Chaos, Solitons & Fractals Vol. 1, No. 5, pp.423–430, 1991
Printed in Great Britain

0960-0779/92$5.00 + 00

Fourier–Galerkin Numerical Technique for Solitary Waves of Fifth Order Korteweg–De Vries Equation

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Abstract—The present paper is aimed to display the performance of the Fourier–Galerkin technique developed earlier by the authors for numerical treatment of localized solutions in $\mathbb{L}^2(-\infty, \infty)$ for higher-order equations. The solitary-wave solution of the Korteweg–de Vries fifth order equation is obtained numerically and compared with approximate results of other authors.

1. INTRODUCTION

Consider the problem of solitary-wave solution to the so-called Korteweg–de Vries equation of fifth order (here, following Boyd [1] the abbreviation "FKDV" is used). The FKDV equation can be presented in several different but equivalent forms. We use here the one proposed in Boyd [1]:

$$u_t + uu_x - u_{xxxxx} = 0. \tag{1.1}$$

Equation (1.1) is a physical model for magneto–acoustic waves, gravity–capillary water waves, and waves in a nonlinear LC circuit with mutual inductance. The only difference from the classical Korteweg–de Vries equation is the presence of a fifth spatial derivative instead of the third one. Unfortunately, the said difference does complicate the matter and as a result, an analytical solution of FKDV is still not available. For this reason different approximate methods are developed (see, e.g. Boyd [1], Nagashima and Kuwahara [2]). Appearance of the higher spatial derivatives may pose some difficulties also when applying approximate methods. The sole purpose of the present paper is to apply to the case the numerical technique proposed in [3] and developed in [4–6] for the application of the method of Fourier in $\mathbb{L}^2(-\infty, \infty)$.

The FKDV equation is particularly suitable because the approximate results of Nagashima and Kuwahara [2] are available and can be used as a basis for comparison.

2. POSING THE PROBLEM

Let us seek a solution of (1.1) of the type of propagating wave $u = u(\xi)$, where $\xi = x - ct$ and $c > 0$ is the phase velocity of the wave. Then (1.1) reduces to

$$-cu' + uu' - u^{\mathrm{V}} = 0 \tag{2.1}$$

where the prime stands for a differentiation with respect to the independent variable ξ. We have a soliton solution (a solitary wave) if the following boundary conditions hold:

$$u(\xi) \rightarrow 0 \quad \text{for} \quad \xi \rightarrow \pm\infty. \tag{2.2}$$

Under these boundary conditions the above ordinary differential equation can be integrated once and rendered to

$$-cu + \tfrac{1}{2}u^2 - u^{\mathrm{IV}} = 0. \tag{2.3}$$

Thus (2.3) and (2.2) form the boundary value problem in $\mathbb{L}^2\ (-\infty, \infty)$ to be solved. The latter is in fact an eigen-value problem for the phase velocity c. So that an answer must be given to the question for which values of the phase velocity c the problem has a nontrivial solution. In Boyd [1] it is shown that if (2.1) has a solution $u_1(x, t)$ for $c = 1$, then for each other $c = a$ there exists a solution $u_2(x, t)$ for which the following relation is valid:

$$u_2(x, t) = au_1(a^{1/4}x, a^{5/4}t). \tag{2.4}$$

This result permits in our further investigations to consider only the case $c = 1$.

3. THE CON SYSTEM

Wiener [7], introduced the system

$$\rho_n = \frac{1}{\sqrt{\pi}}\frac{(ix - 1)^n}{(ix + 1)^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

as a Fourier transform of Laguerre functions. Higgins [8] defined it also for negative n and proved its completeness and orthogonality. The significance of (3.1) for nonlinear problems was revealed in [3], where the product formula was derived:

$$\rho_n\rho_k = \frac{1}{2\sqrt{\pi}}(\rho_{n+k} - \rho_{n-k}) \tag{3.2}$$

and the two real subsequences of odd functions S_n and even functions C_n are introduced according to the formulae:

$$S_n = \frac{\rho_n + \rho_{-n-1}}{i\sqrt{2}}, \quad C_n = \frac{\rho_n - \rho_{-n-1}}{\sqrt{2}}, \quad n = 0, 1, 2, \ldots. \tag{3.3}$$

The explicit expressions for S_n and C_n can be found in [3]. Some other formulae interrelating the CON system employed here to the different families of Boyd's functions are presented in Boyd [9], where is discussed also the rate of convergence.

