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INELASTIC INTERACTION OF BOUSSINESQ SOLITONS

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Two *improved* versions of Boussinesq equation (Boussinesq paradigm) have been considered which are well-posed (correct in the sense of Hadamard) as an initial value problem: the Proper Boussinesq Equation (PBE) and the Regularized Long Wave Equation (RLWE). Fully implicit difference schemes have been developed strictly representing, on difference level, the conservation or balance laws for the *mass*, *pseudoenergy* or *pseudomomentum* of the wave system. Thresholds of possible nonlinear blow-up are identified for both PBE and RLWE.

The head-on collisions of solitary waves of the *sech* type (Boussinesq solitons) have been investigated. They are subsonic and negative (surface depressions) for PBE and supersonic and positive (surface elevations) for RLWE. The numerically recovered sign and sizes of the phase shifts are in very good quantitative agreement with analytical results for the two-soliton solution of PBE.

The *subsonic* surface elevations are found to be not permanent but to gradually transform into oscillatory pulses whose support increases and amplitude decreases with time although the total *pseudoenergy* is conserved within 10^{-10}. The latter allows us to claim that the pulses are solitons despite their "aging" (which is felt on times several times the time-scale of collision).

For *supersonic* phase speeds, the collision of Boussinesq solitons has *inelastic* character exhibiting not only a significant phase shift but also a residual signal of sizable amplitude but negligible *pseudoenergy*. The evolution of the residual signal is investigated numerically for very long times.

1. Introduction

J. Scott Russell's discovery (August, 1834) around "Turning Point" in Union Canal near Edinburgh and later on laboratory systematic experimental investigations on waves (Russell [1838, 1845, 1895], see also Bazin [1865], Daily & Stephan [1951]) led him to pay great attention to a particular type which he called the "solitary wave." This is a wave consisting of a single elevation, of height not necessarily small compared with the depth of the fluid, which, if properly started, may travel for a considerable distance along a uniform canal with little or no change of shape. Russell's "solitary" type may be regarded as an extreme case of Stokes' oscillatory waves of a permanent type, the wavelength being great compared with the depth of the canal, so that the widely separated elevations are practically independent of one another [Stokes, 1847, 1849]. The methods of approximation employed by Stokes, however, become unsuitable when the wavelength much exceeds the depth [Lamb, 1945, art. 252, p. 423].

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Russell's observations and conclusions were at first sight in conflict with Airy's wave theory [Airy, 1845] where "a wave of finite height of length great compared with the depth must inevitably suffer a continual change of form as it advances, the changes being the more rapid the greater the elevation above the undisturbed level" [Lamb, 1945, art. 252]. It was not until the works of Boussinesq [1871a, 1871b, 1872] and Lord Rayleigh [1876] (see also McCowan [1895]) that due credit was given to the Russell's findings. Indeed, Boussinesq provided a theory (see also [Keulegan & Patterson, 1940; Rayleigh, 1876]) where higher-order linear dispersion was introduced in the long-wave limit and the crucial role of the balance between nonlinearity and dispersion was shown, albeit not clearly seen by later authors (for an illuminating paper on the subject, see Ursell [1953]). Particularly relevant is that Boussinesq found analytical expression of *sech* type for the permanent long-wavelength waves which are solutions of the equation he derived. A precursor of Boussinesq in the same problem was Lagrange, according to Lord Rayleigh who also says "I have lately seen a memoir by Boussinesq in which is contained a theory of the solitary wave very similar to mine. So far as our results are common, the credit of priority belongs, of course, to Boussinesq" (p. 279).

Later on, exploring consistently the simplifications of the problem under the assumption of slow evolution in the moving frame (already in Boussinesq derivations) and quoting the work of Boussinesq, as well as acknowledging the great influence of Lord Rayleigh's work [Rayleigh, 1876], Korteweg and de Vries [1895] derived an evolution equation governing the wave profile, now called the Korteweg–de Vries equation (or KdVE). For waves slowly evolving in the moving frame (quasistationary) the Boussinesq equation strictly reduces to KdVE in the right-moving coordinate frame. Naturally, the same analytical solution of *sech* type is then valid also for the KdVE, and that was perhaps the solution which has received the greatest attention during the last two decades. This bias is also the result of great discoveries made by Zabusky & Kruskal [1965] on the one hand, and Gardner *et al.* [1967] on the other.

Korteweg and de Vries found also another solution — the *cnoidal* wave train — that at two appropriate limits reduces to the Boussinesq solitary wave and to the harmonic wave train, respectively. (For accounts of the approximations leading to KdVE and related matters see Bullough [1988], Grimshaw [1986], Sander & Hutter [1991], Ursell [1953]).

Motivated by the study of Fermi, Pasta & Ulam [1955] on energy sharing in nonlinear discrete systems, Zabusky & Kruskal [1965] investigated the interaction of nonlinear waves in the KdVE and introduced the notion of "soliton" for waves that upon (overtaking) collision behave as *particles*. It was in classical mechanics the opposite side of the de Broglie wave-particle dichotomy in quantum mechanics. The first numerical results [Zabusky & Kruskal, 1965] showed that save the phase shift they experienced in the course of collision the interaction of the "solitary" particle-waves appeared elastic (hereafter the coinage "solit-*on*"). Moreover, now we have laboratory experiments with waves that we interpret as the result of particle collisions, bound-states, and so on [Weidman *et al.*, 1992; Linde *et al.*, 1993a; Linde *et al.*, 1993b; Velarde *et al.*, 1994a, 1994b], while in quantum mechanics the collision or scattering of two particles is interpreted by means of Schrödinger waves.

The problem with the KdVE is that it is an equation for slow evolution of a wave system in a frame moving with the characteristic speed, i.e. when the differences between the phase speeds and the characteristic speed are small (within the first order of the appropriate small parameter). In other words, the KdVE is valid only for a wave system that is slowly evolving in the moving frame. The one-way approximation completely rules out the possibility to investigate head-on collisions in the framework of KdVE. An equation describing two-way waves should contain second time-derivatives (as apparently, albeit not in reality, does the Boussinesq equation).

2. Boussinesq Equations: Variations on a Theme

2.1. *Original, proper, improper, and improved equations*

The $(1+2)$D potential flow with free surface is a well-posed initial-value problem for the longitudinal velocity component u and the relative surface elevation h. This system can be reduced to $(1+1)$D by means of an approximate solution for the 2D bulk flow under the assumption that the scale of motion λ is long in comparison with the depth of layer H. If no additional simplifications are effected, the reduced $(1+1)$D system will also be well-posed as an initial-value problem.

From the approximate (but correct) system, Boussinesq succeeded to derive an ill-posed one-equation model through mathematically illicit, albeit physically relevant, assumptions about the approximate relation between $\frac{\partial}{\partial t}$ and $\gamma\frac{\partial}{\partial x}$, where $\gamma = \sqrt{gH}$ is the characteristic velocity of the system (characteristic speed of gravity waves). The said relation is physically pertinent only for a solution that evolves slowly enough in the right-going frame. Thus he was the first to derive an equation containing dispersion in the form of fourth spatial derivative. The equation derived by Boussinesq himself [1971a, 1971b, 1972] we shall refer to as the "Original Boussinesq Equation" (OBE) (Eq. (26) Boussinesq [1872]):

$$\frac{\partial^2 h}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[(gH)h + \frac{3g}{2}h^2 + \frac{gH^3}{3}\frac{\partial^2 h}{\partial x^2}\right]. \qquad (1)$$

One should not be deceived by the presence of second time derivatives in (1). Strictly speaking, this equation is a one-way wave equation due to the assumptions under which it was derived. In addition, the sign of the fourth spatial derivative in (1) is positive which makes it incorrect in the sense of Hadamard. The OBE [Eq. (1)] has the same singularity as the heat equation with reversed direction of time. The ill-posedness of (1) does not prevent it from having **stationary** solutions of physical interest, i.e., its restriction to the sub-space of solutions spanned by the stationary propagating waves is mathematically correct. We shall address the problem of quasistationarity in the moving frame solution later on.

The assumptions related to the moving frame led Boussinesq to establish the approximate relation $h = u\sqrt{Hg^{-1}}$ between the surface elevation and the longitudinal velocity component of velocity. Then Eq. (1) holds also for the velocity component u within the same order of approximation. Upon introducing the dimensionless variables

$$x = \lambda x', \quad h = h_* h', \quad u = \sqrt{gH}u', \quad t = \sqrt{\frac{\lambda}{gH}}t',$$

one arrives at the following dimensionless form of Eq. (1):

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta\frac{\partial^2 u}{\partial x^2}\right],$$
$$\alpha = -\frac{3}{2}\frac{h_*^2}{H^2}, \quad \beta = \frac{1}{3}\frac{H^2}{\lambda^2}, \qquad (2)$$

where the "primes" are omitted without fear of confusion since in what follows we shall use only the dimensionless form. Here α stands for the dimensionless parameter of nonlinearity (amplitude parameter) and β is the dimensionless dispersion parameter. These are supposed to be small quantities in order for Boussinesq derivation to be valid.

