Continuum Models and Discrete Systems Volume 1

Edited by

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Université Pierre et
Marie Curie Paris

Longman Scientific & Technical
1990

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Flows with coherent structures: application of random point functions and stochatic functional series

1 Introduction

Most of the deterministic physical systems evolving from deterministic initial states go random in the wake of instability, e.g., random noises, turbulence, etc. The phenomenon is called "chaotic dynamics" and is observed only for intrinsically non-linear systems. Due to non-linearity infinite hierarchies for the moments (cumulants) occur [21]. The predominant number of theoretical attempts is based on different kinds of perturbation techniques (see, e.g., [23] for a survey). The simplest one consists in discarding the higher-order moments (cumulants) at a certain level and solving the remaining linear system. In hydrodynamic turbulence such an approach gives results only for the case of decaying isotropic homogeneous turbulence [20], [2]. The natural development of the technique is to couple the truncated version of the cascade system for moments by means of relating the higher-order moments (cumulants) to the lower-order ones through additional (sometimes quite arbitrary) relations. These are known as hierarchy techniques [35] and the most popular of them is, perhaps, the Millionshtchikov's [27] quasi-normal assumption called also "zero-fourth-cumulant- approach", for which Ogura [33] shows that in certain cases it yields a negative energy spectrum. Another version of hierarchy technique is the so-called local indpendence assumption that relates the fourth moment to a product of the first- and third-order moments. It also turns out to be incorrect in general. In the cases when it is correct it gives the same results as the mere perturbation technique [1].

A step in the correct direction is the method of functional expansions outlined by Wiener in [44]. Unfortunately, he employed the Gaussian white noise as a basis function and the series (named in [5], Wiener-Hermite) lacked a clear physical meaning. As a result the application to turbulence problems started in [41] was faced with difficult convergence problems and required frequent renormalization [26] because the higher-order kernels representing the deviation from normal distribution increased sharply with time. The way to improve the performance of the functional-series method is to bring the very basis function as close as possible to the physics in order to have an adequate information for the process even from the first-order terms. The chief objective of the present paper is to show that the right choice for the basis function is the class of random point functions.

In recent years a new point of view on turbulence has evolved according to which, along with the small-scale chaotic motions, the turbulence signal consists also of organized (called "coherent") structures with determinsitic average shapes. Existence

of coherent structures appears to be a general property of non-linear dynamic systems with chaotic behaviour of solution and is now a well-recognized phenomena far beyond the frame of turbulence investigations.

In the authors works [6-9] there is launched the idea that the predominant part of the random solution is represented from a (generally marked) random point fucntion, composed by structures of similar deterministic shapes randomly located in the region under consideration (time interval, spatial domain, etc.). This is a heuristic assumption based on the perception that the instability gives rise to disturbances that develop and eventually decay, returning the system approximately in the same initially unstable state and only after that a new disturbance can occur and the scenario is repeated. During its life span a structure is stable to disturbances of the same characteristic length and the secondary instabilities result in smaller-scale disturbances superimposed upon the main one. This notion fits very well in the picture of a turbulent flow with coherent structures and permits one to obtain self-contained models and to predict quantitatively well the multipoint statistical characteristics of the stochastic regimes for a number of non-linear systems: Lorenz system, Burgers [7, 8] and Kuramoto-Sivashinsky [13, 38] equations, plane mixing layer [28-30], near-wall region of plane Poiseuille flow [16, 17] and inlet region of axisymmetric jet [31].

The generalization of the technique goes through developing the sought solutions into Volterra-Wiener functional series with a marked random point basis function. When the simplest Poisson random point function is used for a basis function one arrives at the so-called Poisson-Wiener expansion first considered by Ogura [32] without any applictions to particular physical systems. The Poisson-Wiener expansion is akin to the Wiener-Hermite expansion [11, 26] but (as it is argued above) has a significantly sound physical basis. In the latter the first-order term represents the Gaussian part of the process and the higher-order terms, the deviation from normality. In the former the first-order term represents the contribution of a single structure to the random field while the higher-order kernels of Wiener functionals are interpreted as the shape functions of double, triple, etc., interactions among the coherent structures [8]. Upon acknowledging the viriality of the series under consideration a natural way to rigorously truncate the infinite hierarchy for kernels is available allowing one to discard the terms of higher than certain order which yields asymptotically correct results with the respective degree of the number γ of structures per unit length.

2 Random point functions

Statistical characteristics of a random point function are fully defined by the respective properties of the generating system of random points. The latter is not systematically elucidated in the literature, though certain basic results have been known for more than 30 years. The most complete presentation of the theory of systems of random points without marks can be found in the monograph [3] and we shall follow it in this instance generalizing to the case of systems of marked random points.

Consider a set of physical objects that are completely characterized by a vector parameter $\mathbf{u} \in \mathbb{U}$, where $\mathbb{U}$ is a μ-dimensional vector space. Let us attach to each object its "number" α which may be a real number when the set of objects is not denumerable. Relationship between an object and its number is random, i.e., the mark of an object is a random function of its number. If these objects are dispersed throughout the region Ω of the space one arrives at a particular version of a system of marked points in which the spatial position of an object is given by $\mathbf{x}_\alpha$. The probability to find k objects in the vicinities of k spatial positions $\mathbf{x}_k$, each of these objects with a mark in the vicinity of the value $\mathbf{u}_k$, is

$$dP = F_k(\mathbf{r}_1, \dots, \mathbf{r}_k)\, d^m\mathbf{x}_1 \dots d^m\mathbf{x}_k\, d^\mu\mathbf{u}_1 \dots d^\mu\mathbf{u}_k, \tag{2.1}$$

where

$$F_k(\mathbf{r}_1, \dots, \mathbf{r}_k) = f_k(\mathbf{x}_1, \dots, \mathbf{x}_k; \mathbf{u}_1, \dots, \mathbf{u}_k) P(\mathbf{u}_1, \dots, \mathbf{u}_2) \tag{2.2}$$

and $P_k(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is the multivariate probability density of the mark.