Most of the practically important formulae for the system (3.3) are compiled in [3] and [4], and here we cite only those that are necessary for carrying out the present calculations.

The most important feature of the system—the equality (3.2)—for the real-valued subsequence C_n adopt the form:

$$C_n(x)\ C_m(x) = \sum_{k=0}^{\infty} \beta_{nmk}C_k(x) \tag{3.4}$$

where

$$\beta_{nmk} = 0.5(2\pi)^{-0.5}\{-\delta_{k,n+m+1} + \delta_{k,n+m} + \delta_{k,|n-m|} - \operatorname{sgn}[|n - m| - 0.5]\delta_{k,[|n-m|-0.5]}\}||$$

where $\delta_{i,j}$ is the Kronecker delta and [.] stands for the integer part of a real.

In the same manner the formulae representing the fourth-order derivatives of a member of system into series with respect to the system are derived in [4]:

$$C_n(x)^{(\mathrm{IV})} = \sum_{m=0}^{\infty} \omega_{nm} C_m(x) \tag{3.5}$$

where

$$\begin{aligned}\omega_{nm} = {} & 0.125\delta_{m,n}\,(35n^4 + 70n^3 + 85n^2 + 50n + 12) \\ & + (n+1)(n+2)(n+3)[(n+4)\delta_{m,n+4}/16 - (n+2)\delta_{m,n+3}/2] \\ & + 0.25(n+1)(n+2)[7(n+1)^2 + 7(n+1) + 4]\delta_{m,n+2} \\ & - 0.5(n+1)^2[7(n+1)^2 + 5]\delta_{m,n+1} \\ & - 0.5n^2(7n^2 + 5)\delta_{m,n-1} + 0.125n(n-1)\,(7n^2 - 7n + 4)\delta_{m,n-2} \\ & + n(n-1)(n-2)[(n-3)\delta_{n,m-4}/16 - 0.5(n-1)\delta_{n,m-3}].\end{aligned}$$

4. FOURIER–GALERKIN METHOD

Following [5], from the known spectral techniques we choose the Galerkin method (see for other techniques Canuto *et al.* [10]). The Galerkin method has the advantage of simplicity in implementation in comparison with the spectral collocation method or tau-method (see Canuto *et al.* [10]) which turns out to be crucial for constructing fast and efficient numerical algorithms. The only problem is that it requires explicit formulae expressing the products of members of the CON system into series with respect to the system. The Hermite functions and Laguerre functions do not possess that kind of explicit relation (one should not confuse them with the Hermite and Laguerre polynomials which are not CON systems in $\mathbb{L}^2(-\infty, \infty)$ and $\mathbb{L}^2(0, \infty)$, respectively). The first system for which product formula was derived is the system employed here from [3]. In a sequence of papers Boyd showed the general way of constructing CON systems with product formulae by means of coordinate transformation in Chebishev polynomials (see [9] for references).

Unlike the systems of nonlinear equations for the coefficients in the Fourier–Galerkin expansions which were obtained for Korteweg–de Vries and Kuramoto–Sivashinsky equations, the one for FKDV is a little more complex, in the sense that the number of nonzero elements of the system matrix will be higher—a fact which is determined by the order of the higher derivative. The validity of above statement becomes clear from formula (3.5) where it is seen that the representation of the fourth derivative of C_n is 9-diagonal matrix. For comparison: C_n' yields to a three-diagonal matrix, C_n''—5-diagonal, etc.