Including the effect of surface tension as was done by Korteweg & de Vries [1895] (see also Kaup [1975] for the two-way equation) one arrives once again at the Boussinesq equation but with a modified coefficient of the fourth derivative:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta\frac{\partial^2 u}{\partial x^2}\right],$$
$$\beta = \left(\frac{1}{3} - \frac{\sigma}{\rho g H^2}\right)\frac{H^2}{\lambda^2}, \qquad (3)$$

where σ and ρ are the air-liquid surface tension and density of the fluid respectively.

Equation (3) is correct in the sense of Hadamard (well-posed as an initial-value problem) only when the surface tension is strong enough in order to have $\beta < 0$. When this is the case we call the above equation Proper Boussinesq Equation (or PBE).

Expecting no confusion in the reader we shall call indistinctly OBE and PBE [Eqs. (2) and (3) respectively] Boussinesq equations (BEs). Other authors call it "good" Boussinesq equation [Manoranjan *et al.*, 1984].

The ill-posedness of OBE can be removed if the Boussinesq assumption that $\frac{\partial}{\partial t} \approx \frac{\partial}{\partial x}$ is used in "reverse," i.e., when some spatial partial derivatives are replaced by their temporal counterparts. In the case of one-equation model it amounts to replacing $\frac{\partial^2}{\partial x^2}$ by $\frac{\partial^2}{\partial t^2}$, namely,

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \alpha u^2 + \beta\frac{\partial^2 u}{\partial t^2}\right]. \qquad (4)$$

For the two-equation model, it was first done by Peregrine [1966] and proved crucial for the direct numerical solution for the undular bore. The detailed arguments and derivations for "reversion" of Boussinesq derivations can be found in Benjamin *et al.* [1972], Peregrine [1966], Whitham [1974]. Equation (4) is called by some authors "improved" (IBE), but the latter coinage is too dangerously close to the "improper" BE and we shun to use it. Following the established terminology we call Eq. (4) Regularized Long Wave Equation (RLWE).

As argued by many authors (see, e.g., Bona *et al.* [1980] for the KdVE), the original (with purely spatial derivatives) and the "improved" (RLWE-type) models should be closely interrelated. We sustain the same thesis for the more general two-way equations outlined above.

2.2. *The solitary wave*

All versions of the Boussinesq equations considered here possess solitary-wave solutions of *sech* type. For the BEs [Eqs. (2), (3)], the solution reads

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{c^2-1}{\beta}}\right). \quad (5)$$

[Here the phase velocity is made dimensionless by means of $\gamma = \sqrt{(gH)}$.]

The corresponding solution for Eq. (4) (RLWE) is

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\operatorname{sech}^2\left(\frac{x-ct}{2c}\sqrt{\frac{c^2-1}{\beta}}\right). \quad (6)$$

The mathematically correct Boussinesq equation (namely PBE) is encountered for $\beta < 0$, or, what is the same, $\sigma > 1/3\rho g H^2$, i.e., when one is faced with capillary waves with surface tension being of prime importance in the problem. This case corresponds to subcritical bifurcation of the surface equilibrium state (see Grimshaw, [1986], Hunter & Vanden Broeck [1983]) and then the shape of the surface is a solitary depression. The case of supercritical bifurcation requires $\beta > 0$ (or $\sigma < \frac{1}{3}\rho g H^2$), and corresponds better to the findings of J. Scott Russell. But then (as already mentioned) the mathematically correct equation is RLWE. So, one can see that, in fact, RLWE is the mathematical object that fulfills the programme of Boussinesq for explaining Russell's experiments, rather than the OBE which was derived by Boussinesq himself. Boussinesq ruled out negative waves (depressions) and only considered one-side moving waves $(x - xt)$.

Yet, β of order of unity or larger should not be considered for RLWE in the context of long surface waves because such values violate the main assumption of long wavelength (see, e.g., Hunter & Vanden Broeck [1983]). As already noted, the long-wave assumption enters the model through the requirement $\beta \ll 1$ while the dimensionless length of the waves is supposed to be of the order of unity. Then the *sech* solitons of arbitrary wave speed *are not long waves* in this sense. They are *rather short*, their scale being of the order of $\sqrt{\beta}$. The only "long" *seches* are those moving at phase velocities, c, sufficiently close to the characteristic speed, namely, when $c \approx [1 + O(\beta)]$. But then their amplitude is $O(\beta)$, i.e., they are weakly nonlinear (see Sander & Hutter [1991], Ursell [1953], for detailed discussions of these interrelated properties). Without the weakly-nonlinear assumption (which must be considered as *additional* to the long-wave assumption), one finds for the *sech* solutions that their support is of order β^{-1}. Thus, on the *sech* solution, the higher-order derivatives neglected in Boussinesq's truncation procedure are of the same order of magnitude as the retained fourth-order derivative. Then it is clear that if one is to progress beyond the territory of weakly-nonlinear assumption, one must retain higher-than-fourth order derivatives in the truncation process. Thus the simplified equations cannot be accepted as rigorously derived as far as the solitary (permanent, nonlinear, etc.) waves of *arbitrary wave speeds* are concerned. On the other hand, the crucial idea of incorporating (linear) dispersion in the model (which was the point of Boussinesq against Airy) was sound and led to the discovery of *seches* which testified that a local balance between the linear dispersion and non-linear steepening of a wave is possible [Ursell, 1953]. There can be mentioned numerous works on the generalization of KdVE or BEs, e.g., Hunter & Vanden Broeck [1983], Kawahara [1972], Nagashima & Kawahara [1981], Pomeau *et al.* [1988] which deal with incorporation of the effects of higher-than-fourth-order terms for the dispersion, but we shall not dwell on this matter here.

In retrospect one can concede the right to Airy's conclusion that when no dispersion exists, the permanent localized solutions with long wavelengths are not possible. We have a case opposite to Boussinesq's: a mathematically correct problem, albeit of no physical relevance! Experiments did show solitary-like behaviour and highly crested wave trains, whose study finally led to the development of a new chapter in mathematical physics thanks in particular to the works of Gardner *et al.* [1967], Zabusky & Kruskal [1965], among others. However, solitons are ideal entities, very much like hard spheres used to define a perfect gas, and like the hard spheres, which have only purely repulsive force and no attraction, they cannot allow for bound states and condensation. Besides, the soliton-bearing equations are structurally unstable and the integrability property is lost when the slightest dissipation is added which is always the case in real life. For specific discussion of this problem and the existence of "dissipative" solitons, see Christov & Velarde [1994a, 1994b].

An analytical two-soliton solution is derived in Toda & Wadati [1973] only for the improper OBE by means of Hirota's bilinear technique. It has the form

$$\begin{aligned} u &= \frac{6}{\alpha}\left[\frac{\psi_{xx}}{\psi} - \frac{\psi_x^2}{\psi^2}\right], \\ \psi &= 1 + A_1 e^{k_1(x-c_1t)} + A_2 e^{k_2(x-c_2t)} \\ &\quad + A_3 e^{(k_1+k_2)x-(k_1c_1+k_2c_2)t}, \end{aligned} \tag{7}$$

where $c_i^2 \equiv 1 + \beta k_i^2$ and

$$A_3 = -\frac{(c_1k_1 - c_2k_2)^2 - (k_1 - k_2)^2 - \beta(k_1 - k_2)^4}{(c_1k_1 + c_2k_2)^2 - (k_1 + k_2)^2 - \beta(k_1 + k_2)^4} A_1 A_2 .$$

For $t \to \pm\infty$ the two-soliton solution degenerates into the respective one-soliton solutions, namely,

$$\frac{\alpha}{6} u(x,t) \to k_i^2 \operatorname{sech}^2\left[\frac{k_i}{2}(x - c_i t + \delta_i^{\pm})\right], \quad t \to \pm\infty . \tag{8}$$

But the important difference between the two-soliton solution and the sheer sum of two one-soliton solutions is the occurrence of phase shift which is defined for each soliton as $\delta_i = \delta_i^- - \delta_i^+$. The "faster" (or the "steeper") soliton (the one with the larger wave number) experiences the smaller phase shift according to the formula

$$\begin{aligned} k_1\delta_1 &= -k_2\delta_2 = \log\Phi , \\ \Phi &\equiv -\frac{(c_1k_1 - c_2k_2)^2 - (k_1 - k_2)^2 - \beta(k_1 - k_2)^4}{(c_1k_1 + c_2k_2)^2 - (k_1 + k_2)^2 - \beta(k_1 + k_2)^4}. \end{aligned} \tag{9}$$

As shown in Toda & Wadati [1973], $0 < \Phi < 1$, which means that the logarithm always exists and has negative value. Then the phase shift (the "phase lag") for the right-going soliton in OBE is always negative, i.e., it is being accelerated temporarily during the collision and appears at a position farther ahead in the direction of motion, rather than the expected position of arrival if there were no interaction with another soliton. The same is valid to the left-going soliton for which the phase shift is negative and it re-emerges after collision in a place ahead in the direction of its motion.