A random point function is the following sum

$$x(\mathbf{x}) = \sum_\alpha h(\mathbf{x}-\mathbf{x}_\alpha; \mathbf{u}_\alpha) \equiv \int_\Omega \int_{\mathbb{U}} h(\mathbf{x}-\xi; \mathbf{u})\, \omega(\xi; \mathbf{u})\, d^m\xi\, d^\mu\mathbf{u},$$

where $\mathbf{x}_\alpha$ is the random spatial coordinate at which there appears a structure of shape h, defined by the random mark $\mathbf{u}_\alpha$. Function ω is called a random density fucntion (r.d.f.):

$$\omega(\mathbf{r}) \equiv \omega(\mathbf{x}; \mathbf{u}) = \sum_\alpha \delta(\mathbf{x}-\mathbf{x}_\alpha)\, \delta(\mathbf{u}-\mathbf{u}_\alpha) \equiv \sum_\alpha \delta(\mathbf{r}-\mathbf{r}_\alpha). \tag{2.3}$$

Here $\mathbf{r} = (x^1, \dots, x^m, u^1, \dots, u^\mu)$ is a vector from the outer product of the spaces $\mathbb{R}$ and $\mathbb{U}$, and $\delta(.)$ is the Dirac delta function

In the pioneer works [34, 22] there is outlined a way how to derive the correlation properties of $\omega(\mathbf{x})$ from the statistical characteristics of the generating system of random points through connecting the charcteristic functional of the random density function to the p.g.f. of the system of random points. Similarly, one can derive the formulae for the relationship between the moments m_n^ω of the marked r.d.f. and the probability distributions F_n of the system of marked points [11]

$$m_1^\omega(\mathbf{r}) = F_1(\mathbf{r}), \tag{2.4}$$

$$m_2^\omega(\mathbf{r}_2, \mathbf{r}_2) = F_1(\mathbf{r}_1)\, \delta(\mathbf{r}_1-\mathbf{r}_2) + F_2(\mathbf{r}_1, \mathbf{r}_2),$$

$$m_3^\omega(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = F_1(\mathbf{r}_1)\, \delta(\mathbf{r}_1-\mathbf{r}_2)\, \delta(\mathbf{r}_1-\mathbf{r}_3)$$

$$+ 3\{\delta(\mathbf{r}_1-\mathbf{r}_2)\, F_2(\mathbf{r}_1, \mathbf{r}_3)\}_s + F_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3),$$

$$m_4^\omega(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = F_1(\mathbf{r}_1)\, \delta(\mathbf{r}_1-\mathbf{r}_2)\, \delta(\mathbf{r}_1-\mathbf{r}_3)\, \delta(\mathbf{r}_1-\mathbf{r}_4),$$

$$+ 4\{\partial(\mathbf{r}_1-\mathbf{r}_2)\, \delta(\mathbf{r}_1-\mathbf{r}_3)\, F_2(\mathbf{r}_3, \mathbf{r}_4)\}_s$$

$$+ 3\{\delta(\mathbf{r}_1-\mathbf{r}_2)\, \delta(\mathbf{r}_3-\mathbf{r}_4)\, F_2(\mathbf{r}_1, \mathbf{r}_3)\}_s$$

$$+ 6\{\delta(\mathbf{r}_3-\mathbf{r}_4) F_3(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)\}_s + F_4(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4), \text{ etc.}$$

It is obvious that the class of marked random point functions is wide enough to include the models of a number of physical phenomena. In the predominant part of practically important cases, however, the random structure is unknown, which raises the problem of adequate approximation of the statistics of the system of random points. One of the ways to do that is to consider the class of functions of compound type for which the geometrical position at which the random objects appear are not strongly correlated (they do not form clusters) on neither statistical level, i.e., they are in a sense well stirred in the region Ω under consideration (see the reasoning in [15] for the spherical suspensions):

$$f_n = \gamma^n Q_n(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{u}_1, \dots, \mathbf{u}_n),\ n \geq 1,\ Q_n \simeq O(1) \tag{2.5}$$

and the marks are statistically independent [9], [11]:

$$P_n(\mathbf{u}_1, \dots, \mathbf{u}_n) = P_1(\mathbf{u}_1) \dots P_1(\mathbf{u}_n). \tag{2.6}$$

For $Q \equiv 1$ the geometrical positions are non-correlated and one arrives at the well-known compound Poisson process [43]. That hints at the name for the subclass of the marked random point funciton for which the relations (2.5), (2.6) hold, namely - "compound point fucntion". It is to be stressed that even this class of functions is wide enough to serve as an adequate basis for successful approximation of quite a large number of real physical processes.

3 Functional expansions

The first to employ the random point funciton in Wiener expansions was Ogura [32], who introduced the Poisson basis function and rendered the series orthogonal. It appears that in Poisson's case the Charlier polynomials help to attain that. They can be generalized [9, 14] so as to obtain orthogonal series even for the compound Poisson random function, namely,

$$C_\omega^{(0)} = 0, \quad C_\omega^{(1)}(\mathbf{r}_1) = \omega(\mathbf{r}_1) - \gamma P(\mathbf{u}_1), \tag{3.1}$$

$$C_\omega^{(2)}(\mathbf{r}_1, \mathbf{r}_2) = \omega(\mathbf{r}_1)\,\omega(\mathbf{r}_2) - \delta(\mathbf{r}_1-\mathbf{r}_2)\,\omega(\mathbf{r}_1)$$

$$-\gamma\,[P(\mathbf{u}_2)\,\omega(\mathbf{r}_1) + P(\mathbf{u}_1)\,\omega(\mathbf{r}_2)] + \gamma^2 P(\mathbf{u}_1)\,P(\mathbf{u}_2),$$

$$\cdots$$

$$C_\omega^{(n)}(\mathbf{r}_1, \dots, \mathbf{r}_n) = \Delta_\omega^{(n)}(\mathbf{r}_1, \dots, \mathbf{r}_n)$$

$$-\gamma\binom{n}{n-1}\left\{\Delta_\omega^{(n-1)}(\mathbf{r}_1, \dots, \mathbf{r}_{n-1})\,P(\mathbf{u}_n)\right\}_s$$

$$+(-\gamma)^{n-2}\binom{n}{1}\left\{\Delta_\omega^{(2)}(\mathbf{r}_1\mathbf{r}_2)\,P(\mathbf{u}_3)\dots P(\mathbf{u}_n)\right\}_s$$

$$+(-\gamma)^{n-1}\binom{n}{2}\left\{\omega(\mathbf{r}_1)\,P(\mathbf{u}_2)\dots P(\mathbf{u}_n)\right\}_s + (-\gamma)^n P(\mathbf{u}_1)\dots P(\mathbf{u}_n),$$

where the following notation [9] is used:

$$\Delta_\omega^{(n)}(\mathbf{r}_1, \dots, \mathbf{r}_n) \equiv \omega(\mathbf{r}_1)\,[\,\omega(\mathbf{r}_2) - \delta(\mathbf{r}_1-\mathbf{r}_2)\,]\dots \tag{3.2}$$

$$\times\,[\,\omega(\mathbf{r}_n) - \delta(\mathbf{r}_1-\mathbf{r}_n) - \dots - \delta(\mathbf{r}_{n-1} - \mathbf{r}_n)\,].$$