It is easily shown that (2.3) admits even functions as solutions and hence we develop the solution to be u into series only with respect to the subsequence of functions C_n, namely,

$$u(x) = \sum_{n=0}^{\infty} a_n C_n(x). \tag{4.1}$$

Then, for the terms entering (2.3), we obtain

$$u^{(\mathrm{IV})}(x) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_m \omega_{mn} C_n(x) \tag{4.2}$$

$$u^2(x) = \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\sum_{n=0}^{\infty} a_{m_1} a_{m_2} \beta_{m_1 m_2 n} C_n(x). \tag{4.3}$$

Since for the Galerkin method, the sets of trial and test functions coincide with the set C_n, then upon introducing (4.1)–(4.3) into (2.3), combining the terms with the like functions C_n, and taking the respective coefficients to be equal to zero (due to the independence of members of subsequence C_n and its completeness in the subspace of even $\mathbb{L}^2(-\infty,\infty)$ functions), we obtain the following infinite nonlinear algebraic system for the unknown coefficients a_n:

$$-a_n + 0.5 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} a_{m_1} a_{m_2} \beta_{m_1 m_2 n} - \sum_{m=0}^{\infty} a_m \omega_{mn} = 0 \qquad (4.4)$$

$$n = 0, 1, 2, \ldots$$

It is obvious that we can solve only finite versions of (4.4) and the number N of unknown coefficients at which the vector $\mathbf{a} = \{a_n\}$ is truncated corresponds to the number of equations of (4.4) that are retained.

The results obtained for the Korteweg–de Vries and Kuramoto–Sivashinsky equations in [4–6], show that the method of Fourier–Galerkin is suitable and effective when solving the nonlinear equations. As far as the solution of the systems of algebraic equations of the type (4.4) is concerned, the applied there method of Brent (see More and Cosnard [11]) turned out suitable as well and we apply it here to the FKDV too.

5. NUMERICAL RESULTS

The general consequence of the algorithm is as follows:

(i) We begin with the case $N = 0$ when the system reduces to just one equation for the unknown $\mathbf{a}_0$, which has two solutions:

$$a_0^{(1)} = 0 \quad \text{and} \quad a_0^{(2)} = 8.355$$

Here shows up the bifurcation character of the problem under consideration. In this simple case we are fortunate to solve the intricate problem for existence of a nontrivial solution yet at the stage $N = 0$.

(ii) Having obtained the solution for certain $N = M$, the latter is used as an initial condition for calculations with $N = M + 1$ coupling it simply with the initial condition $a_{m+1} = 0$. After the convergence of the numerical procedure of the Brent method is attained, the $(M + 1)$th approximation is completed.

(iii) The calculations are terminated when the first $K + 1$ unknowns a_i, $i = 0, 1, 2, \ldots, K$ cease to change with increasing the number of equations, and more specifically, when the following criterion is satisfied:

$$\sum_{i=0}^{K} |a_i^{(M)} - a_i^{(M+1)}| < \varepsilon \, \|\mathbf{a}\|$$

where $\|\mathbf{a}\| = \max(|a_0|, |a_1|, \ldots, |a_M|)$ is the uniform norm of the solution and ε is a small quantity. In our calculations we took $\varepsilon = 10^{-4}$, $K = 7$ and the convergence in the said sense was obtained for $N = 20$.

The results of the numerical experiments carried out for different N are given in Table 1 and the obtained approximate forms of the soliton are given in Fig. 1.

It is obvious that the problem is invariant under the transformation $x_1 = \beta x$ since $\mathbf{G}(x_1) \equiv \mathbf{F}(\beta x) \in L^2(-\infty, \infty)$. This fact was employed in [3], [5] and [6] when optimizing the numerical procedure and the respective algorithm for selecting the optimal β was developed. Without going into much detail we note here that for the case under consideration we obtained $\beta_{\text{optimal}} \approx \sqrt{10}$ for $N_\infty = 12$. The results from the respective numerical experiments are presented by Table 2 and shown in Fig. 2. The practical convergence is close to exponential for large n as estimated in [9].