The reader should note here that Toda & Wadati called "phase shift" the quantity $k_i\delta_i$. We call "phase shift" the quantity δ_i which is in fact the displacement of the center of a soliton with respect to its projected position, i.e., the displacement of the center of the "phase pattern" called soliton. It is clear that the latter is the phase shift which can be observed in experiments.

Although in Toda & Wadati [1973] only the improper equation ($\beta > 0$) was considered, one finds that the two-soliton solution is valid also for the proper equation, provided that when the wave number is calculated the opposite sign ($\beta < 0$) is taken, i.e., $k_i = \sqrt{1 - c_i^2}$. The only difference is that Φ is now not always positive. We tabulated the function Φ for various values $c = c_1 = c_2$ and discovered that the sign of Φ changes approximately at $c = c_1 = c_2 = 0.867$ while Φ being always bigger than unity (the solitons are lagging in comparison with their projected positions). In fact $\Phi \to +\infty$ while c approaches 0.867 from above, and $\Phi \to -\infty$ while c approaches the same value from bellow. It goes beyond the frame of the present work to investigate in detail the zeros of the denominator of (9) for the case when $c_1 \neq c_2$.

2.3. *Boussinesq paradigm*

Boussinesq's was certainly a great mind and he correctly understood the basic theoretical aspect underlying Scott Russell's discovery. Yet Boussinesq's theory can only be considered in a purely heuristic manner. We believe the latter is enough for our purpose here.[1] Our stand is that the intrinsic physical relevance of a certain class of models should be sound and hopefully with robust property, beyond their dubious foundation, and not crucially depending on the existence or nonexistence of analytical solutions. This conviction, concerning in particular Boussinesq's theory, is the motivation of the present work in which by direct numerical simulation we show that, for instance, the main

[1]Another great mind, Lord Rayleigh, also appreciated Scott Russell's discovery. In a different problem, Bénard convection, he also clearly appreciated the relevant finding and provided a theory that later on was found not applicable to Bénard's experiment. Yet Bénard convection and Lord Rayleigh's paradigmatic work on buoyancy-driven convection are today outstanding model problems for nonlinear science.

properties of the two-soliton interaction, e.g. the sign and size of phase shift, are "universal" within what we shall from now on call the "Boussinesq paradigm." Thus we shall not concern ourselves with the quantitative correspondence of Boussinesq's theory to water waves, but rather shall deliberately consider here the Boussinesq equation(s) as a universal *out-of-context model problem* (a paradigm) containing nonlinearity and dispersion in appropriate balance which allows propagation of both left- and right-going waves of permanent form. The latter allows for investigation of the head-on collision of solitons. For extension of the Boussinesq Paradigm to "dissipative" solitons, see Christov & Velarde [1994a, 1994b], Nepomnyashchy & Velarde [1994].

It is clear now that for $\beta > 0$ the *seches* exist only if $|c| > 1$, i.e. they are "supersonic" or "superluminous" traveling signals (OBE and RLWE). Respectively for $\beta < 0$ they exist if $|c| < 1$, being "subsonic" (PBE). Then the OBE supersonic solitons cannot be simulated directly because of the linear instability of the equation (as it is incorrect in the sense of Hadamard). A similar situation occurs with the RLWE subsonic solitons when the corresponding sign of β leads to incorrect problems in the sense of Hadamard. This sets up the limitation of the numerical studies of collisions of Boussinesq solitons, namely subsonic case for PBE and supersonic, for RLWE.

3. Hamiltonian Representation

Before turning to numerical implementation, let us show the Hamiltonian form of the problem under consideration. For the BEs it was shown in Manoranjan *et al.* [1984, 1988]. It is interesting to note that the original Boussinesq system (first order in β truncation of the equations governing the free-surface ideal flow in a shallow layer) could not be put into a formally Hamiltonian form (energy might not be positive definite). As mentioned in Kaup [1975] it has "... too much nonlinearity." As a result, the representation of the one-equation Boussinesq model as a Hamiltonian system does not coincide with the original $O(\beta)$ system of two equations.

Here we extend the results of Manoranjan *et al.* [1984, 1988] to the case of RLWE but in terms of a somewhat different auxiliary function first introduced in Christov [1994]. Another generalization is that while the works [Manoranjan *et al.*, 1984, 1988] are concerned with infinite intervals we consider a finite spatial interval which is the case in numerical implementations. The shape of a localized solution approaches at each infinity a constant and hence all its derivatives decay automatically to zero, and when treating the problem analytically one may also impose boundary conditions on the second, third, etc. derivatives that follow from the condition for decay. These are called asymptotic boundary conditions. However, in numerics and in physics as well, one never has a true infinity and the exact form of the boundary conditions is of crucial importance. Indeed, any conservation or balance law is a property of the boundary value problem as a whole, not only of the equation itself.

Upon introducing an auxiliary function q one shows that the BE follows from the system

$$u_t = q_{xx}\,, \tag{10}$$

$$q_t = u - \frac{dU(u)}{du} - \beta u_{xx}\,, \quad U(u) = -\frac{\alpha}{3}u^3\,. \tag{11}$$

The last system admits conservation laws if one of the following sets of boundary conditions are imposed:

$$u = 0\,, \quad q = 0 \quad \text{for} \quad x = -L_1,\, L_2\,, \tag{12}$$

or

$$u = 0\,, \quad q_x = 0 \quad \text{for} \quad x = -L_1,\, L_2\,, \tag{13}$$

where $-L_1$, L_2 are the values of the spatial coordinate at which we truncate the infinite interval ("actual infinities").

Defining the mass M, the pseudomomentum P, and energy E of the system as

$$\begin{aligned} M &\stackrel{\text{def}}{=} \int_{-L_1}^{L_2} u\,dx\,, \\ P &\stackrel{\text{def}}{=} \int_{-L_1}^{L_2} uq_x\,dx\,, \\ E &\stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \frac{1}{2}[u^2 + q_x^2 + U(u) + \beta u_x^2]dx\,, \end{aligned} \tag{14}$$

one can show that the following conservation/balance laws hold:

$$\frac{dM}{dt} = 0\,, \quad \frac{dE}{dt} = 0\,, \quad \frac{dP}{dt} = \left[\frac{\beta}{2}u_x^2\right]\Bigg|_{-L_1}^{L_2} \equiv F\,. \tag{15}$$

The first can be called a conservation law for the mass of the wave, and the second, conservation of the energy. Following Maugin [1992], Maugin & Trimarco [1992] we call the third equality a balance law for the pseudomomentum P, with F being the pseudoforce. In the elasticity applications envisaged in Maugin [1992], Maugin & Trimarco [1992], the energy is the kinetic energy plus stored elastic energy. In fluid dynamics applications the interpretation is not so unambiguous and depends on whether the function u is thought of as velocity or as surface elevation. In order to be on the safe side we call E pseudoenergy.

The conservation laws ensure that the mass and the pseudoenergy remain constant during evolution of the solution. For the case of asymptotic boundary conditions the balance law suffices to claim that the pseudomomentum remains constant as well. Numerically speaking, however, the boundary conditions are not asymptotic and it may happen that $u_x \neq 0$ at the actual infinities. Only for strictly symmetric cases, when the collision of two identical solitary waves is considered, is pseudomomentum conserved after a full cycle of reflections from the boundaries $x = -L_1, L_2$.