This is called the factorial of marked r.d.f. In [11] it is shown that

$$\langle \Delta_\omega\,(\mathbf{r}_1, \dots, \mathbf{r}_n) \rangle = F_n(\mathbf{r}_1, \dots, \mathbf{r}_n) \tag{3.3}$$

which allows one to calculate the ensemble averages of the products of the generalized Charlier polynomials of compound Poisson function and to show that they are centred random variables and orthogonal in a stochastic sense, namely:

$$\langle C_\omega^{(n)} \rangle = 0, \tag{3.4}$$

$$\langle C_\omega^{(n)}\, C_\omega^{(k)} \rangle = 0 \ \text{ for } \ n \neq k \tag{3.5}$$

and

$$\langle C_\omega^{(n)}(\mathbf{r}_1, \dots, \mathbf{r}_n)\, C_\omega^{(n)}(\mathbf{s}_1, \dots, \mathbf{s}_n) \rangle$$

$$= n!\,\gamma^n \left\{ \langle C_\omega^{(1)}(\mathbf{r}_1)\, C_\omega^{(1)}(\mathbf{s}_{i_1}) \rangle \dots \langle C_\omega^{(1)}(\mathbf{r}_1)\, C_\omega^{(1)}(\mathbf{s}_{i_1}) \rangle \right\}_s$$

$$= n!\,\gamma^n \left\{ \delta(\mathbf{r}_1 - \mathbf{s}_{i_1}) \dots \delta(\mathbf{r}_n - \mathbf{s}_{i_n}) \right\}_s .$$

Turning to the general compound functions one finds that the series cannot be rendered orthogonal by means of Charlier polynomials, but we mention here that it is still convenient to use them for reasons that go beyond the frame of the present work. Let us denote for the sake of brevity

$$M^{(2)}(\mathbf{r}, \mathbf{s}) \equiv \langle C_\omega^{(1)}(\mathbf{r})\, C_\omega^{(1)}(\mathbf{s}) \rangle \tag{3.6}$$

$$= \gamma\, P(\mathbf{u})\,\delta(\mathbf{r}-\mathbf{s}) + \gamma^2 P(\mathbf{u})\,P(\mathbf{v})\ [\,Q_2(\mathbf{r}, \mathbf{s}) - 1\,],$$

$$M^{(3)}(\mathbf{r}, \mathbf{s}, \mathbf{t}) \equiv \langle C_\omega^{(1)}(\mathbf{r})\, C_\omega^{(1)}(\mathbf{s})\, C_\omega^{(1)}(\mathbf{t}) \rangle \tag{3.7}$$

$$= \gamma\, P(\mathbf{u})\,\delta(\mathbf{r}-\mathbf{s})\,\delta(\mathbf{r}-\mathbf{t}) + 3\gamma^2 \left\{ P(\mathbf{u})\,P(\mathbf{w})\,\delta(\mathbf{r}-\mathbf{s})\ [\,Q_2(\mathbf{r}, \mathbf{s}) - 1\,] \right\}_s$$

$$+\gamma^3 P(\mathbf{u})\,P(\mathbf{v})\,P(\mathbf{w})\,[\,Q_3(\mathbf{r}, \mathbf{s}, \mathbf{t}) - Q_2(\mathbf{r}, \mathbf{s}) - Q_2(\mathbf{s},\mathbf{t}) - Q_2(\mathbf{t},\mathbf{r}) + 2\,],$$

where $\mathbf{s} = (y^1, \dots, y^m, v^1, \dots, v^\mu)$, $\mathbf{t} = (z^1, \dots, z^m, w^1, \dots, w^\mu)$ are vectors of the type of $\mathbf{r}$ and, $\mathbf{v}, \mathbf{w}$ are other notations for the marks. The formulae for the multivariate moments in the genral compound case are rather cumbersome and for this reason are not cited here. They are obtained from the summed products of $M^{(2)}$ and $M^{(3)}$, etc.

Consider now the following fucntional of marked r.d.f.

$$G^{(n)}[\omega] = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \int_{\mathbb{U}} \cdots \int_{\mathbb{U}} K^{(n)}(\mathbf{x}-\mathbf{x}_1, \dots, \mathbf{x}-\mathbf{x}_n; \mathbf{u}_1, \dots, \mathbf{u}_n) \tag{3.8}$$

$$\times\, C_\omega^{(n)}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{u}_1, \dots \mathbf{u}_n)\, d^m \mathbf{y}_1 \dots d^m \mathbf{y}_n\, d^\mu \mathbf{u}_1 \dots d^\mu \mathbf{u}_n ,$$

where $C_{\omega}^{(n)}$ are generealized Charlier polynomials of nth order, and $K^{(n)}$ are non-random kernels. Following [37] we call $G_{\omega}^{(n)}$ Wiener G-functional. For the case of the compound Poisson function the functionals (3.24) are centred stochastic variables and orthogonal in a statistical sense: $(\langle G^{(n)}[\omega] G^{(k)}[\omega] \rangle = 0$ for $k \neq n)$. For the general case, that does not hold, but employing the generalized Charlier polynomials reduces the complexity of the manipulations.

Each continuous functional $F[\omega(.)]$ of marked r.d.f. $\omega(\mathbf{r})$ can be expanded into a Volterra-Wiener functional series

$$F[\omega(.); \mathbf{x}] = \sum_{n=0}^{\infty} G_F^{(n)}[\omega]. \tag{3.9}$$

The series (3.9) contain the full statistical information about the random function $F(\mathbf{x})$ and allows one to calculate all averaged characteristics: correlations, spectra, etc. One of the most important properties of Volterra-Viener series is the viriality which means that a G-functional of nth order contributes to the averaged characteristics a quantity of order of $\gamma^{\tilde{n}}$, where $\tilde{n} \geq n$ and γ is the mean number of random points. The latter possesses a dimension and in the three-dimensional case is rendered dimensionless by the mean volume V of a structure introducing the "volume fraction" of structures $c = V\gamma$. The characteristic measures of a structure are not prior defined and the asymptotic truncation of the series with the small parameter $c \leq 1$ bears heuristic character.

The procedure for identification of the kernels (Fourier coefficients) $K^{(n)}$ from function $F(\mathbf{x})$ is developed in [7, 8, 15 and 14] and consists in multiplying by the respective Charlier polynomials $C_{\psi_\omega}^{(1)}(0)$, $C_{\psi_\omega}^{(2)}(0, \zeta)$, ... , and taking an ensemble average to obtain the required equations for the first, second, etc., kernels.

4 Poisson case. Application to Burgers turbulence

There are indications that even in such simple dynamic system as the Burgers equation [4]:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

the non-linearity shows itself through transforming the random initial condition into a random train of coherent structures. It is interesting to compare the predictions of the present method to results of numerical simulations of initial-condition Burgers turbulence [18, 39].

The problem in applying the random point approximation here is that the intensity γ (the number of structures per unit length) is a function of time, as a result of the dissipation of solution. Hence a similar solution is sought and the variables are scaled in such a way as to be homogeneous. So we introduce the transformation

$$t = t,\ \chi = \frac{x}{\sqrt{\nu t}},\quad u = \sqrt{\frac{\nu}{t}}\, K(\chi, t), \tag{4.2}$$

and substituting into (4.1) obtain

$$t\frac{\partial K}{\partial t} - \frac{1}{2}(K + \chi\frac{\partial K}{\partial \chi}) + K\frac{\partial K}{\partial \chi} = \frac{\partial^2 K}{\partial \chi^2}. \tag{4.3}$$

Unlike the solution of (4.1) the amplitude of the solution of (4.3) does not decrease with time and one can assume that certain steady and homogeneous (with respect to the spatial variable) random process is attained for large times. Hence we confine ourselves here only to the stationary case and neglect in (4.3) the time derivative. The solution then can be approximated by a Volterra-Wiener series with respect to the random density function

$$\psi(\xi) = \sum_{\alpha} \delta(\xi - \chi_\alpha),\quad \varepsilon = \gamma\sqrt{\nu t} \tag{4.4}$$

whose intensity $\varepsilon = \langle \psi(\xi) \rangle$ is dimensionless independent of time quantity. Consider now the Poisson random function. Then the series takes the form

$$K(\chi) = B_0 + \int_{-\infty}^{\infty} B_1(\chi-\xi)\, C_{\psi}^{(1)}(\xi)\, d\xi \tag{4.5}$$

$$+ \int\int_{-\infty}^{\infty} B_2(\chi-\xi_1, \chi-\xi_2)\, C_{\psi}^{(2)}(\xi_1, \xi_2)\, d\xi_1\, d\xi_2 + \dots,$$

where $C_{\psi}^{(n)}$ are the respective Charlier polynomials. Without losing the generality we consider a centred random function for which $B_0 \equiv \langle K \rangle = 0$. Following the general procedure of kernel identification (see Section 3) and retaining for the sake of simplicity only the first two kernels B_1 and B_2 we obtain:

$$\varepsilon[-\frac{1}{2}(B_1 + \chi\frac{dB_1}{d\chi}) + B_1\frac{dB_1}{d\chi} - \frac{d^2B_1}{d\chi^2}] \tag{4.6}$$

$$= 2\varepsilon^2 \frac{d}{d\chi} \int_{-\infty}^{\infty} B_2(\chi,\zeta)\,[B_1(\zeta) + B_2(\chi,\zeta)]\,d\zeta + O(\varepsilon^3).$$

$$\varepsilon^2 \Big\{ -\frac{1}{2}[B_2(\chi,\chi-\zeta) + \chi \frac{\partial B_2(\chi,\xi-\zeta)}{\partial\chi}] \qquad (4.7)$$

$$+ 2B_2(\chi,\chi-\zeta)\frac{\partial}{\partial\chi}[B_1(\chi) + B_1(\chi-\zeta)] + \frac{\partial}{\partial\chi}[B_1(\chi)B_1(\chi-\zeta)]$$

$$+ 2\frac{\partial}{\partial\chi}B_2^2(\chi,\chi-\zeta) - 2\nu\frac{\partial^2}{\partial\chi^2}B_2(\chi,\chi-\zeta)\Big\} = O(\varepsilon^3).$$

It is interesting to mention here that the last equation is effectively one-dimensional since it contains no derivatives of function B_2 with respect to the second argument ξ and the latter plays the role of a parameter.

System (4.6), (4.7) is to be solved under the requirement that the kernels are with summable squares over the infinite region which ensures, in the frame of random point approximation, the finiteness of the energy of the random solution. Therefore

$$\int_{-\infty}^{\infty} B_1^2(\chi)\,d\chi < +\infty, \qquad \int_{-\infty}^{\infty}\!\!\int B_2^2(\chi,\zeta)\,d\chi\,d\zeta < +\infty. \qquad (4.8)$$

It is easily shown that under the natural assumption of smoothness of the kernels (4.8) yield the following

$$B_1(\chi) \to 0, \quad \chi \to \pm\infty, \qquad (4.9a)$$

$$B_2(\chi,\zeta) \to 0, \quad \chi \to \pm\infty; \ \forall\zeta, \quad \zeta \to \pm\infty; \ \forall\chi. \qquad (4.9b)$$

which means that a localized solution is sought. The physical interpretation of the kernels is now obvious. As far as the first kernel satisfies the original equation it is in fact the shape of a single coherent structure from the random train. The second kernel is respectively the contribution of the interaction of the solitons. Finally, the term containing B_2 in (4.7) accounts for the influence of that interaction on the very shape of a particular coherent structure.

The boundary value problem (4.7), (4.9) is closed for the two unknowns - the kernels B_1 and B_2 - and contains the underlying idea of the present approach - to truncate the cascade system for kernels, not for the moments (cumulants). Unlike the Wiener-Hermite expansions the truncated versions of Poisson-Wiener expansion are "non-linearly closed" in a sense that the equation for B_1 contains B_1^2. This is a very important property and the solution to the first equation with $B_2 = 0$ is [7]:

$$B_1(\chi) = -\frac{4\chi}{2+\chi^2}, \text{ i.e., } u(x,t) = -\frac{4\nu x}{2\nu t + x^2}. \qquad (4.10)$$

This solution for the kernel together with the assumption that the random density function is Poisson allows one to calculate the statistical characteristics and to compare them with the numerical experiment. In Figure 1 is shown the comparisons with the numerical experiment of [18], [39] which virtually coincide with each other for the correlation coefficient. The agreement is very good. In Figure 2 is presented the third-order normalized correlation function, compared to the numerical simulation [39]. The third-order cumulant is quite a specific quantity and the very good comparison speaks in favour of random point approximation. Then in Figures 3 and 4 are depicted two different normalized fourth-order two-point moments. The second one, which stems from the product of squares of velocities in the two respective points, shows a significant deviation from the numerical simulation [39] and this shows the deviation from the Poisson behaviour in the higher orders of statistical interconnection. Figure 5 shows the spectrum for which the comparison is good.

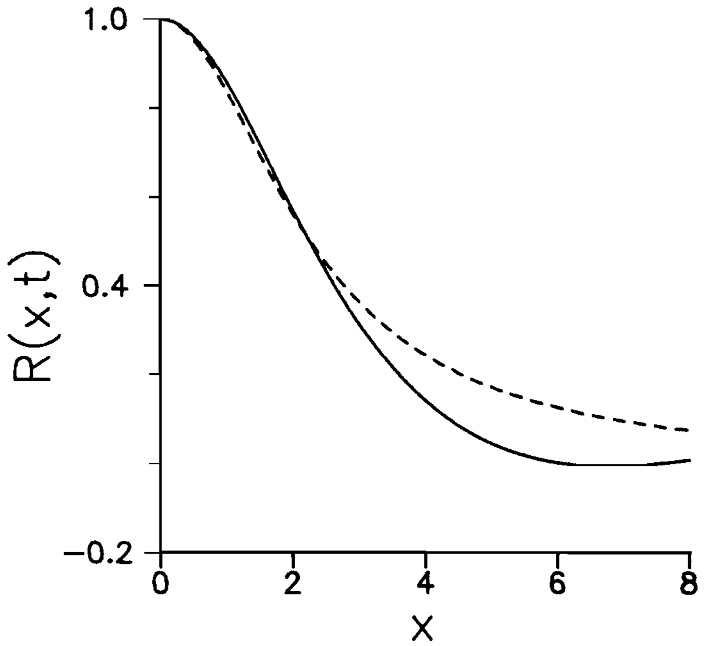

Figure 1. Correlation funciton of Burgers turbulence: - - - random point approximation; ——— numerical simulation [39].

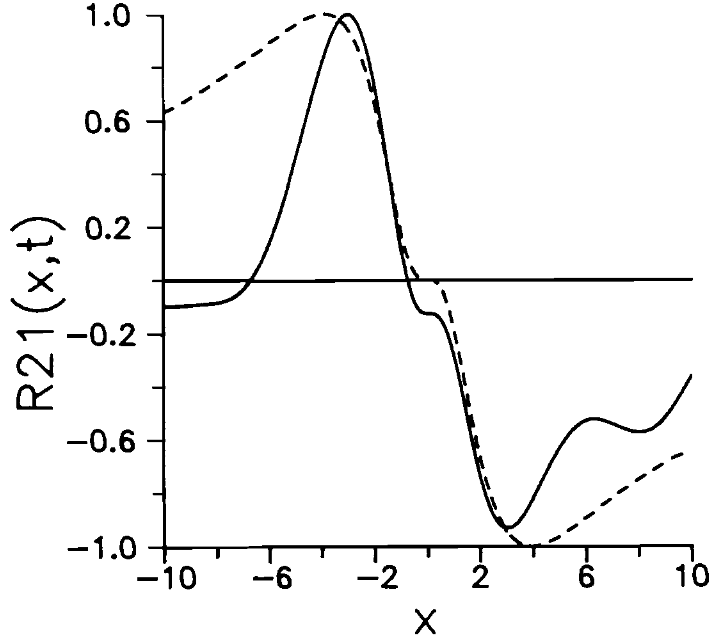

Figure 2. Two point third-order correlation function $R_{21} = \langle u^2(x+\xi)u(\xi) \rangle$ (for legend, see Figure 1.)

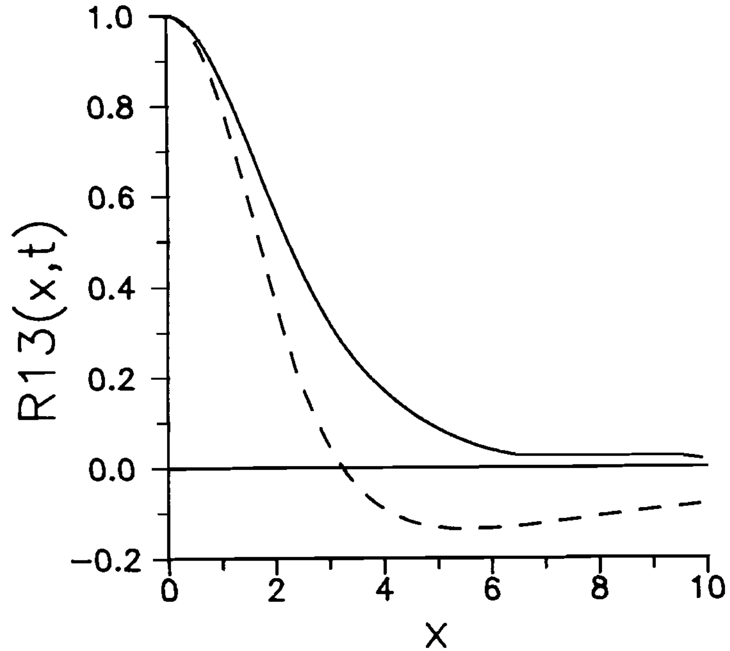

Figure 3. Two point fourth-order correlation function $R_{13} = \langle u(x+\xi)u^3(\xi) \rangle$ (for legend, see Figure 1.)

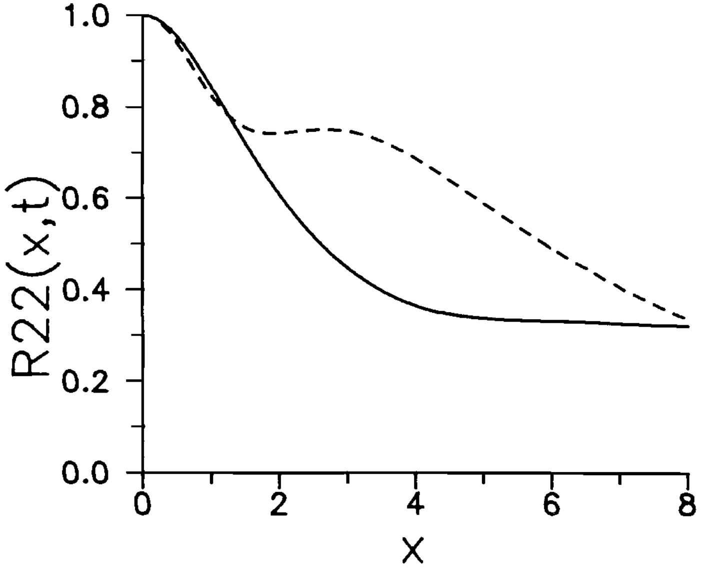

Figure 4. Two point fourth-order correlation funciton $R_{22} = \langle u^2(x+\xi)u^2(\xi) \rangle$ (for legend, see Figure 1.)

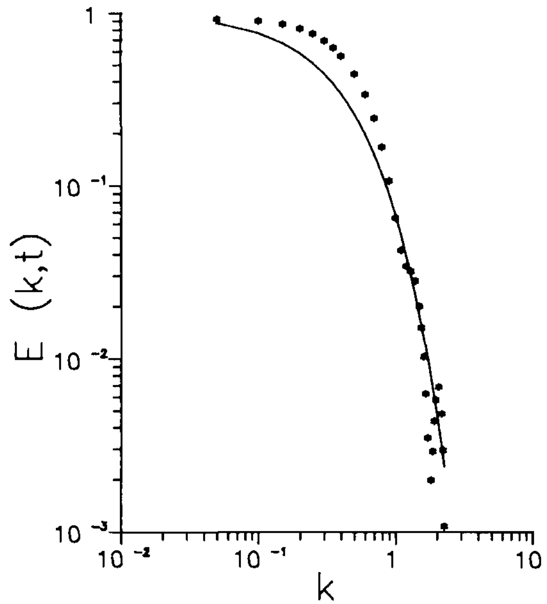

Figure 5. The spectrum of Burgers turbulence: ——— random point approximation; * * * numerical simulation [39].

It is interesting to note that assuming a Poisson random train for the experimental data one can identify the shape of a single structure which is shown in Figure 6. The comparison with the analytic solution (4.10) is impressive.

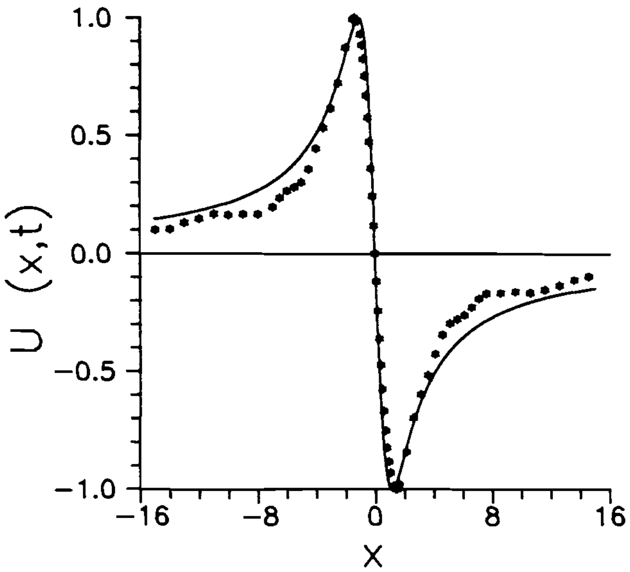

Figure 6. The identified shape of a single coherent structure for Burgers turbulence (for legend, see Figure 5.)

The conclusion of the present section is that even the simplest random point function (Poisson) and roughest approximation (first order) are capable of good quantitative prediction of multipoint statistic characteristics for moderate separation of points.

5 Lorenez attractor

One of the simplest and most studied examples of stochastic behaviour of a deterministic non-linear system is the system of ODEs originally investigated by Lorenz [24] in connection with the free convection in a horizontal layer. In terms of dimensionless variables it has the form

$$\dot{x} = -\sigma(x-y), \quad \dot{y} = -xz + rx - y, \quad \dot{z} = xy - bz \,, \tag{5.1}$$

where r is the Rayleigh number, σ is Prandtl number and b is a combination of the characteristic lengths of the waves in the horizontal directions. Lorenz studied the evolution of the attractors of (5.1) with increase of r for given $\sigma = 10$ and $b = 8/3$ and showed that for $r > 24.74$ all three stationary points of the system became unstable. Furthermore, his numerical solution to (5.1) apparently went random for $r = 28$, but the trajectories were attracted to a certain lower-dimension manifold, which was later on named "strange attractor" [36]. In [19] it was discovered that the random solution of Lorenz system exhibits random behaviour even for Rayleigh numbers as low as $r = r_{cr} = 13.926$ when the homoclinic orbit is born and then it has

the shape of a random train of homoclinics which obviously is a random point function. Unfortunately, this random solution is metastable and eventually degenerates into periodic orbits. Therefore it cannot serve as a basis for comparisons when investigating the properties of the random point regimes. It becomes necessary to conduct a precise numerical simulation to answer the question of whether the regime for higher r does resemble the shape of the random point function. A couple of different schemes (including the original Lorenz one) of different order of approximation are employed and it is found that the average results vary insignificantly, i.e., reliable "experimental" data are gathered that can be used for comparison to the random point predictions.

Taking advantage of the simple shape of the solution and defining the centre of a structure as the point of maximum of the third unknown z the random times of occurrence of a structure are identified for each particular realization [12]. An ensemble of 120,000 realizations is compiled and then the mean number of structures per unit time γ and the two-point probability density f_2 of the system of random centres of the structures are calculated. It appears that $\gamma = 1.327$. In Figure 7 the result obtained is shown in terms of the normalized quantity $Q = \gamma^{-2} f_2$. It is seen (as it should have been expected for a random process) that for very large separation distances Q approaches unity. Still the rate of convergence is rather slow and a train of steep local maxima take place. All this means that though the simulated solution is of no doubt stochastic it is still in a sense quasi-periodic. Nevertheless it can serve as a featuring example of the method proposed here. Simultaneously, the time-averaged quantities are calculated.

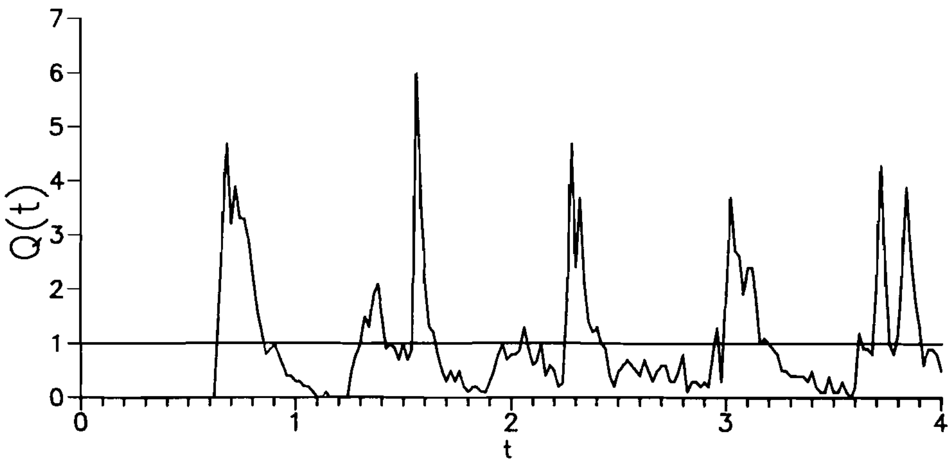

Figure 7. The two-point probability distribution funciton for Lorenz system obtained from numerical experiment [12].

We seek now for a marked random point solution

$$x(t) = \sum_\alpha a_\alpha x_h(t-t_\alpha), \quad y(t) = \sum_\alpha a_\alpha y_h(t-t_\alpha), \tag{5.2}$$

$$z(t) = \sum_\alpha z_h(t-t_\alpha),$$

where the mark a is a random variable which can take only two different values: +1 or -1. This specific set of values is hinted by the symmetry of (5.1) and as far as there exist no particular reasons to prefer the positive or negative solution for x and y, the probability density of a should be symmetric:

$$P(a) = 0.5\,\delta(a-1) + 0.5\,\delta(1+a). \tag{5.3}$$

We assume that the first condition for perfect disorder (2.6) holds and neglect as well the higher-order terms in the Volterra-Wiener series, i.e., the plain random point approximation is considered. Because of its bifurcation nature the boundary value problem is hard to tackle even for the first-order kernel. Substituting (5.2) into (5.1) and applying the above presented machinery we arrive at the following closed system for the first-order kernels:

$$\dot{x}(t) = -\sigma(x_h - y_h),$$

$$\dot{y}(t) = -z_h x_h + r x_h - y_h - \gamma x_h(t) \int_{-\infty}^{\infty} z_h(t-\tau)\, Q(\tau)\, d\tau, \tag{5.4}$$

$$\dot{z}(t) = x_h y_h - b z_h - \gamma \int_{-\infty}^{\infty} [\,\dot{z} + b z_h - x_h y_h\,]_{(t-\tau)}\, Q(\tau)\, d\tau,$$

and the solution is sought among the processes with finite energy which, similarly to the Burgers equation, renders the problem inverse and requires special numerical approaches. In [10] we developed a new method called "variational imbedding" consisting in replacing the original incorrect boundary value problem with a correct one for the Euler-Lagrange equations for minimizing the quadratic functional of the original set of equations. The algorithm implementing the method of variational embedding was verified solving the homoclinic problem for the original Lorenz system obtained in [10]; the homoclinic solution compares quantitatively very well with that reported in [40].

For the case under consideration (r = 28) the method of variational imbedding solves the problem of least square approximation with the Poisson random point function and provides thus an opportunity, on some heuristic basis, to assess the localized solution for higher Rayleigh numbers r.

In Figure 8 we show the obtained least square approximation to the localized solution of (5.4) which is in fact the "coherent structure" for this simplest case. In Figure 9 are presented the correlation coefficients of the first two unknowns, x and y, calculated from the Poisson train of coherent structures of above-described shape for the said r = 28. This exhibits an excellent agreement with the numerical experiment which means that the shape in Figure 8 is quite close to the real one. In order to verify this finding the approximations for the coherent structures are obtained also for r = 51 and r = 100. The respective comparisons with the numerical simulation

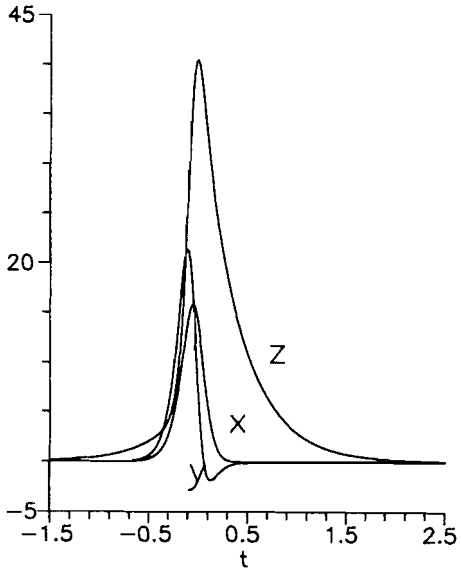

Figure 8. The shape of coherent structure for Lorenz system for r = 19 [10].

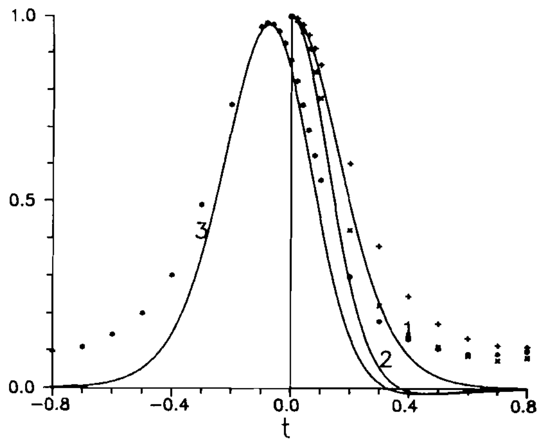

Figure 9. Predicted from the random point approximation correlation coefficients for r = 28: and comparison with numerical experiment [11] (1) + + + - $\langle x(t+\tau)x(\tau)\rangle$; (2) × × × - $\langle y(t+\tau)y(\tau)\rangle$; (3) * * * - $\langle x(t+\tau)y(\tau)\rangle$.

are presented in Figures 10 and 11. The quality of agreement is essentially the same as that which confirms the conclusion of the previous section. Significantly worse is the case with the third unknown z for which the two different numerical simulations [11, 25] suggest a non-monotonic shape of the correlation while the plain Poisson

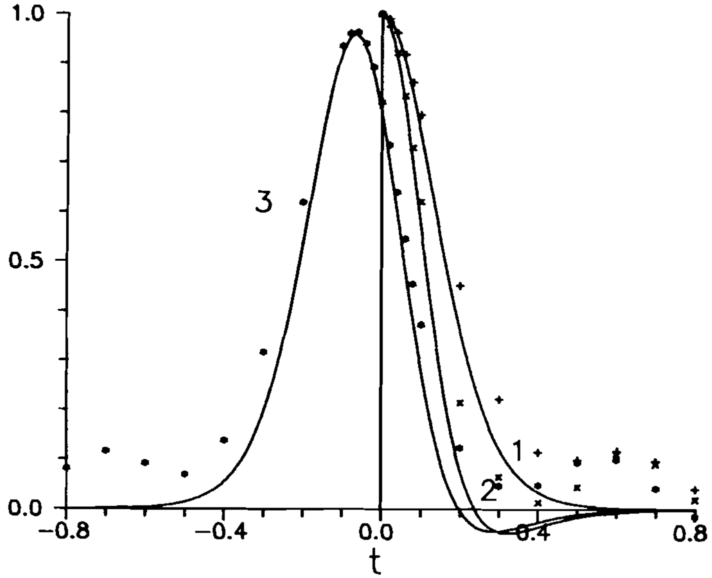

Figure 10. Predicted from the random point approximation correlation coefficients for $r = 51$ (for legend, see Figure 9).

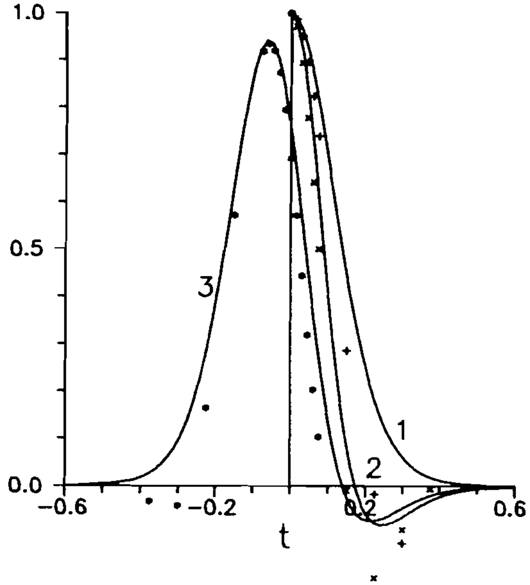

Figure 11. Predicted from the random point approximation correlation coefficients for $r = 100$ (for legend, see Figure 9).

random point approximation predicts a monotonic one quite similar to those in Figure 9 to 11. This discrepancy is resolved to a certain extent if the experimental two-point probability density from Figure 7 is used in conjunction with the structure shape of z from Figure 8 when calculating the correlation of the random point function z (see Figure 12). The failure of the simple Poisson method is attributed to the fact that the structures significantly overlap each other, at least when the third funciton z is considered, and then the deviations from Poisson behaviour of the random function are noticeable.

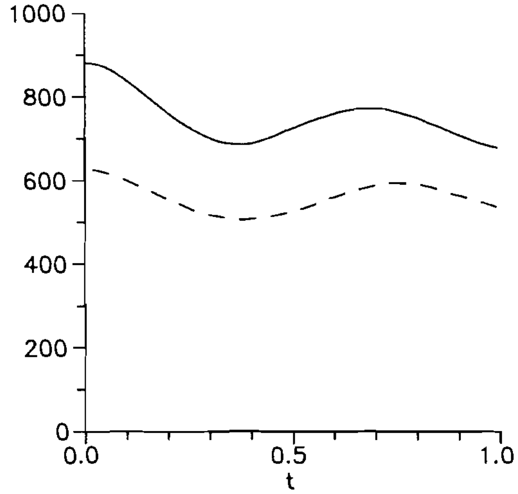

Figure 12. The second moment $M_z^2 = \langle z(t+\tau)z(\tau) \rangle$ of third unknown for $r = 28$ with the two-point probability density incorporated: —— random point prediction; - - - numerical simulation [11].

The conclusion of this section is that the Poisson random point approximation is capable of good quantitative prediction for the average characteristics and the correlations with moderate separation of arguments even in more complicated systems with strong deviations from the Poisson pattern. In a sense it can serve for the express assessment of the structural regimes of a non-linear system and only when more sophisticated statistical characgeristics are required must one use more complicated models.

6 Conclusions

It is argued that the onset of chaos in determinsitic systems due to instability yields regimes that are adequately represented by random point functions. The Volterra-Wiener functional expansion is employed with a basis function which is the compound random density fucntion generated by the system of marked random points. The way of obtaining the infinite hierarchies for kernels is outlined and the problems of truncation are discussed. The considered functional series are virial and each higher-order term contributes to the average characteristics a quantity proportional to the respective power of the dimensionless number of points (concentration) per unit volume (area, length). It gives an asymptotically strict procedure for truncating the hierarchy and the nth order kernels are interpreted as n-tuple interactions among the inclusions (or coherent structures).

The performance of the new technique is displayed on two typical examples: Lorenz system and Burgers equation. The Poisson limiting case for the statistics of basis function is considered. The statistical characteristics (multipoint moments) predicted from the random point approximation for moderate separation of points are in good quantitative agreement with the available numerical simulations for both systems. For the Lorenz system we have sketched the way in which an information about multipoint probability densities of the generating system of random points can be acknowledged. The random point approximation proves to be a new, capable tool for the quantitative modelling of the turbulent behaviour of non-linear systems.

The random point approximation as an approach to decoupling the infinite hierarchies of statistical characteristics has a major advantage in comparison with the other techniques, because it gives all the multipoint mometns of arbitrary order, i.e., it gives a complete though approximate statistical description of the field, which is impossible for semi-empirical theories, for example.

Acknowledgements

This work was supported in part by Ministry of Science and Higher Education of Bulgaria under Grant 1052.

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