Table 1. Developing the solution by increasing the number of equations for scale factor $\beta = 1$

	N					
a_i	0	1	6	10	20	22
a_0	8.355	5.262	4.562	4.414	4.388	4.388
a_1		0.948	1.174	1.161	1.136	1.136
a_2			−0.041	−0.140	−0.167	−0.168
a_3			−0.299	−0.508	−0.529	−0.529
a_4			−0.225	−0.487	−0.489	−0.489
a_5			−0.102	−0.347	−0.321	−0.320
a_6			−0.025	−0.205	−0.148	−0.148
a_7				−0.102	−0.018	−0.018

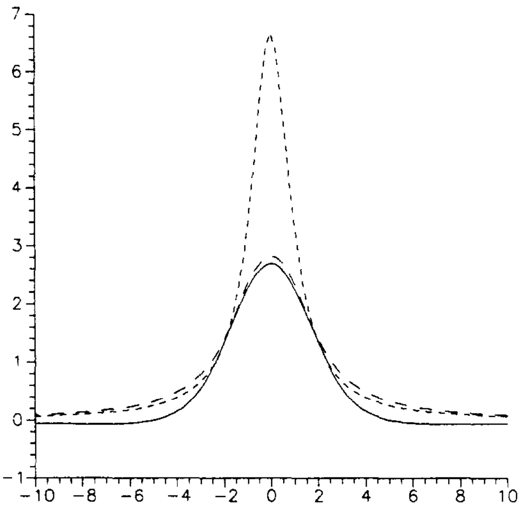

Fig. 1. Developing the calculated shape of solitons of the FKDV equation with increasing the number of modes $N + 1$: - - - ($N = 0$); — — — ($N = 3$); —— ($N = 11$).

Table 2. Developing the solution by increasing the number of equations for scale factor $\beta_{\text{optimal}} = \sqrt{10}$

	N					
a_i	0	1	6	8	10	15
a_0	3.343	1.857	0.837	0.835	0.835	0.835
a_1		−1.308	−0.782	−0.781	−0.780	−0.780
a_2			0.667	0.666	0.666	0.666
a_3			−0.449	−0.500	−0.500	−0.500
a_4			0.311	0.313	0.313	0.313
a_5			−0.154	−0.155	−0.155	−0.155
a_6			0.064	0.056	0.057	0.057
a_7				−0.016	−0.016	−0.016

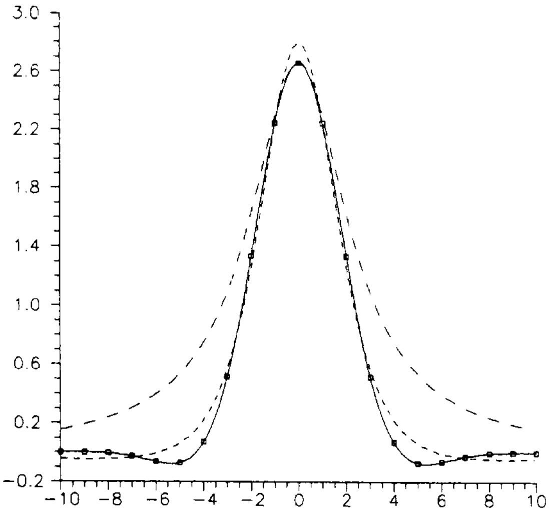

Fig. 2. Developing the calculated shape of solitons of the FKDV equation for scaling factor $\beta_{\text{optimal}} = \sqrt{10}$ with increasing the number of modes $N + 1$: — — — ($N = 0$); - - - ($N = 3$); —— ($N = 11$), and comparison with Nagashima–Kuwahara solution (□ □ □).

6. ESTIMATION OF THE ERROR. COMPARISON WITH NAGASHIMA–KUWAHARA SOLUTION

In order to check the accuracy of the Fourier–Galerkin method, define the maximal relative error for two successive approximations $\mathbf{a}_N = (\mathbf{a}_0, \mathbf{a}_1, \ldots, \mathbf{a}_N)$, computed for $N = M$ and $N = M + 1$ as follows:

$$E_R(M) = \max_x \left| \sum_{k=0}^{M} (a_k^{(M)} - a_k^{(M+1)})\, C_k(x) \right|.$$

$E_R(M)$ is in fact the maximum with respect to x of the difference between two consecutive approximations. Obviously if the series of Fourier–Galerkin converges with the increase of M, then $E_R(M)$ is to converge to zero.

For FKDV equation, $E_R(M)$ decreases rapidly with the increase of M.

In [5] was also defined the maximal absolute error of the Mth approximation for the series (3.4), similar to the definition of Boyd [12], which in our case takes the form:

$$E_A(M) = \max_x \left| \sum_{k=0}^{M} (a_k^M - a_k^{N\infty}) C_k(x) \right|.$$

The evolution of the relative and absolute error for FKDV with the increase of M is shown in Table 3.

It is seen that for optimal $\beta = 10^{0.5}$, selected in the previous section, even $M = 10$ sufficiently well represents the equation, and the error in comparison with $M = 0$ has decreased about 100 times.

The evolution of the error shows in fact the rate of convergence of the Fourier–Galerkin method but at any rate these are "inner criteria" which do not give exact information about the deviation of the calculations from the exact solution.

In Nagashima and Kuwahara [2] an approximate solution of (2.3) is obtained which has the form:

$$u(z) = A_0 \exp(-(z/2.5)^2)(a_0 + a_1 z^2 + a_2 z^4 + a_3 z^6 + a_4 z^8) \tag{6.1}$$

Table 3. Relative (E_R) and absolute (E_A) errors as functions of truncation number M for β_{optimal}

M	$E_R(M)$	$E_A(M)$
1	1.1850	2.0010
2	0.5511	1.2250
3	0.3834	0.6878
4	0.2081	0.3128
5	0.0764	0.1033
6	0.0440	0.0514
7	0.0287	0.0328
8	0.0139	0.0245
9	0.0023	0.0046
10	0.0005	0.0009

where

$$A_0 = 2.657587;\ a_0 = 1.000794;\ a_1 = -0.006762;$$
$$a_2 = -0.001356;\ a_3 = 2.5202 \circ 10^{-5};\ a_4 = -4.7826 \circ 10^{-6}$$

The solution (6.1) is convenient for comparison because it is a compact analytical expression. Let us note in passing that the solution (6.1) obviously does not satisfy equation (2.3) at infinity. That is why the solution obtained by Nagashima–Kuwahara is representative only in a limited interval around zero. The same shortcoming is exhibited also in the Fourier–Galerkin method proposed here since the basic set of functions is not adequate to show the FKDV behavior at infinity. The advantage of the spectral method is the possibility of increasing the number of modes N to achieve an arbitrary precision for an arbitrarily large finite interval.

In Fig. 2 the comparison of the numerical solutions obtained by us with those from Nagashima and Kuwahara [2] is shown (naturally, only a fixed finite interval is concerned). The coincidence of the values of the solution extremums is also indicative. The first three of them are denoted by Nagashima and Kuwahara [2] by U_0, U_1 and U_2:

$$U_0 = 2.657586 \qquad U_1 = -0.0810444 \qquad U_2 = 0.00352966$$

while obtained here values for $N = 20$ are:

$$U_0 = 2.657573 \qquad U_1 = -0.0809546 \qquad U_2 = 0.00321076$$

7. CONCLUSIONS

The results of this work suffice to claim that the method of Fourier–Galerkin developed in previous works of the authors [3–6] is effective when applied to differential equations of a higher order such as FKDV. The results obtained show that the increase of the order of the differential equation does not complicate significantly the method in the course of implementation. It does not require a large expenditure of calculation time—all computations are carried out on a PC–IBM AT and do not take more than a couple of minutes. This is mentioned here in order to stress the point that the method proposed is very convenient for problems of this type.

The good quantitative comparison of our results with those of Nagashima and Kuwahara [2] confirms the conjecture that FKDV does indeed provide a solution of the type of solitary wave.

Acknowledgements—This research was supported in part by the Ministry of Science and Higher Education of Bulgaria under grant 1052.

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