SIAM J. SCI. STAT. COMPUT.
Vol. 11, No. 4, pp. 631-647, July 1990

003

A FOURIER-SERIES METHOD FOR SOLVING SOLITON PROBLEMS*

C. I. CHRISTOV† AND K. L. BEKYAROV‡

Abstract. A Fourier–Galerkin method with an earlier proposed complete orthonormal system of functions in $L^2(-\infty, \infty)$ as the set of trial functions is developed and displayed for the problem of calculating the shape of the one-soliton solution of the Korteweg–de Vries equation. The convergence of the method is investigated through comparison with the analytic solution, which appears to be very good. The truncation and discretization errors are assessed pointwise. The technique developed is also applied to the soliton problem for the so-called Kuramoto–Sivashinsky equation and the obtained soliton shape is compared to the existing difference solution. The quantitative agreement between the Fourier-series-method result and the numerical one is good. In the present paper, however, the soliton solution is obtained for a significantly wider range of phase velocities, which suggests that the spectrum might be continuous. The new technique can also be applied to a variety of other problems involving identification of homoclinic solutions.

Key words. solitons, spectral methods, Fourier–Galerkin method, Korteweg–de Vries, Kuramoto–Sivashinsky

Introduction. In recent years, the problem of calculating shapes of solitons has attracted considerable attention due to its application in different fields of modern physics (see, e.g., [1], [2]). At this time the available techniques for calculating solitons lack generality and, as a rule, bear semianalytical character. Each of these techniques proves effective only for the particular class of equations for which it is devised. In turn the numerical approaches based on straightforward difference approximations are faced with formidable challenges. The first one is rooted in the inverse nature of the boundary value problem in an unbounded region, forcing us to employ shooting procedures that are intrinsically highly unstable. One of the effective means of overcoming that difficulty appears to be the method of variational imbedding [3], [4] based on rendering the original inverse problem to a higher-order but correct boundary value problem. The second challenge is connected with the choice of the "actual" infinity for the difference approximations and often shows itself through the occurrence of artificial eigenvalue problems that are not characteristic for the original problem in an unbounded region.

A method free from said shortcoming is that of Fourier-series expansion with respect to certain complete orthonormal (CON) system of functions in $L^2(-\infty, \infty)$ space. However, the governing equations for solitons are, as a rule, nonlinear. For the employed CON system, this places the very stringent requirement of possessing, for the product of two members of the system, a representation in series with respect to the system. It should be noted that for the well-known sets of Hermitian functions in $L^2(-\infty, \infty)$ and Laguerre functions in $L^2[0, \infty)$, such a representation is not available (see, e.g., [5]).

The most systematic way to devise a CON system with the required properties is, perhaps, to map through an algebraic function the infinite interval into $[-1, 1]$ and then to use the CON set of Chebyshev polynomials. This idea was initially sketched

* Received by the editors August 22, 1988; accepted for publication (in revised form) May 31, 1989. This research was supported in part by the Ministry of Culture Science and Education of Bulgaria under grant 1052.

† Laboratory of Hydrometeorological Informatics, Hydrometeorological Service, Bulgarian Academy of Sciences, boul. Lenin 66, Sofia 1184, Bulgaria.

‡ Department of Atmosphere Physics, Institute of Geophysics, Bulgarian Academy of Science, "Akad. G. Bonchev" str. block 3, Sofia 1113, Bulgaria.

by Grosch and Orszag [6] for the semi-infinite interval and was generalized by Boyd [7], who gave the appropriate mapping for the entire interval $(-\infty, \infty)$ map and built the rigorous basis beneath the said technique. Boyd [8] coined the term "rational Chebyshev functions" for the newly devised set of functions (which we prefer to call Boyd functions in order to stress the specifics in the unbounded intervals) and presented a scrupulous analysis of the convergence region of the series with respect to them.

Another system suitable for nonlinear problems in infinite intervals is proposed in the earlier work [9] and developed further in [10]. Unlike the above-mentioned works of Boyd, in the works of Christov and Bekyarov the emphasis is put on the performance of the spectral method in the infinite interval for intrinsically nonlinear problems.

In the present paper the numerical technique for application of the Fourier method based on the CON system from [9] is developed. The method is featured through solving the soliton problem for the Korteweg-de Vries (KdV) equation and for the equation of weakly nonlinear approximation in falling down a wall of thin viscous capillary films (sometimes called the Kuramoto-Sivashinsky equation). Although some authors use the term "soliton" only for special kinds of solitary waves, in the present work this term is used as a synonym for the term "solitary wave."

1. Posing the problem. For the sake of simplicity consider the following form of the Korteweg-de Vries equation (see [2, Chap. 3]):

$$u_t - 6uu_x + u_{xxx} = 0, \tag{1.1}$$

and seek a solution of the type of propagating wave $u = u(\xi)$, where $\xi = x - at$, and $a > 0$ is the phase velocity of the wave. Then (1.1) reduces to

$$-au' - 6uu' + u''' = 0,$$

where the prime stands for a differentiation with respect to the independent variable ξ. We have a soliton solution (a solitary wave) if the following boundary conditions hold:

$$u(\xi) \to 0 \quad \text{for } \xi \to \pm\infty. \tag{1.2}$$

Under these boundary conditions the above ordinary differential equation can be integrated once and rendered to

$$-au - 3u^2 + u'' = 0. \tag{1.3}$$

Thus (1.3) and (1.2) form the boundary value problem to be solved. Fortunately, the latter possesses an analytical solution for each $a > 0$:

$$u = -\frac{a}{2}\operatorname{sech}^2[\sqrt{a/2}\,\xi] \tag{1.4}$$

that can be used for checking the accuracy of the numerical schemes proposed.

Another interesting one-dimensional nonlinear equation of evolution arises in the weakly nonlinear approximation for the shape of the free surface of thin film of viscous liquid falling down a vertical plane when the capillary forces are significant. The rich phenomenology of this flow made it one of the most popular and spurred a formidable amount of research papers. It is not the purpose of the present work to go into the details connected with that flow and we refer the reader to [11] and [12] for a comprehensive review of the experimental and theoretical approaches, respectively. For our purposes it is enough to cite here that in the frame of the weakly nonlinear approximation [13] the following dimensionless equation for evolution of the scaled film thickness φ in a moving frame can be derived [14], [15]:

$$\frac{\partial \varphi}{\partial t}+6\varphi\frac{\partial \varphi}{\partial x}+\frac{\partial^2 \varphi}{\partial x^2}+\frac{\partial^4 \varphi}{\partial x^4}=0. \tag{1.5}$$

The same equation is arrived at in [16] when modeling chemically reacting fronts and flames. Now (1.5) is called the Kuramoto-Sivashinsky equation (for brevity the K-S equation). A thorough investigation of the different regimes is given in [17], where the bifurcation catalogue of the attractors of the K-S equation is compiled on the basis of several hundred numerical experiments.