Let us now turn to the Hamiltonian representation of RLWE. Introducing the same auxiliary function q we show that Eq. (3) follows from the system

$$u_t = q_{xx}\,, \tag{16}$$

$$q_t - \beta q_{txx} = u - \frac{dU(u)}{du}\,. \tag{17}$$

The last system is of a lower order with respect to x and only requires one condition at each end. One has the choice between the following two boundary conditions:

$$q = 0 \quad \text{or} \quad q_x = 0 \quad \text{for} \quad x = -L_1, L_2\,. \tag{18}$$

Respectively the pseudomomentum and pseudoenergy read

$$\begin{aligned} P &\stackrel{\text{def}}{=} \int_{-L_1}^{L_2} u(q_x - \beta q_{xxx})dx\,, \\ E &\stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \frac{1}{2}[u^2 + q_x^2 + U(u) + \beta u_t^2]dx\,. \end{aligned} \tag{19}$$

The conservation laws for mass and pseudoenergy have the same form as for OBE while the balance law for pseudomomentum now adopts the form

$$\frac{dP}{dt} = \frac{1}{2}[q_x^2 - \beta q_{xx}^2]\Big|_{-L_1}^{L_2} \equiv F\,. \tag{20}$$

The first boundary condition (18) provides for the conservation of pseudoenergy but not of mass. The second boundary condition secures the conservation of both mass and pseudoenergy. On the other hand, the first one yields also $u = 0$ at $x = -L_1, L_2$ provided that the velocity has trivial values at the boundaries at the initial moment of time. In this instance it corresponds better to the real physical situation. Numerically speaking, the second boundary condition yields on each iteration to a linear problem of Neumann type that has ill-conditioned matrix unless some special care is taken. For this reason we consider here the first boundary condition. In fact the lack of conservation of the mass of wave is felt only if the *seches* draw very near to the boundaries. This is not a real limitation here since for RLWE we are only concerned with collisions among the solitons rather than with reflection from the boundaries. Anyway, it is a shortcoming of the RLWE model.

4. Difference Scheme

For the Proper Boussinesq Equation (PBE) a conservative fully implicit scheme was created in Christov [1994]. Here we extend it to RLWE.

Consider the set function $u_i \stackrel{\text{def}}{=} u(x_i)$ on the regular mesh in the interval $[-L_1, L_2]$ with spacing h, i.e.,

$$x_i = -L_1 + (i-1)h\,, \quad h = \frac{L_1 + L_2}{N-1}\,, \tag{21}$$

where N is the total number of grid points in the said interval.

We leave the problem of linearization to a later moment and show first the strictly conservative scheme which is inevitably nonlinear, namely

$$\begin{aligned} \frac{q_i^{n+\frac{1}{2}} - q_i^{n-\frac{1}{2}}}{\tau} = \frac{u_i^{n+1} + u_i^{n-1}}{2} &- \alpha\frac{(u_i^{n+1})^2 + u_i^{n+1}u_i^n + (u_i^n)^2}{3} \\ &- \beta\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{\tau^2}\,, \end{aligned} \tag{22}$$

$$\frac{u_i^{n+1} - u_i^n}{\tau} = \frac{q_{i+1}^{n+\frac{1}{2}} - 2q_i^{n+\frac{1}{2}} + q_{i-1}^{n+\frac{1}{2}}}{h^2}\,, \tag{23}$$

with boundary condition

$$q_1^{n+\frac{1}{2}} = q_n^{n+\frac{1}{2}} = 0\,. \tag{24}$$

We can prove that the difference approximation $\mathcal{E}$ of the pseudoenergy,

$$\mathcal{E}^{n+\frac{1}{2}} = \frac{1}{2}\sum_{i=2}^{N-1}\left[\frac{(u^{(n+1)})_i^2 + (u^n)_i^2}{2} - \alpha\frac{(u^{(n+1)})_i^3 + (u^n)_i^3}{3} + \frac{1}{2}\left(\frac{q_{i+1}^{n+\frac{1}{2}} - q_i^{n+\frac{1}{2}}}{h}\right)^2 + \frac{1}{2}\left(\frac{q_i^{n+\frac{1}{2}} - q_{i-1}^{n+\frac{1}{2}}}{h}\right)^2\right], \tag{25}$$

is conserved by the difference scheme in the sense that $\mathcal{E}^{n+1} = \mathcal{E}^n$. The balance law for pseudomomentum is also strictly satisfied on the difference level within the round-off error of the calculations with double precision.

In order to iteratively implement the nonlinear scheme we introduce an "inner" iteration for the sought functions

$$q^{n+\frac{1}{2},k}, \quad u^{n+1,k}$$

and approximate the nonlinear term as

$$-\alpha\frac{u_i^{n+1,k}u_i^{n+1,k-1} + u_i^{n+1,k}u_i^n + (u_i^n)^2}{3}.$$

The "inner" iterations start from the initial conditions

$$u_i^{n+1,0} = u_i^n \quad \text{and} \quad q_i^{n+\frac{1}{2},0} = q_i^{n-\frac{1}{2}}, \tag{26}$$

and terminate at $k = K$ when

$$\frac{\max|u_i^{n+1,K} - u_i^{n+1,K-1}|}{\max|u_i^{n+1,K}|} \leq 10^{-11}. \tag{27}$$

The value 10^{-11} is selected well above the round-off error 10^{-15} for computations with double precisions. It is important to note that in all cases described below the number of iterations was $K = 4$–6. After the inner iterations converge, the solution for the "new" time stage $n+1$ of the nonlinear conservative difference scheme is obtained, namely $u_i^{n+1} \stackrel{\text{def}}{=} u_i^{n+1,K}$.

The conservative scheme is stable in time provided that the "inner" iterations converge.

5. Results and Discussion

For the sake of definiteness we select in our experiments $\alpha = -3$ and $|\beta| = 1$. It is important to note, however, that the dependent and independent variables of the equations that belong to the Boussinesq paradigm can be re-scaled so as to reduce each case to the one considered here. Thus the only independent parameters that govern the evolution of a wave system are the phase speeds of the initial *seches*.

We begin with the results for PBE. Its most important trait is that the analytical solution for particle-like waves is only known for *subsonic* phase velocities. Respectively, the amplitudes are negative (i.e., the solitons are "depressions" on the water surface). Unlike the KdV case the faster solitons of PBE are of smaller amplitudes. This defines the essential feature of the Boussinesq solitons in this case.

When the phase velocities are close to the characteristic velocity (i.e., the weakly nonlinear limit), the phase shift is insignificant and the interaction of the two "particles" is completely *elastic*. The results for $c \geq 0.99$ are very instructive for understanding the advantage of a conservative scheme. The amplitudes of *seches* in this case are so small that if the energy was not strictly conserved by the scheme, even the slightest "leakage" of energy during the calculations would have led to eventual linear dispersion of the solution and disappearance of the permanent shapes (*seches*). We can also add here that the calculations for PBE were conducted for very large times and after multiple reflection from the boundaries, and the solitons perfectly preserved their shapes and individuality.

In Fig. 1 is presented a typical case of head-on collision of two equal *sech* Boussinesq solitons governed by PBE. The interaction is perfectly *elastic* save the significant positive phase shift. The solitons are being retarded because of the collision and re-emerge in positions that are behind the positions they would have reached if there were no collision. Qualitatively the same is the case in Fig. 2, but the phase lag is more than twice larger. In Fig. 3 and Fig. 4 are depicted two cases with unequal solitons. It is seen that the smaller soliton (here it is the faster one!) experiences larger phase lag. Table. 1 presents the systematic investigation of the phase shift as compared to the analytical formula (9). The quantitative agreement is very good. In this instance, we may state that the interactions presented in Figs. 1–4 are just the corresponding two-soliton solutions as obtained by the direct simulation.

What is more important is that our calculations allow one to establish the threshold of nonlinear

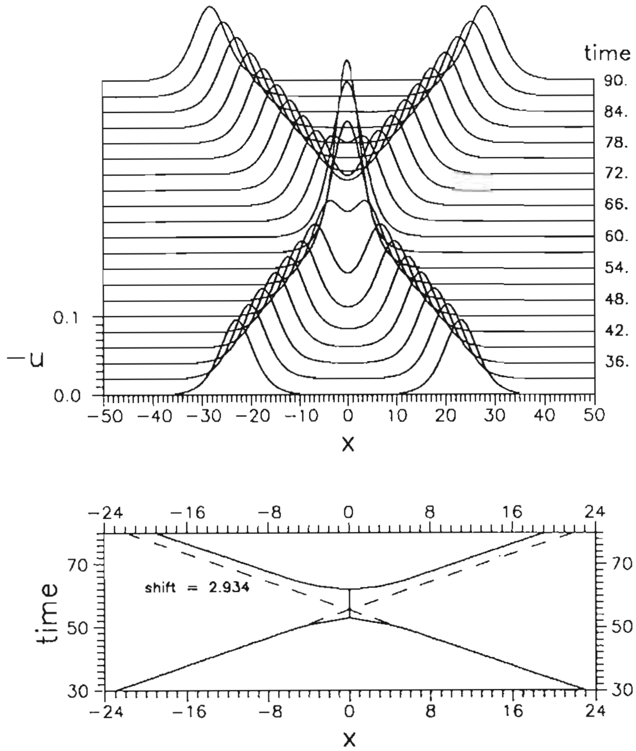

Fig. 1. Collision of two equal negative *seches* (depressions) in the PBE for $c_1 = -c_1 = 0.9$: (a) shape of wave function; (b) trajectories of centers of solitons.

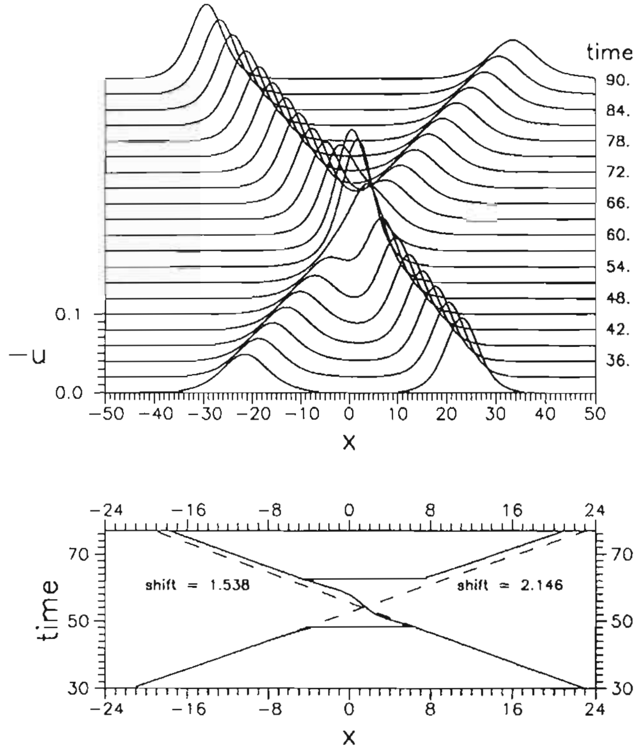

Fig. 3. Collision of two unequal *seches* in PBE for $c_1 = 0.95$, $c_2 = -0.9$: (a) shape of wave function; (b) trajectories of centers of solitons.

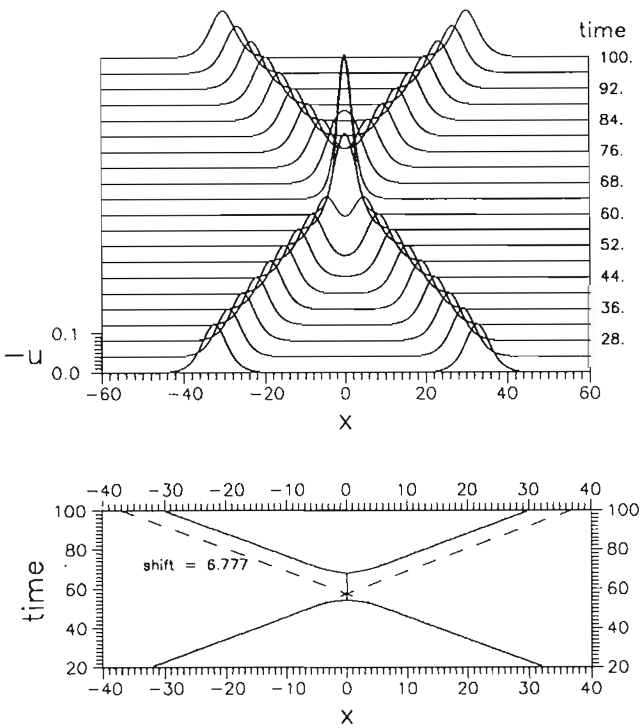

Fig. 2. Collision of two equal *seches* in PBE near the threshold of nonlinear blow-up for $c_1 = -c_2 = 0.87$: (a) shape of wave function; (b) trajectories of centers of solitons.

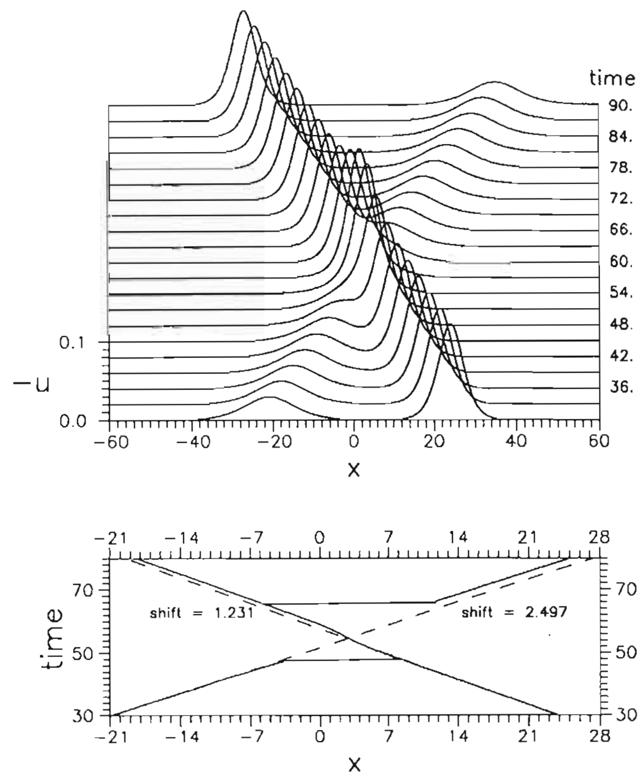

Fig. 4. Collision of two *seches* of significantly different amplitudes in the PBE for $c_1 = 0.97$, $c_2 = -0.87$: (a) shape of wave function; (b) trajectories of centers of solitons.

Table 1. Comparison of the Phase Shifts for PBE ($\beta < 0$).

c_1	s_1 Eq. (9)	s_1 (num.)	c_2	s_2 Eq. (9)	s_2 (num.)
0.97	0.859	0.911	−0.97	0.859	0.911
0.90	2.791	2.934	−0.90	2.791	2.934
0.87	6.716	6.777	−0.87	6.716	6.777
0.867	9.456	9.290	−0.867	9.456	9.290
0.95	2.117	2.146	−0.90	1.517	1.538
0.97	2.457	2.497	−0.87	1.212	1.231

blow-up [Turitzyn, 1993a, 1993b]. We discovered that the smallest phase velocity for which the calculations were possible with two equal solitons was $c = 0.867$. All the three digits of this result are in perfect agreement with the threshold of validity of the two-soliton solution as established here on the basis of the formula (9) (see Sec. 2.2). This means that the analytical two-soliton solution (9) of the PBE ceases to exist at the threshold of the nonlinear blow-up of solution.

It is interesting to check for the existence of permanent waves of positive amplitudes in PBE. This case was treated in Christov & Maugin [1991, 1994], Christov [1994] starting from different initial shapes for the positive bump. The numerical results show, for instance, that if a *sech* is taken with a positive amplitude, it cannot preserve its shape. The respective evolution is depicted in Fig. 5 and Fig. 6, for short time and long time respectively. One sees that a signal of smaller amplitude is emitted from the main hump and the reminder gradually transforms into an oscillatory pulse. Due to the recoil from the small signal the pulse is accelerated to the characteristic speed and then it continues its propagation with unit phase velocity. The evolution of the pulse is peculiar in the sense that although it retains its individuality, it is not a stationary creature. Its support increases with time while its amplitude decreases. This kind of behavior was behind the coinage "Big-Bang" pulses introduced by Christov & Maugin [1991, 1994]. Note that the total energy is conserved in our calculations up to 10^{-12}. We did actually perform numerical experiments with collisions of "Big-Bang" pulses and they did behave as solitons in the sense that they retained their individuality during the collision, and so did the total energy of the system. Since the definition of a "mass-center" of a pulse is somewhat ambiguous one cannot strictly assess the phase shift, if any. For the mass center defined on the basis of the modulus of the wave shape, we discovered no appreciable phase shift. All this suffices to claim that the pulses **are** solitons in the usual sense, although they are in fact "aging" solitons. During the cross-section of the collision they do behave as particles while the aging effect has much longer time scales

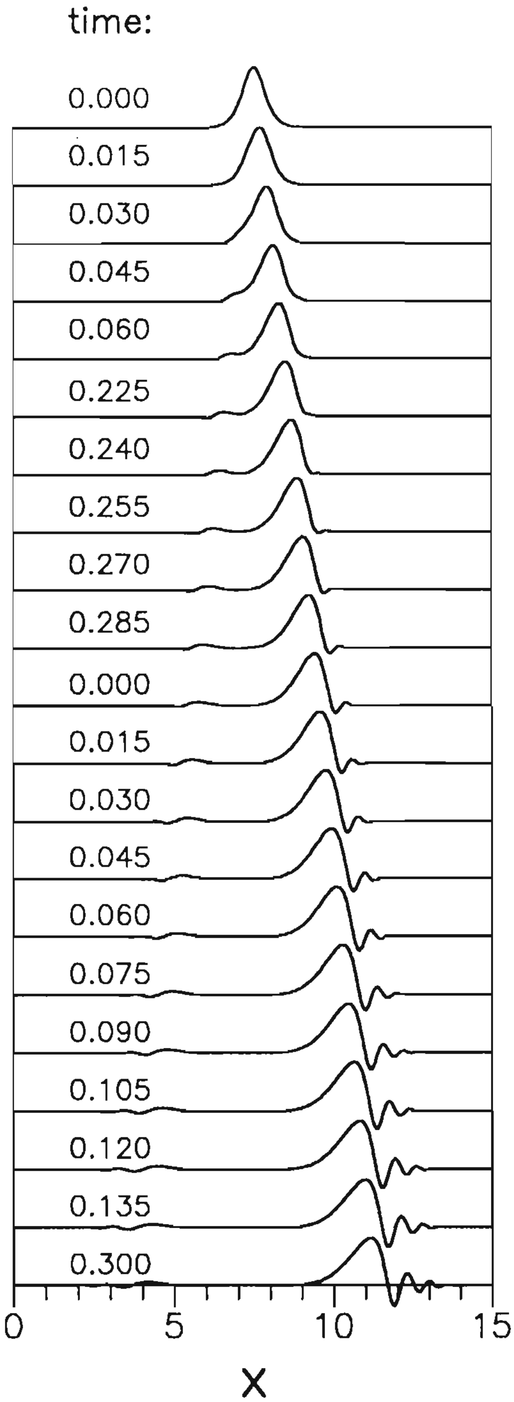

Fig. 5. Short-time evolution of a positive hump of *sech*-shape moving with phase speed $c = 0.9$.

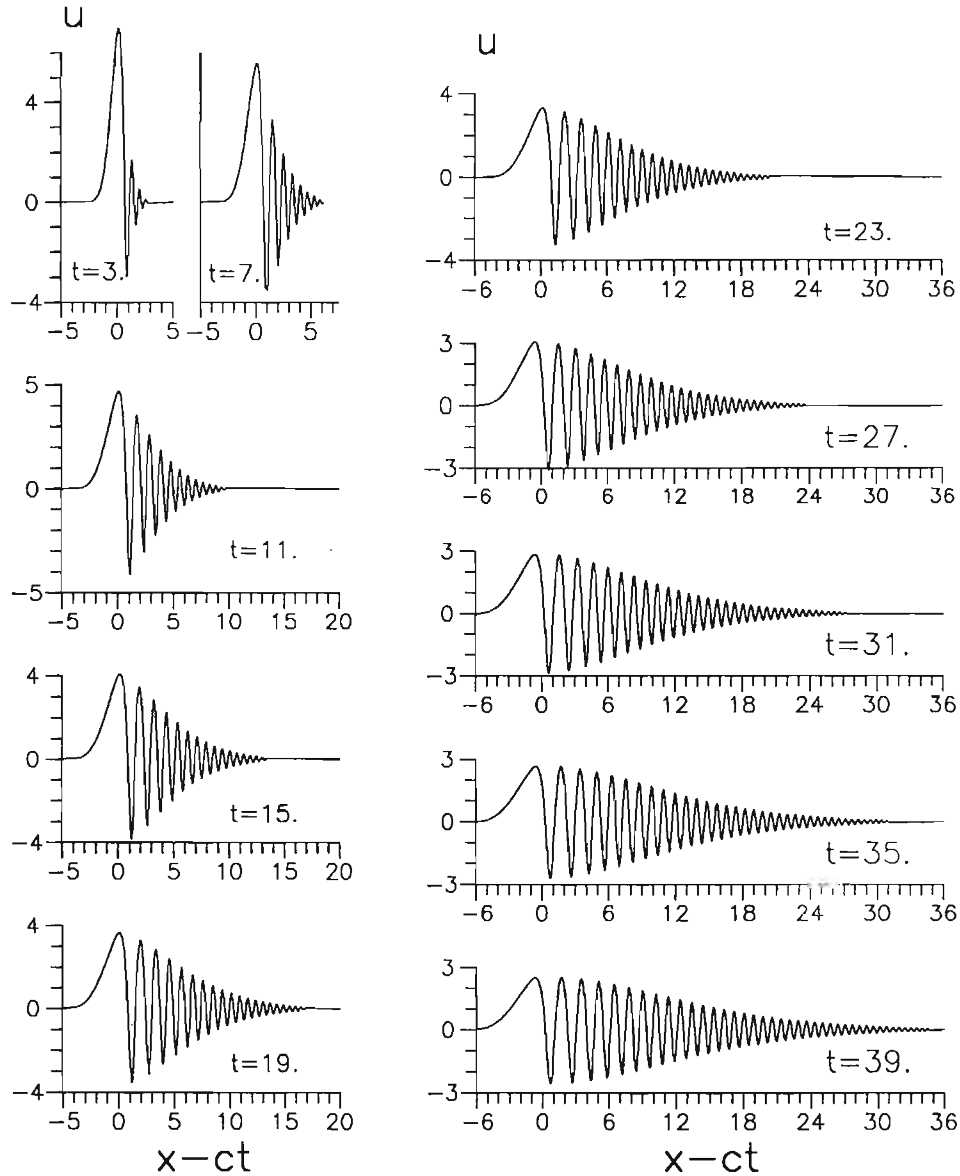

Fig. 6. Long-time evolution of the right-going pulse from Fig. 5.

and is felt on very long times, say one or two orders of magnitude larger than the time scale of the interaction. On the other hand, the aging (with possible extinction) is clearly a characteristic of "dissipative" solitons [Christov & Velarde, 1994a, 1994b]. Here, due to conservation of pseudoenergy, the extinction is strictly impossible and the aging means transformation of shape in time.

The "reversed" behavior of solitons of PBE is a clear warning that the latter represents a limited model. As argued above, within the Boussinesq Paradigm the particular equation is a matter of choice, so that we investigate also the solitonic properties of *sech* solutions of RLWE.

The RLWE solitons exist for *supersonic* phase speeds. Once again we start with an investigation of the weakly nonlinear case, e.g., when $c \approx 1.05$. Figure 7 gives an impression about the dynamics in this case. One sees that the interaction is perfectly elastic.

The signs of residual signal (*inelastic* collision) appeared to be first noticeable for $c \approx 1.2$ (see Fig. 8). In order to clarify the issue, we investigated systematically the inelastic collision and discovered that the threshold of nonlinear blow-up of the solutions is $c \leq 1.64$, i.e. for larger c the long-time evolution was impossible. Indeed, we encountered it for all cases for which $c > 1.64$, i.e. the

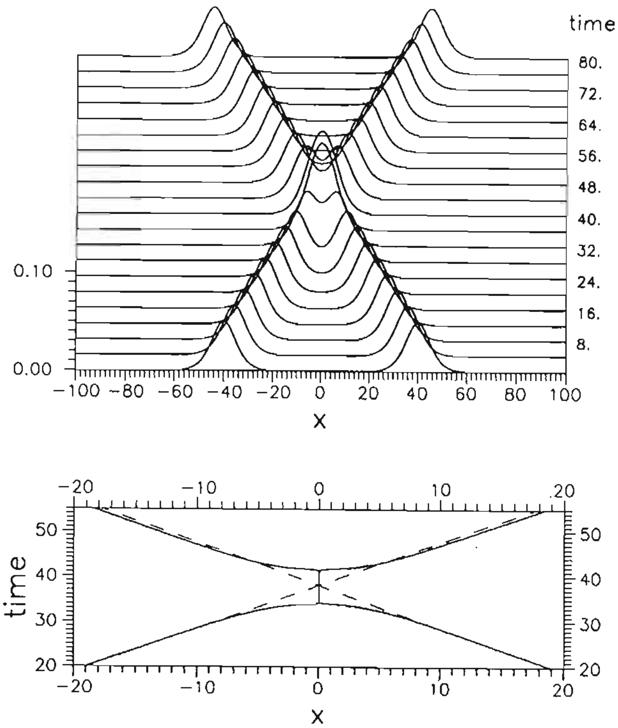

Fig. 7. Perfectly elastic interaction in RLWE for slightly supersonic phase velocities $c_1 = -c_2 = 1.05$: (a) shape of wave function; (b) trajectories of centers of solitons.

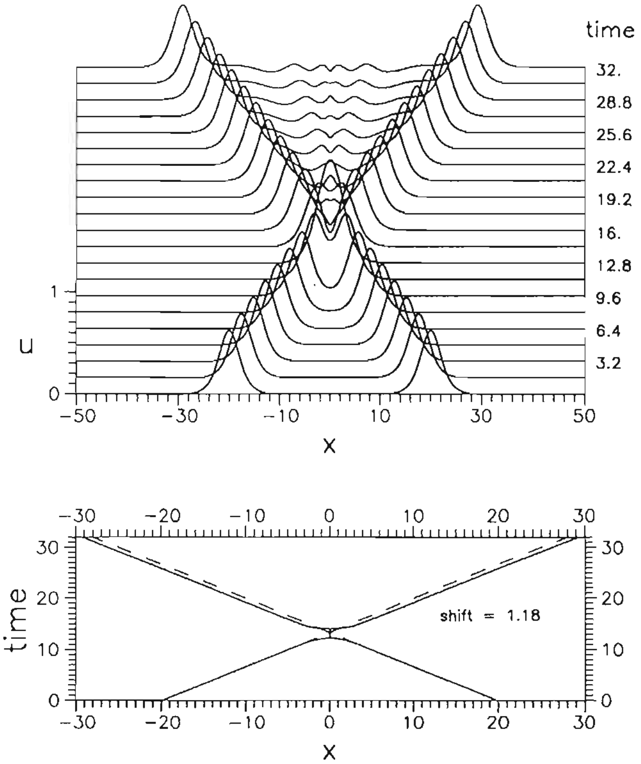

Fig. 9. The inelastic interaction in RLWE near to the threshold of nonlinear blow-up, $c_1 = -c_2 = 1.5$: (a) shape of wave function; (b) trajectories of centers of solitons.

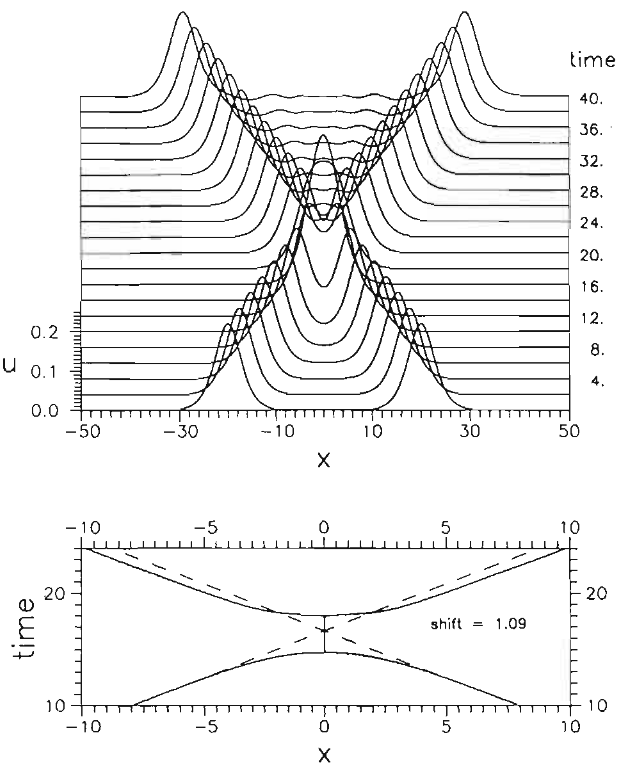

Fig. 8. The first signs of inelastic interaction in RLWE for slightly supersonic phase velocities $c_1 = -c_2 = 1.2$: (a) shape of wave function; (b) trajectories of centers of solitons.

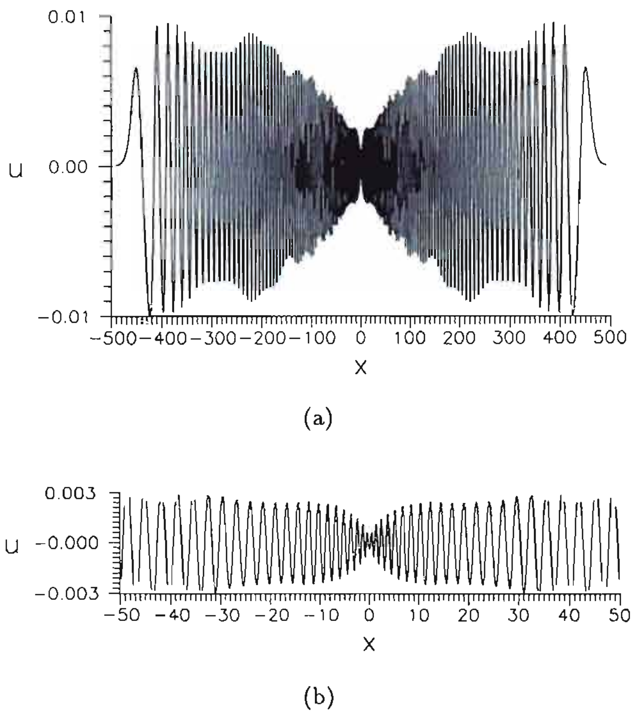

Fig. 10. The shape of residual signal in RLWE for the case of Fig. 9 for extremely large dimensionless time, $t = 500$ (100000 time steps): (a) the full residual signal; (b) shape of signal in the vicinity of point of collision.

additional signal that develops in the collision site adopts a shape which is dangerous enough in a sense that it can grow according to the provision of the theorem of Turitzyn [1993a, 1993b]. Technically speaking there develops a portion of the solution which if considered separately has negative pseudoenergy.

For this reason, when tracking the long-time behavior of the residual signal the phase velocity $c = 1.5$ was specially selected to be nonlinear enough within the limits where long-time evolution is still possible. Figure 9 exhibits a residual signal of a larger amplitude than the case for Fig. 8 and the phase shift is also larger. However, the residual signal created at the site of bygone collision must be of very small pseudoenergy content, because the total pseudoenergy of the system is conserved in our scheme, at the time when the *seches* re-emerge from the collision virtually unchanged in shape (and hence almost at their original pseudoenergy levels). To be specific, the pseudoenergy of the residual in Fig. 6 appears to be 0.4% of the total pseudoenergy of the initial system of waves.

We discovered that the residual signal is comprised of a small *breather* in the geometric center of the collision and two "Big Bang" pulses being constantly "fed" by the breather (see the profiles for larger times in Fig. 9). After the collision was over and the *seches* separated sufficiently, we removed them and solved numerically with the rest of the profile taken as initial condition. We went to dimensionless times as large as 500 (10^6 time steps for the case in Fig. 9) and used spatial resolution of 10^4 nodes. The shape of the residual is shown in Fig. 10(a), where the most striking feature is the nonuniform wavelength of the signal (the "Big-Bang" feature). The shape with shorter wavelength near the origin of the coordinate system is shown in Fig. 10(b) which is a zoom of Fig. 10(a). Note that the total pseudoenergy of the signal in Fig. 10 is exactly equal to the pseudoenergy of the residual in Fig. 9, although the signal in Fig. 10 looks much more complicated and the first intuitive guess is to say that it has much larger energy content. This is one of the apparently paradoxical

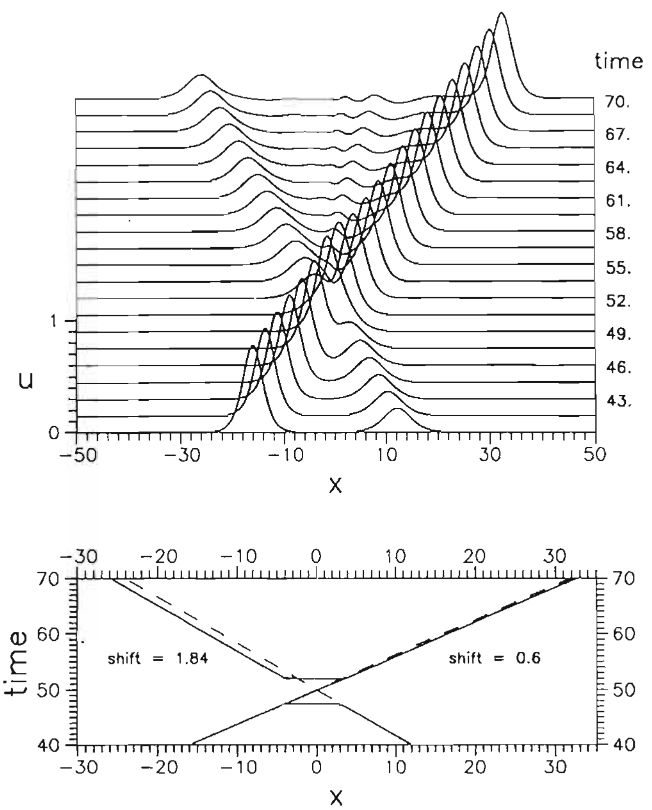

Fig. 11. Asymmetric RLWE collision for a case without blow-up $c_1 = 1.6$, $c_2 = -1.2$: (a) shape of wave function; (b) trajectories of centers of solitons.

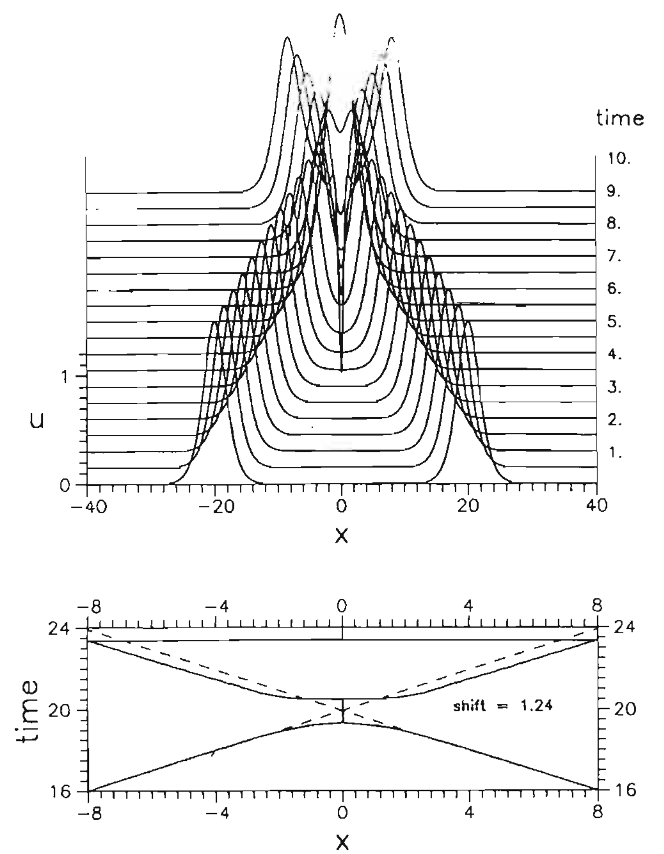

Fig. 12. Symmetric RLWE collision for moderate supersonic phase velocities $c_1 = -c_2 = 2$: (a) shape of wave function; (b) trajectories of centers of solitons.

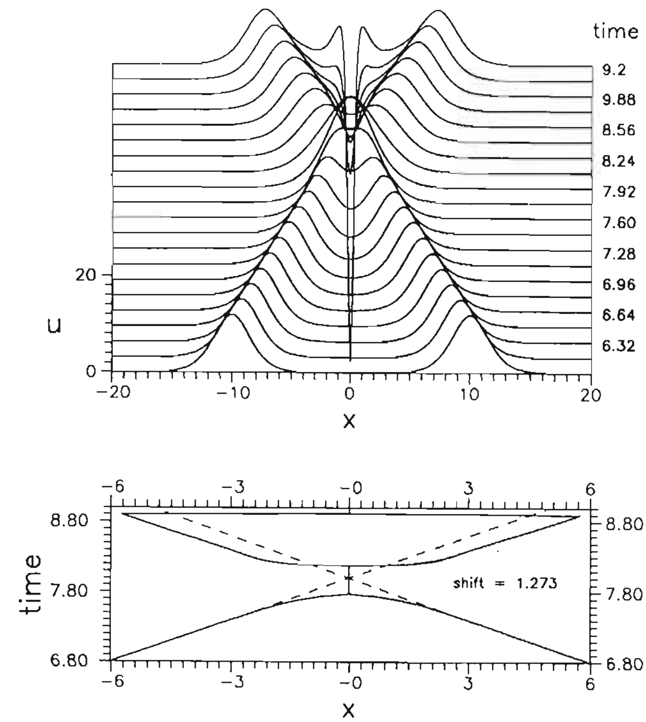

Fig. 13. Symmetric RLWE collision for very large supersonic phase velocities $c_1 = -c_2 = 5$: (a) shape of wave function; (b) trajectories of centers of solitons.

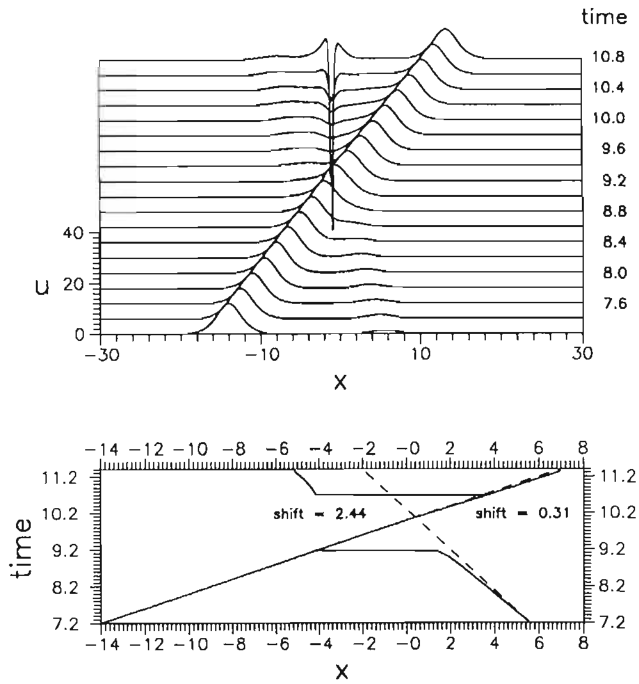

Fig. 15. The inelastic collision for two significantly different large phase speeds $c_1 = 5$, $c_2 = -2$: (a) shape of wave function; (b) trajectories of centers of solitons.

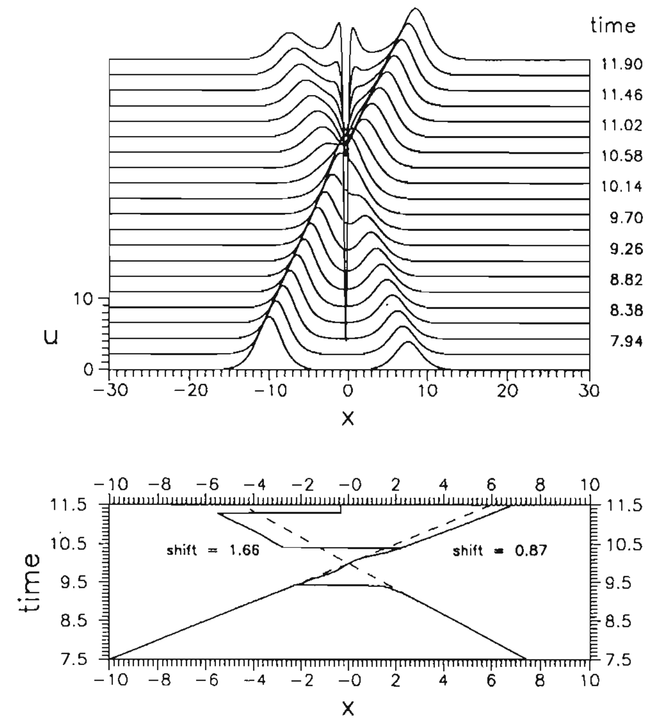

Fig. 14. Asymmetric inelastic collision for large phase speeds $c_1 = 4$, $c_2 = -3$: (a) shape of wave function; (b) trajectories of centers of solitons.

features of systems whose energy functional is not positive definite.

The first numerical results revealing the inelastic properties of RLWE solitons were reported by Bogolubsky [1977] who tackled a case roughly corresponding to our Fig. 9 by means of an explicit scheme. The finding of Bogolubsky [1977] was verified by Iskander & Jain [1980], whose scheme was already implicit but still not conserving. The lack of conservativeness of the schemes of these cited works left the door open for suspicions that the inelasticity of collision could be a numerical artifact. