4 u}{\partial x^4}\right) dxdt-\int \frac{\partial \Phi}{\partial t}\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^4 u}{\partial x^4}\right)_0^T dx$$

i. e.

$$B_1=\int\int \frac{\partial^2 \Phi}{\partial t^2}\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^4 u}{\partial x^4}\right) dxdt$$

and

$$B_2=\int\int \Phi\left(-\frac{\partial^6 u}{\partial t^2 \partial x^4}+\frac{\partial^8 u}{\partial x^8}\right) dxdt=\int\int \frac{\partial^4 \Phi}{\partial x^4}\left(-\frac{\partial^2 u}{\partial t^2}+\frac{\partial^4 u}{\partial x^4}\right) dxdt$$

and hence

(16)
$$0=\int\int \Phi\left(\frac{\partial^4 u}{\partial t^4}-2\frac{\partial^6 u}{\partial t^2 \partial x^4}+\frac{\partial^8 u}{\partial x^8}\right) dxdt=B_1+B_2=-[u, \Phi].$$

The existence of the generalized solution follows directly from Reece theorem because (14) defines a scalar product and therefore a functional. In order to prove the uniqueness we consider the difference $v=u_1-u_2$ between two supposed solutions u_1 and u_2. It is obvious that $v \in H(D)$. On the other hand (15) holds for v too. Then taking simply $\Phi \equiv v$ we have

(17)
$$[v, v]=0 \;\therefore\; v \equiv 0.$$

5. The Essence of Imbedding. So far it has been shown that the Euler-Lagrange equation (9) possesses a unique solution under the boundary conditions (2), (3), (6), (7), (10), (11). That equation is a necessary condition for the existence of a stationary point of the functional I_r and the uniqueness of its solution means that the stationary point is only one. As far as the functional I_r is quadratic then its minimum (appearing to be unique due to the uniqueness of the stationary point) is equal to zero, i. e. this is the minimum attained on the solution of (5) with the respective boundary conditions, i. e. we succeeded to find the solution to the Euler-Lagrange equation for the minimization of the functional I. As this solution is unique then the same reasoning concerning the uniqueness of the minimum of I applies and, hence, we have obtained the unique solution of the original incorrect problem (1)-(3). The gist of the idea is that the eight-order elliptic boundary value problem is correctly posed while the original problem is not.

6. Numerical Implementation. The boundary value problem discussed here is elliptic and may be solved by means of the convergence method that is outlined for the biharmonic problem in [4]. The crucial point is in splitting the boundary conditions (12a) and here once again the idea of [4] to add time derivative and mixed time/fictitious time derivative in (12a) is applied and to effect the splitting is performed in accordance with the scheme of stabilizing correction [7].

Department of Fluid Mechanics
Institute of Mechanics and Biomechanics
Bulgarian Academy of Sciences
Sofia, Bulgaria

REFERENCES

1 Тихонов, А. Н., В. Я. Арсенин. Методы решения некорректных задач. Наука, Москва, 1974. 2 Коздоба, Л. А., П. К. Круковский. Методы решения обратных задач теплопереноса. Наукова Думка, Киев, 1982. 3 Lattés, R., J.-L. Lions Méthode de quasi-réversibilité et applications, Dunod, Paris, 1967. 4 Cristov, C. I. Compt. rend. Acad. bulg. Sci 39, 1986, 12, 23-26. 5 Christov. C. I. Compt. rend. Acad. bulg. Sci. 40, 1987, 2, 21. 6 Цлаф, Л. Я. Вариационное исчисление и интегральные уравнения. Наука, Москва, 1966. 7 Яненко, Н. Н. Метод дробных шагов решения многомерних задач математической физики. Наука, Новосибирск, 1967.