Consider once again a solution of the type of propagating wave $\varphi=\varphi(\xi)$, $\xi=x-ct$ when (1.5) is rewritten as follows:

$$-c\varphi'+6\varphi\varphi'+\varphi''+\varphi^{(4)}=0$$

and when under the boundary conditions for soliton solutions

$$\varphi(\xi)\to 0 \quad \text{for} \quad \xi\to\pm\infty \tag{1.6}$$

we can integrate once and obtain

$$-c\varphi+3\varphi^2+\varphi'+\varphi'''=0. \tag{1.7}$$

Unlike for the Korteweg-de Vries equation for the last boundary value problem (1.6), (1.7) an analytic solution is not yet available. Rather, different numerical techniques are applied [4], [12], [18], [19] and the shape of the one-soliton solution is calculated fairly reliably, but the question of which is the type of spectrum (continuous or discrete) for the eigenvalue parameter c is still unresolved. Having the Fourier method verified in the case of the KdV equation, we apply it to boundary value problem (1.6), (1.7) and the soliton solution obtained compares very well with the finite-difference solution [4]. The interesting finding here supporting the difference-method results of [4] is that we were able to find the approximate solutions for the soliton problem for each value of celerity c, which we tried up to 20. This allows us to state the hypothesis that the spectrum for which solitons of the K-S equation exist is continuous $0<c<\infty$.

It should be mentioned that we consider only smooth classical solutions to (1.3) or (1.7), which yields that under conditions (1.2) and (1.6) these solutions should belong to the $L^2(-\infty,\infty)$ space. So we occupy ourselves in what follows with solutions for which the following requirement holds:

$$\int_{-\infty}^{\infty}\varphi^2(\xi)\,d\xi<+\infty \quad \text{or} \quad \int_{-\infty}^{\infty}u^2(\xi)\,d\xi<+\infty. \tag{1.8}$$

2. The CON system. Wiener [20, p. 35] introduced the system

$$\rho_n=\frac{1}{\sqrt{\pi}}\frac{(ix-1)^n}{(ix+1)^{n+1}}, \qquad n=0,1,2,\cdots \tag{2.1}$$

as a Fourier transform of Laguerre functions. Higgins [21] defined it also for negative n and proved its completeness and orthogonality. The significance of (2.1) for nonlinear problems is revealed in [9], where the product formula is derived:

$$\rho_n\rho_k=\frac{i}{2\sqrt{\pi}}(\rho_{n+k}-\rho_{n-k}) \tag{2.2}$$

and the two real subsequences of odd functions S_n and even functions C_n are introduced according to the formulae

$$S_n = \frac{(\rho_n + \rho_{-n-1})}{i\sqrt{2}}, \quad C_n = \frac{(\rho_n - \rho_{-n-1})}{\sqrt{2}}, \quad n = 0, \pm 1, \cdots. \tag{2.3}$$

The sequence can be reduced just to its portion with positive values of index n due to the symmetry property:

$$S_{-n} = S_{n-1}, \qquad C_{-n} = -C_{n-1}. \tag{2.4}$$

The explicit expressions for S_n and C_n can be found in [9]. A simple representation in terms of trigonometric functions has been brought to our attention by Geshev [22]:

$$\begin{aligned} S_n(x) &= (-1)^{n+1} \frac{\sin (n+1)\vartheta + \sin n\vartheta}{\sqrt{2}}, \\ C_n(x) &= (-1)^n \frac{\cos (n+1)\vartheta + \cos n\vartheta}{\sqrt{2}}, \end{aligned} \tag{2.5}$$

where $\vartheta = 2 \operatorname{arctg}(x)$. These and other formulae interrelating the CON system employed here to the different families of Boyd's functions are presented in [23].

Most of the practically important formulae for the system (2.3) are compiled in [9] and [10], and here we cite only those that are necessary for carrying out the present calculations.

The most important feature of the system, namely, equality (2.2) for the real-valued subsequences S_n, C_n adopt the form

$$\begin{aligned} S_n S_m &= \sum_{k=0}^{\infty} \alpha_{nmk} C_k, \\ \alpha_{nmk} &= \frac{1}{2\sqrt{2\pi}} \left\{ \delta_{k,n+m+1} - \delta_{k,n+m} + \delta_{k,|n-m|} - \operatorname{sgn}\left[|n-m| - \frac{1}{2} \right] \delta_{k,[|n-m|-1/2]} \right\}, \end{aligned} \tag{2.6.SS}$$

where $\delta_{i,j}$ is Kronecker delta and $[\cdot]$ stands for the integer part of a real. Respectively,

$$\begin{aligned} C_n C_m &= \sum_{k=0}^{\infty} \beta_{nmk} C_k, \\ \beta_{nmk} &= \frac{1}{2\sqrt{2\pi}} \left\{ -\delta_{k,n+m+1} + \delta_{k,n+m} + \delta_{k,|n-m|} - \operatorname{sgn}\left[|n-m| - \frac{1}{2} \right] \delta_{k,[|n-m|-1/2]} \right\}, \end{aligned} \tag{2.6.CC}$$

and

$$\begin{aligned} S_n C_m &= \sum_{k=0}^{\infty} \gamma_{nmk} S_k, \\ \gamma_{nmk} &= \frac{1}{2\sqrt{2\pi}} \{ -\delta_{k,n+m+1} + \delta_{k,n+m} + \operatorname{sgn}(n-m) \delta_{k,|n-m|} - \operatorname{sgn}(n-m) \delta_{k,|n-m|-1} \}. \end{aligned} \tag{2.7.SC}$$

In the same manner the formulae representing derivatives of a member of the system into series with respect to the system are derived:

$$C'_n = \sum_{m=0}^{\infty} \theta_{nm} S_m, \qquad S'_n = -\sum_{m=0}^{\infty} \theta_{nm} C_m, \tag{2.8a}$$

where

$$\theta_{mn}=\frac{n}{2}[\delta_{m,n}-\delta_{m,n-1}]-\frac{n+1}{2}[\delta_{m,n+1}-\delta_{m,n}].$$

(2.8b)
$$C''_n=\sum_{m=0}^{\infty}\chi_{nm}C_m,\qquad S''_n=\sum_{m=0}^{\infty}\chi_{nm}S_m,$$

where

$$\chi_{nm}=-\frac{1}{4}n(n-1)\delta_{m,n-2}+n^2\delta_{m,n-1}-\frac{n^2+(2n+1)^2+(n+1)^2}{4}\delta_{n,m}$$
$$+(n+1)^2\delta_{m,n+1}-\frac{1}{4}(n+1)(n+2)\delta_{m,n+2}.$$

(2.8c)
$$C'''_n=\sum_{m=0}^{\infty}\varphi_{nm}S_m,\qquad S'''_n=-\sum_{m=0}^{\infty}\varphi_{nm}C_m,$$

where

$$\varphi_{nm}=\frac{1}{8}n(n-1)(n-2)\delta_{m,n-3}-\frac{3}{8}n(n-1)(2n-1)\delta_{m,n-2}$$
$$+\frac{3}{8}n(5n^2+1)\delta_{m,n-1}+\frac{3}{8}(n+1)[5(n+1)^2+1]\delta_{m,n+1}$$
$$-\frac{4n^3+4(n+2)^3+(2n+1)[n^2+(2n+1)^2+(n+1)^2]}{8}\delta_{m,n+1}$$
$$-\frac{3}{8}(n+1)(n+2)(2n+3)\delta_{m,n+2}+\frac{1}{8}(n+1)(n+2)(n+3)\delta_{m,n+3}.$$

3. Fourier series for the Korteweg–de Vries equation. In order to display the performance of the Fourier-series method for solving soliton problems, we begin with the Korteweg-de Vries equation. Since the sole purpose of the present work is to check the applicability of the new CON system to the case of intrinsically nonlinear problems, we are not concerned with the problem of which of the spectral techniques will work most efficiently: Galerkin, collocation, or tau version (see [24] for a comprehensive review of the spectral methods), and as so we do not create different algorithms to implement each of them for the purposes of comparison. Rather we choose the Galerkin scheme, because in our case it is cheaper in implementation due to the fact that the matrix is sparse. In fact, the part of the matrix that is responsible for the linear terms is a band matrix with seven nontrivial diagonals (see (2.8c)), whereas the part responsible for the nonlinear terms has only four nontrivial diagonals (see (2.6) and (2.7)) that are not adjacent, however. It is clear that for equations with more complicated nonlinear terms it is better to use the collocation (pseudospectral) technique, since the latter is much easier to program and the respective algorithms are easily verified. The only reason to decide against it in the present paper is that the pseudospectral method always converts the differential equation into a system of nonlinear algebraic equations with dense matrix. So we resort here to the Fourier–Galerkin method.

It is easily shown that (1.3) admits even functions as solutions and hence we develop the solution to be u into series only with respect to the even subsequence of functions C_n, namely,

(3.1)
$$u(x)=\sum_{n=0}^{\infty}a_nC_n(x).$$

Then for the terms entering (1.3), we obtain

$$u''(x)=\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}a_m\chi_{mn}C_n(x) \tag{3.2}$$

and

$$u^2(x)=\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n=0}^{\infty}a_{m_1}a_{m_2}\beta_{m_1m_2n}C_n(x). \tag{3.3}$$

Since for the Galerkin method the sets of trial and test functions coincide with the set C_n, then upon introducing (3.1)-(3.3) into (1.3), combining the terms with the like functions C_n, and taking the respective coefficients to be equal to zero (due to the independence of members of subsequence C_n and its completeness in the subspace of even $L^2(-\infty,\infty)$ functions), we obtain the following infinite nonlinear algebraic system for the unknown coefficients a_n:

$$-a\cdot a_n-3\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}a_{m_1}a_{m_2}\beta_{m_1m_2n}+\sum_{m=0}^{\infty}a_m\chi_{mn}=0,\qquad n=0,1,2,\cdots. \tag{3.4}$$

For the sake of definiteness we take $a=1$, which does not cause a loss of generality, since substituting $u(x)=av(ax)$ we can exclude the parameter a from the equation.

It is obvious that we can solve only finite versions of (3.4) and the number N of unknown coefficients at which the vector $\mathbf{a}=\{a_n\}$ is truncated corresponds to the number of equations of (3.4) that are retained. The chief purpose of the present work is to show the efficiency of the Fourier-Galerkin method with the CON basis set from [9] for nonlinear problems, as is done in [23] for the case of linear equation with polynomial coefficients. The sole trait of efficiency of a spectral method is the capability to give good approximation with a sufficiently small number $(N+1)$ of functions, used in the series (see, e.g., [24, Chap. 1] and this will be the central issue of what follows.

For this reason we are not concerned here with the problems of the particular numerical implementation of the procedure for solving the nonlinear algebraic system representing the truncated version of (3.4). These problems require a detailed treatment if they are to be tackled and this goes far beyond the framework of the present paper. For now we need only a sufficiently rapid robust procedure for solving nonlinear systems. Moreover, here we solve only one-dimensional problems when the required computational time is small. We found satisfactory the pseudo-Newton's widely used algorithm of Brent (see [25]). The problem of efficiency of the numerical procedure shall, however, inevitably arise when multidimensional soliton problems yielding vast algebraic systems are to be considered.

The general consequence of the algorithm is as follows:

(i) We begin with the case $N=0$ when the system reduces to just one equation for the unknown a_0, which has two solutions

$$a_0^{(1)}=0 \quad\text{and}\quad a_0^{(2)}=-0.835543.$$

Here is shown the bifurcation character of the problem under consideration. In this simple case we are fortunate to solve the intricate problem for existence of a nontrivial solution at the stage $N=0$. This will be not the case for some other equations and, in general, we must try with increasing N in order to find a nontrivial solution.

(ii) Having obtained the solution for certain $N=M$, it is used as an initial condition for calculations with $N=M+1$ coupling it simply with the initial condition $a_{M+1}=0$. After the convergence of the numerical procedure of the Brent method is attained, the $(M+1)$th approximation is completed.

(iii) The calculations are terminated when the first $K+1$ unknowns a_i, $i = 0, 1, 2, \cdots, K$ cease to change with increasing the number of equations, and more specifically, when the following criterion is satisfied:

$$|a_i^{(M)} - a_i^{(M+1)}| < \varepsilon \|\mathbf{a}\|, \qquad i = 0, 1, \cdots, K, \tag{3.5}$$

where $\|\mathbf{a}\| = (a_0^2 + \cdots + a_M^2)^{1/2}$ is the Euclidean norm of the solution and ε is a small quantity. In our calculations we took $\varepsilon = 10^{-3}$, $K = 6$ and the convergence in the said sense was obtained for $N = 17$. Table 1 gives an insight into the manner in which the convergence is attained. It is seen that the convergence is very rapid and for $N = 6$ even a_3 changes only with quantity of approximately 0.005.

In order to check the accuracy of the Fourier method, we used two different methods. The first consists of developing the exact solution (1.4) into a series with respect to the system using the Simpson formula with fourth-order approximation for evaluation of the respective integrals taken over the region $x \in [0, 20]$ with 501 grid points. The coefficients obtained in this manner comprise the first column of Table 1. We can easily see the excellent agreement for the first 10 coefficients $a_0, \cdots, a_9$ of the exact solution, and calculated with $N = 10$, the approximate solution. This is a certificate for good performance of the method proposed.

The second way of verifying the method is the comparison with the exact solution for the soliton shape itself. Such a comparison is depicted in Fig. 1, and the convergence for the shape is so rapid that even the approximate solution with $N = 7$ cannot be discerned from the exact one. The most striking thing, however, is that the solution with $N = 1$ (two terms in the truncated series) differs less than with 0.025 from the exact solution at the time when the size of solution is 0.5, i.e., the difference is within five percent from the maximal value. The latter means that in certain cases the present method can serve as a method for express assessment of the solution shape that requires solving just a couple of nonlinear algebraic equations.

It is interesting to assess the error of the Fourier-Galerkin method. The analytic solution (1.4) of the nonlinear KdV equation gives us the unique opportunity to

TABLE 1

Developing the solution by increasing the number of equations, and comparison with the respective coefficients obtained by developing the analytic solution into Fourier series.

	Analytic solution	$N=0$	$N=1$	$N=4$	$N=6$	$N=9$
a_0	−0.7925	−0.8355	−0.8101	−0.7994	−0.7930	−0.7925
a_1	−0.1760		−0.163	−0.1688	−0.1731	−0.1760
a_2	0.0190			0.0137	0.0203	0.0190
a_3	0.0588			0.0371	0.0567	0.0588
a_4	0.0500			0.0176	0.0444	0.0501
a_5	0.0319				0.0237	0.0321
a_6	0.0167				0.0078	0.0167
a_7	0.0067					0.0066
a_8	0.0009					0.0008
a_9	−0.0010					−0.0009
Maximal absolute error (3.8)		0.167	0.0430	0.0243	0.0046	0.0016

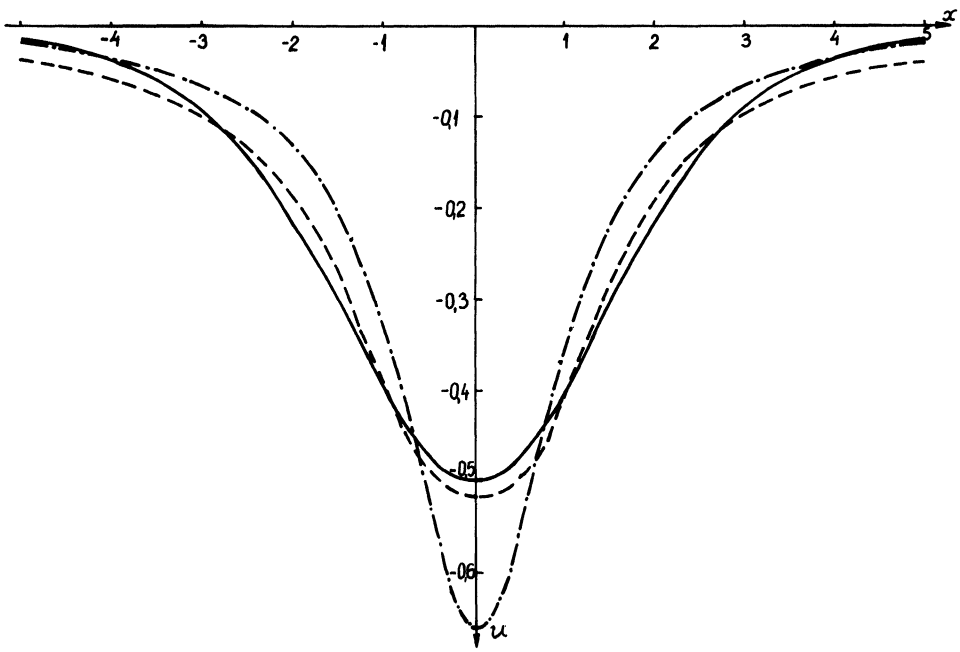

FIG. 1. *Developing the calculated shape for the solitons of the Korteweg-de Vries equation by increasing the number of modes* $N+1$: $-\cdot-$ ($N=0$); $---$ ($N=1$) *and comparison with the analytical solution*: ——.

quantitatively assess the part of the absolute error due to the truncation of the series—the truncation error (see, e.g., [8], [24] for definition):

$$E_T(x;\, N) \equiv u(x) - \sum_{n=0}^{N} (a_n S_n + \bar{a}_n C_n). \tag{3.6}$$

Taking a_n, $\bar{a}_n$ from the first column of Table 1 (i.e., the "true" ones) we obtain the pure contribution to the error due to the truncation of the series. This error as a function of the spatial coordinate x is depicted in Fig. 2.

Following [8] we also consider the so-called discretization error:

$$E_D(x;\, N) \equiv \sum_{n=0}^{N} [(a_n - a_n^{(N)}) S_n + (\bar{a}_n - \bar{a}_n^{(N)}) C_n], \tag{3.7}$$

where the quantities denoted by a superscript N are the solution of the truncated system, whereas those without a superscript are the coefficients of the series for the analytic solution (see the first column in Table 1). Discretization error as a function of x is shown in Fig. 3.

In most of the cases, an analytical solution is not available and hence the two kinds of errors above cannot be calculated explicitly. For this reason the absolute error (which is not a simple sum of the truncation and discretization errors) is the only one that can be assessed in the numerical computations:

$$E_\Delta(x;\, N) \equiv u(x) - \sum_{n=0}^{N} (a_n^{(N)} S_n + \bar{a}_n^{(N)} C_n), \qquad e_\Delta(N) = \max_x |E_\Delta(x;\, N)|, \tag{3.8}$$

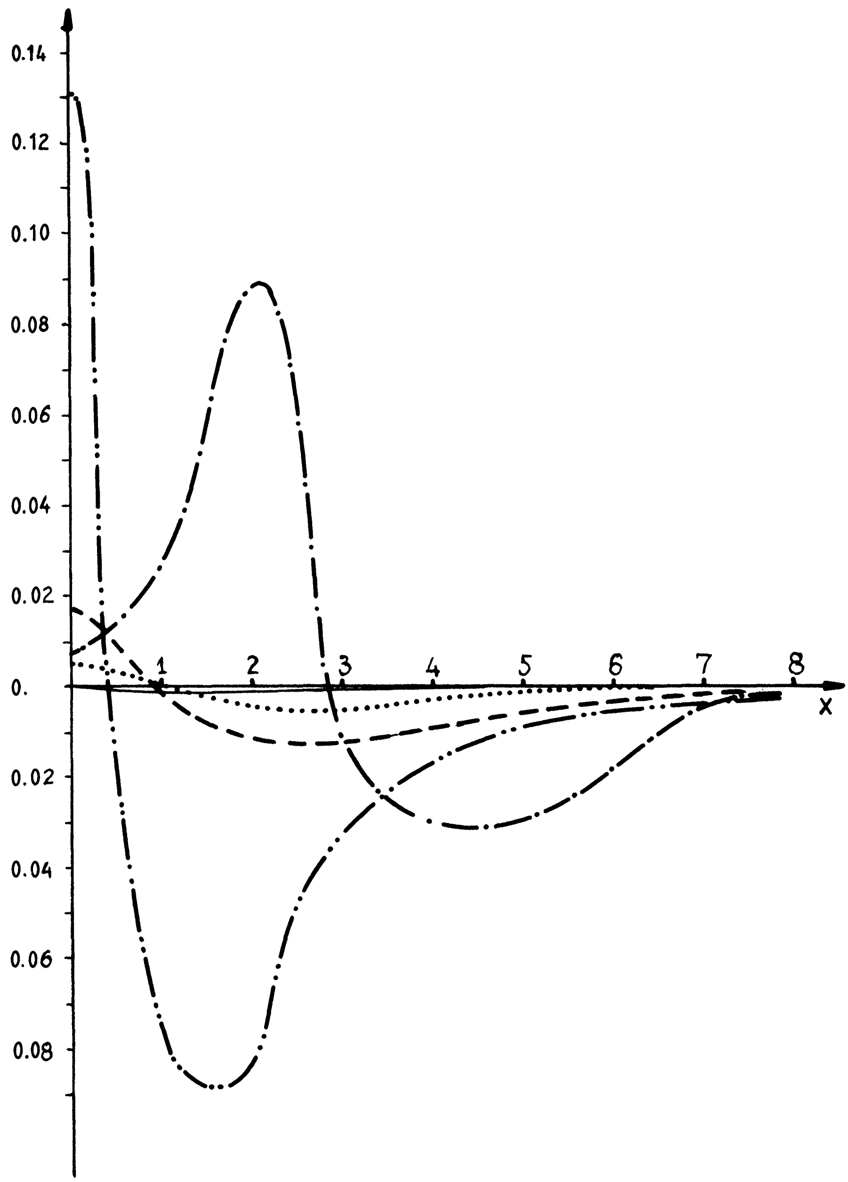

FIG. 2. *The truncation error for the Korteweg–de Vries equation as a function of x for different N*: – · · – ($N=0$); – · – ($N=1$); – – – ($N=4$); · · · · · ($N=6$); —— ($N=9$).

where $u(x)$ is either the analytic solution (when available) or the numerical spectral solution with certain sufficiently large number N_∞ of terms pertained, i.e.,

$$u(x)=\sum_{n=0}^{N_\infty}\left(a_n^{(N_\infty)}S_n+\bar{a}_n^{(N_\infty)}C_n\right). \tag{3.9}$$

The last row of Table 1 presents the maximal with respect to the x value $e_\Delta(N)$ of the absolute error. The rapid decrease of the absolute error with the increase of the number of the retained terms in the series is well seen. The latter is very important since the solution (1.4) decays exponentially at infinity, whereas the functions of the

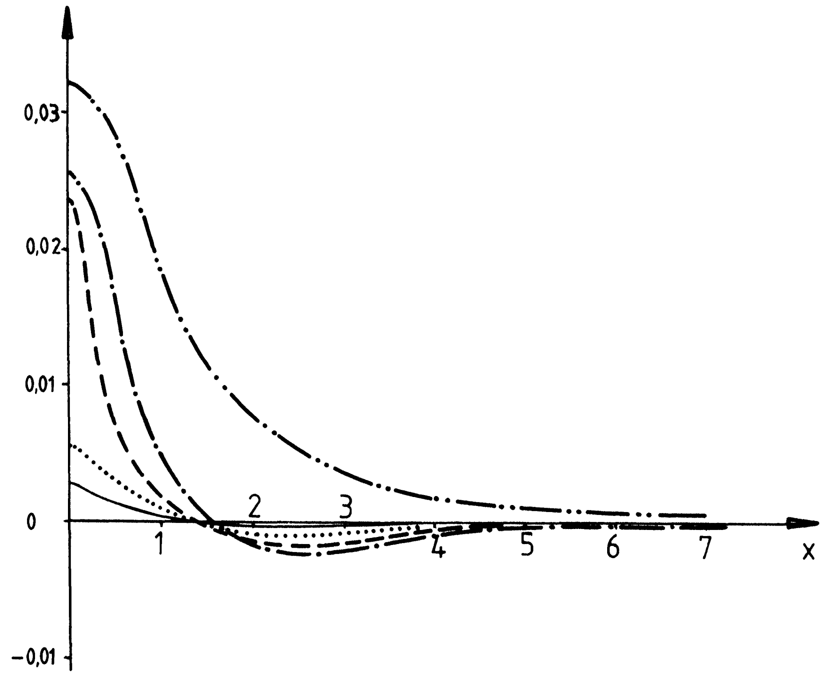

FIG. 3. *The discretization error for the Korteweg–de Vries equation as a function of x for different N*: $-\cdot\cdot-\cdot\cdot-$ ($N=0$); $-\cdot-$ ($N=1$); $---$ ($N=4$); $\cdot\cdot\cdot\cdot\cdot\cdot$ ($N=6$); —— ($N=9$).

employed CON system decay as x^{-2}. There is no doubt that at $x\to\infty$ the approximate spectral solution poorly represents the real behavior of the solution. In that region, however, the magnitude of the solution is small and does not bear significant practical importance. That is, the method proposed is adequate enough for practical purposes, even in situations where the asymptotic behavior of the sought solution differs significantly from the respective behavior of the functions from the basis set.

4. Optimization of the method. The delightful results of the previous section are a matter of luck in a sense, because the characteristic measure of the support of the sought function turns out to be close to that of the employed set of functions, and more specifically for the first couple of members. If it happens that this is not the case and those two characteristic lengths are not close enough, a significantly greater number of terms might be needed to secure acceptable approximation.

Fortunately, the unboundedness of the considered interval always allows us to bring the mentioned characteristic lengths in correspondence, since if $F(x)\in L^2(-\infty,\infty)$ then $F(\beta x)\in L^2(\infty,\infty)$, where $\beta>0$ is real. The idea to scale the independent variables is also employed in [6]–[8], and for the scale factor it is shown that its optimal value may even depend on the number of terms retained in the truncated spectral series. It is simpler to scale the independent variable prior to the calculations and thus render the characteristic length of the sought solution in accordance with the scales length of functions S_0, C_0. It is important to mention here that the adequate choice of β in [9] allowed us to reduce for the Burgers equation the required nontrivial coefficients a_i just to one: the coefficient a_0. That was possible since the soliton problem there reduced to a first-order ordinary differential equation. When higher-order equations are considered such a drastic reduction is not to be expected but still the solution can be significantly improved. The problem is to devise a quantitative criterion for discerning the better solutions. Qualitatively speaking, a solution is better when the higher-order coefficients represent a smaller share from the norm of the vector of coefficients a_i. One of the possible versions a criterion securing the presence of that property is the following:

$$I(\beta)=\sum_{n=0}^{N}|a_n(\beta)|n^2=\min. \tag{4.1}$$

In the last formula, the coefficients $a_i(\beta)$ are the solution of the system

$$-a\cdot a_n-3\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}a_{m_1}a_{m_2}\beta_{m_1m_2n}+\frac{1}{\beta^2}\sum_{m=0}^{N}\chi_{mn}a_n=0, \tag{4.2}$$

$$n=0,1,2,\cdots.$$

The reason for seeking the value $\beta=\beta_{\min}$ for which the minimum of I is attained is that this can be done for a relatively small value of N and only after that to run the final calculations with higher N. The quest for minimum turns out to be a very inexpensive procedure, since for each new value β the solution for the previous one serves as an initial condition and the iterative procedure [25] converges rapidly.

The minimum of $I(\beta)$ is sought in the interval $0.5\leqq\beta\leqq 25$ by means of the method of the golden section [26]. For $N=5$ the sought value is $\beta_{\min}\approx 2.84$. Here we mention that a couple of different criteria have been checked, e.g.,

$$I(\beta)=\sum_{n=0}^{N}a_n^2(\beta)n^2=\min \tag{4.3}$$

or

$$I(\beta)=\sum_{n=0}^{N}a_n^2(\beta)n^4=\min, \tag{4.4}$$

and what is amazing is that the optimal scale factor is always $\beta_{\min}\approx 2.8$. Table 2 gives an insight into the way in which the solution for a_i depends on β for the case $N=4$. It is seen that a_2 promptly decreases near the optimal value of β. Figure 4 shows the solution calculated only on the basis of the first two coefficients a_0, a_1 with two different values of β. It is interesting to note that for the optimal value $\beta=2.8$, the solution virtually coincides with the analytic one and cannot be discerned in the figure.

Here we note that calculating the scale factor β in accordance with the adopted criterion does not necessarily yield for a fixed N a solution with least value for the

TABLE 2

The dependence of a_i on i for different scale factors β when $N=4$.

β	a_0	a_1	a_2	a_3	a_4	Maximal error, compared to $N=10$
1	−0.7994	−0.1688	0.0137	0.0371	0.0176	0.0243
2	−0.5640	0.1086	0.0568	0.0112	−0.0002	0.0033
2.8	−0.4513	0.1806	−0.0004	−0.0080	−0.0002	0.0021
4	−0.3475	0.2043	−0.0643	0.0088	−0.0015	0.0003
5	−0.3377	0.2136	−0.0587	0.0091	−0.0027	0.066

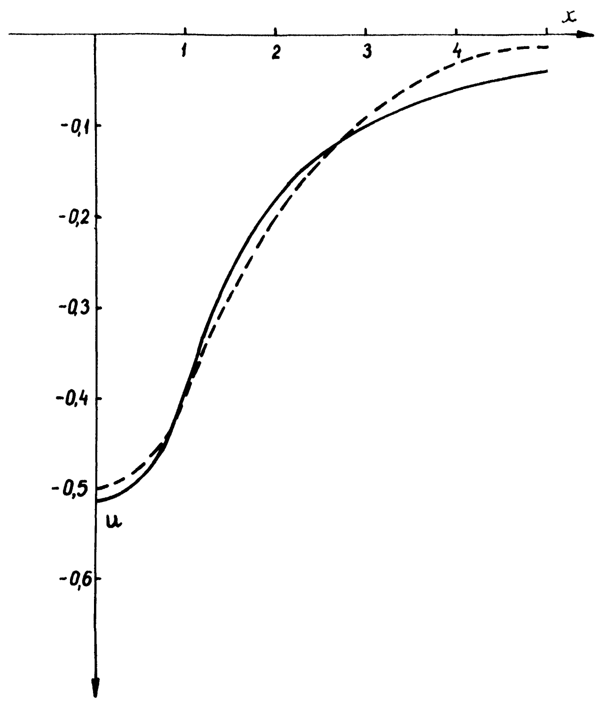

FIG. 4. *The shape of the soliton for the Korteweg-de Vries equation as given by the first two terms* a_0, a_1 *with different* β: --- ($\beta=2.8$) *and analytic solution*; —— ($\beta=1$).

maximal absolute error. This is easily seen in Table 2, where the least error is attained for $\beta=4$. The purpose of the optimization proposed here is to minimize the number N of needed terms in series that are enough to give quantitatively adequate approximation.

5. Solitons for the Kuramoto–Sivashinsky equation. Deriving confidence from the successful attempt with the Korteweg-de Vries equation, here we apply the proposed spectral method for calculating the soliton solution of (1.7). Some preliminary results in this direction have been obtained in [27]. The situation here is more complicated for two reasons: a higher-order derivative is present, and the solution is neither an even nor an odd function. As such, the solution is sought in the following truncated series:

$$\varphi(x)=\sum_{n=0}^{N}[h_n(x)S_n(x)+\bar{h}_n(x)C_n(x)]. \tag{5.1}$$

Introducing the latter into (1.7), after standard manipulations, we arrive at the nonlinear system for coefficients

$$\begin{aligned}&\sum_{m=0}^{N}(\theta_{nm}+\varphi_{nm})-ch_n+6\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}\gamma_{m_1m_2n}h_{m_1}\bar{h}_{m_2}=0,\\ &-\sum_{m=0}^{N}(\theta_{nm}+\varphi_{nm})-c\bar{h}_n+3\sum_{m_1=0}^{N}\sum_{m_2=0}^{N}(\alpha_{m_1m_2n}h_{m_1}h_{m_2}+\beta_{m_1m_2n}\bar{h}_{m_1}\bar{h}_{m_2})=0.\end{aligned} \tag{5.2}$$

System (5.2) contains $2N+2$ equations for the $2N+2$ unknown coefficients $h_0, \bar{h}_0, \cdots, h_N, \bar{h}_N$. The general scheme of the algorithm is the same as in § 3. We stress that in this case the convergence with respect to the number N is slower and can be seen in Table 3. If we consider only the first seven pairs h_i, $\bar{h}_i$, the convergence within three percent ($\varepsilon = 0.03$) is attained for $N = 20$. The slow convergence is easily explained by the fact that the solution has a more complicated form (see Fig. 5), showing a row of local minima and maxima. It turns out that in the last case the adequate choice of scaling factor β is much more important than in the previous case, where the solution has monotone shape. Here we do not employ the full-scale technique for defining the optimal β. Rather, we estimate from obvious considerations that $\beta = 2$ is to be good enough for "compressing" the soliton to fit the length scale of the S_0, C_0. Indeed, Table 4 and Fig. 6 convince us that the results are considerably improved. First, the accuracy reached with $N = 20$ is $\varepsilon = 0.001$ (30 times better) and this time even $N = 0$ gives fully acceptable approximation for the soliton. For $N = 3$ the obtained accuracy corresponds to $N = 6$ with $\beta = 1$, and $N = 14$ matches the performance of $N = 20$. We should mention that the results above are obtained for $c = 1$.

TABLE 3

Developing the solution with N for $c = 1$.

	i \ $N+1$	7	8	9	11	15	20
h_i	0	0.1936	0.0325	−0.1474	0.0787	0.0297	0.0813
	1	0.0572	0.1104	0.1459	0.0707	0.0969	0.0764
	2	−0.0531	0.0101	0.0880	−0.0114	0.0066	−0.0140
	3	−0.0857	−0.0669	−0.0229	−0.0554	−0.0602	−0.0625
	4	−0.0684	−0.0859	−0.0880	−0.0580	−0.0739	−0.0625
	5	−0.0361	−0.0662	−0.0970	−0.0401	−0.0519	−0.0370
	6	−0.0108	−0.0349	−0.0717	−0.0197	−0.0176	−0.0074
	7		−0.0105	−0.0373	−0.0054	0.0127	0.0145
	8			−0.0112	0.0011	0.0314	0.0250
	9				0.0022	0.0377	0.0256
	10				0.0010	0.0344	0.0199
	11					0.0258	0.0117
	12					0.0159	0.0038
	13					0.0075	−0.0021
	14					0.0021	−0.0053
$\bar{h}_i$	0	0.6634	0.6859	0.6267	0.6493	0.6713	0.6632
	1	0.0723	0.1036	0.1407	0.0562	0.0901	0.0710
	2	−0.0745	−0.0726	−0.0616	−0.0913	−0.0786	−0.0832
	3	−0.0713	−0.0887	−0.1027	−0.0708	−0.0774	−0.0706
	4	−0.0381	−0.0563	−0.0776	−0.0141	−0.0306	−0.0199
	5	−0.0135	−0.0247	−0.0409	0.0279	0.0097	0.0188
	6	−0.0026	−0.0069	−0.0148	0.0443	0.0297	0.0344
	7		−0.0008	−0.0026	0.0409	0.0322	0.0315
	8			0.0003	0.0279	0.0249	0.0186
	9				0.0138	0.0150	0.0032
	10				0.0041	0.0065	−0.0038
	11					0.0013	−0.0182
	12					0.0159	0.0038
	13					−0.0010	−0.0212
	14					−0.0004	−0.0181
		Maximal absolute error compared to the case $N+1=20$					
		0.1714	0.1168	0.1821	0.0212	0.0091	0

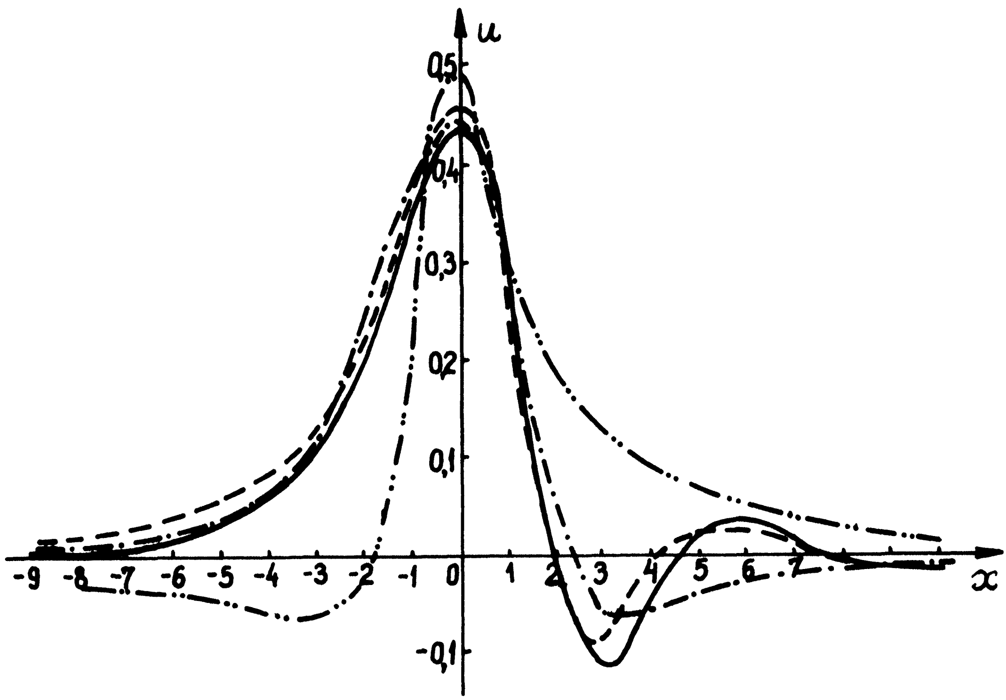

FIG. 5. *Developing the calculated shape of the film soliton for* $c = 1$ *with number of modes* $N + 1$: $-\cdot\cdot\cdot-$ ($N = 0$); $-\cdot\cdot-$ ($N = 6$); $---$ ($N = 19$) *and comparison with the difference solution of* [4] ——.

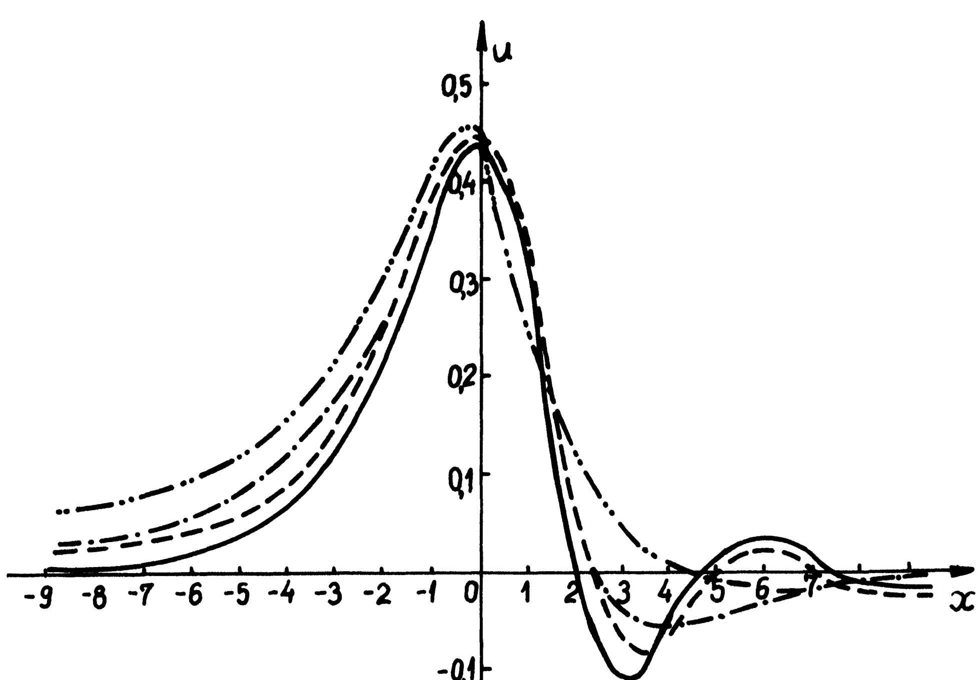

FIG. 6. *Developing the calculated shape of the film soliton for* $c = 1$ *and the scaling factor* $\beta = 2$ *with the number* $N + 1$ *of modes*: $-\cdot\cdot\cdot-$ ($N = 0$); $-\cdot\cdot-$ ($N = 3$); $---$ ($N = 14$) *and comparison with the difference solution of* [4] ——.

Now, after being convinced that the proposed method is highly effective for calculating the shapes of solitons, we try to answer the question of what for spectrum exists for the parameter c. In [4] it is found that the difference solution to the problem based on the notion of variational imbedding exists for $0.4 \leqq c \leqq 4.9$ and the results strongly suggest that the soliton is to be expected for the entire open interval $c \in (0, \infty)$. It is not computed in [4] only because some measures for the grid are to be taken for $c \ll 1$ and $c \gg 1$. The present results confirm that conclusion and the shape of the film soliton is obtained for a wide range of governing parameters c (see Fig. 7). We can see the intricate shape for very high $c \geqq 10$. The latter can be thought of, however, only as preliminary results as far as the physics of the phenomenon is concerned, but is still a very good achievement of the method of Fourier in $L^2(-\infty, \infty)$.

6. Conclusions. The present paper deals with developing the numerical aspects of a new technique for the Fourier method in $L^2(-\infty, \infty)$ with a CON basis system of functions proposed earlier. The necessary formulae are compiled and the systems for coefficients of the series are obtained in the frame of the Galerkin approach for two famous nonlinear equations: Korteweg-de Vries and Kuramoto-Sivashinsky for which

TABLE 4
Developing the solution with N for $c = 1$ and the scale factor $\beta = 2$.

	i \ $N+1$	2	4	7	12	15	20
	0	−0.0180	0.2083	0.2351	0.2162	0.2128	0.2112
	1	0.1452	−0.0656	−0.1109	−0.0939	−0.0939	−0.0916
	2		−0.0923	−0.0703	−0.0691	−0.0664	−0.0672
	3		−0.0317	0.0106	0.0062	0.0677	0.0065
	4			0.0357	0.0311	0.0299	0.0297
	5			0.0246	0.0195	0.0167	0.0174
	6			0.0080	0.0009	−0.0012	−0.0002
h_i	7				−0.0101	−0.0098	−0.0091
	8				−0.0120	−0.0089	−0.0089
	9				−0.0087	−0.0036	−0.0041
	10				−0.0044	0.0014	0.0009
	11				−0.0012	0.0004	0.0004
	12					0.0040	0.0042
	13					0.0025	0.0031
	0	0.4378	0.4922	0.4632	0.4698	0.4674	0.4689
	1	−0.1032	−0.0911	−0.0846	−0.0930	−0.0955	−0.0957
	2		−0.0159	−0.0125	−0.0116	−0.0118	−0.0115
	3		0.0129	0.0152	0.0202	0.0223	0.0224
	4			0.0027	0.0025	0.0046	0.0047
	5			−0.0057	−0.0125	−0.0122	−0.0126
	6			−0.0037	−0.0120	−0.0136	−0.0135
$\bar{h}_i$	7				−0.0039	−0.0058	−0.0053
	8				0.0031	0.0024	0.0033
	9				0.0055	0.0071	0.0075
	10				0.0040	0.0078	0.0072
	11				0.0015	0.0061	0.0041
	12					0.0036	0.0004
	13					0.0016	−0.0025
		Maximal absolute error compared to the case $N+1=20$					
		0.2291	0.0280	0.0211	0.0180	0.0011	0

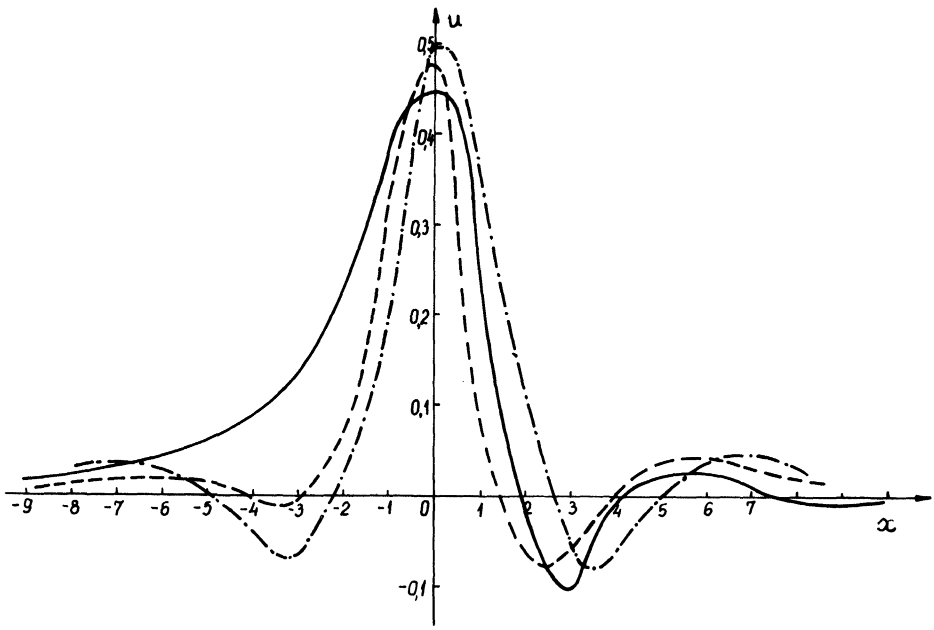

FIG. 7. *Evolution of the soliton shape with the parameter c*: —— ($c = 1$); --- ($c = 10$); -·- ($c = 20$).

soliton solutions do exist. The performance of the Fourier-Galerkin method is checked through comparing the approximate solution for KdV to the known analytic solution. The different kinds of errors (truncation, discretization, and absolute one) are calculated as functions of the spatial coordinate x. Some means for optimization of the Fourier method based on the notion of scaling the independent variable are discussed. Then the mathematical technology is applied to the K-S equation and the soliton solution is obtained for a variety of values of nonlinear eigenvalue parameters c. The shapes of solitons compare well with known difference solutions.

The results obtained suggest that a reliable and robust numerical technique is devised for calculating the shape of solitons occurring as solutions for certain nonlinear differential equations of evolution.

Acknowledgments. The authors are indebted to the referees for helpful suggestions.

REFERENCES

[1] M. A. ABLOWITZ AND H. SEGUR, *Solitons and the Inverse Scattering Transform*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 1981.

[2] P. L. BHATNAGAR, *Nonlinear Waves in One-Dimensional Dispersive Systems*, Clarendon Press, Oxford 1979.

[3] C. I. CHRISTOV, *A method for identification of homoclinic trajectories*, in Proc. 14th Spring Conference Union Bulgarian Mathematicians, Sunny Beach, Bulgaria, 1985, pp. 571-577.

[4] C. I. CHRISTOV AND V. P. NARTOV, *The use of a method of variational imbedding for calculating solitons in falling-liquid-film flow*, in Proc. Conference "Anniversary of academician L. Tchakalov," Samokov, Bulgaria, 1986, pp. 135-142. (In Russian.)

[5] H. BATEMAN, *Higher Transcendental Functions*, Vol. 2, McGraw-Hill, New York, Toronto, London, 1953.

[6] C. E. GROSCH AND S. A. ORSZAG, *Numerical solution of problems in unbounded regions: Coordinate transformations*, J. Comput. Phys., 25 (1977), pp. 273-296.

[7] J. P. BOYD, *The optimization of convergence for Chebyshev polynomial methods in an unbounded domain*, J. Comput. Phys., 45 (1982), pp. 43-79.

[8] ———, *Spectral methods using rational basis functions on an infinite interval*, J. Comput. Phys., 69 (1987), pp. 112-142.

[9] C. I. CHRISTOV, *A complete orthonormal system of functions in* $L^2(-\infty, \infty)$ *space*, SIAM. J. Appl. Math., 42 (1982), pp. 1337-1344.

[10] ———, *A method for treating the stochastic bifurcation of plane Poiseuille flow*, Ann. Univ. Sofia, Fac. Math. Mech., Libre—Mechanique, vol. 76/1982 (1987), pp. 87-113.

[11] S. V. ALEKSEENKO, V. E. NAKORYAKOV, AND B. G. POKUSAEV, *Wave formation on vertical falling liquid films*, Internat. J. Multiphase Flow, 11 (1985), pp. 607-627.

[12] A. PUMIR, P. MANNEVILLE, AND Y. POMEAU, *On the solitary waves running down an inclined plane*, J. Fluid Mech., 135 (1983), pp. 27-50.

[13] D. J. BENNEY, *Long waves in liquid films*, J. Math. Phys., 45 (1966), pp. 150-155.

[14] S. P. LIN, *Finite amplitude side-band stability of a viscous film*, J. Fluid Mech., 63 (1974), pp. 417-429.

[15] G. SIVASHINSKY AND D. M. MICHELSON, *On irregular wavy flow of a liquid down a vertical plane*, Progr. Theoret. Phys., 63 (1980), pp. 2112-2114.

[16] Y. KURAMOTO, *Diffusion-induced chemical turbulence*, in Dyn. Synerg. Syst., Proc. Internat. Symposium Synerg., Biefield, Berlin, 1979, pp. 134-146.

[17] M. HYMAN AND B. NIKOLAENKO, *The Kuramotò-Sivashinsky equation: A bridge between* PDE'*s and dynamical systems*, Phys. D, 18 (1986), pp. 113-126.

[18] V. YA. SHKADOV, *Solitary waves in viscous films*, Izv. Akad. Nauk SSSR, Mekhanika Zhidkosti i Gasa, 1 (1977), pp. 63-66. (In Russian.)

[19] O. YU. TZVELODUB, *Stationary propagating waves on films falling down an inclined plane*, Izv. Akad. Nauk SSSR, Mekh. Zhidk. Gaza, 4 (1980), pp. 142-146. (In Russian.)

[20] N. WIENER, *Extrapolation, Interpolation and Smoothing of Stationary Time Series*, Technology Press MIT and John Wiley, New York, 1949.

[21] J. R. HIGGINS, *Completeness and Basis Properties of Sets of Special Functions*, Cambridge University Press, London, New York, Melbourne, 1977.

[22] P. GESHEV, Private communication, Institute of Theoretical and Applied Mechanics, Siberian Division of Academy of Sciences of the USSR, Novisibirsk, 1986.

[23] J. P. BOYD, *The orthogonal rational functions of Higgins and Christov and algebraically mapped Chebyshev polynomials*, J. Approx. Theory, (1989), in press.

[24] C. CANUTO, M. Y. HUSSAINI, A. QUARTERONI, AND T. A. ZANG, *Spectral Methods in Fluid Dynamics*, Springer-Verlag, New York, Berlin, Heidelberg, London, Paris, Tokyo, 1988.

[25] J. J. MORÉ AND M. Y. COSNARD, BRENTM, *A Fortran Subroutine for the Numerical Solution of a System of Nonlinear Equations*, Collected Algorithms from ACM, Algorithm No. 554, Association for Computing Machinery.

[26] D. J. WILD, *Optimum Seeking Methods*, Prentice-Hall, Englewood Cliffs, NJ, 1964.

[27] C. I. CHRISTOV AND K. L. BEKYAROV, *A Fourier-series method for calculating solitons*, University Annual, Applied Mathematics, Sofia, Bulgaria, 22 (1986), pp. 199-209. (In Russian.)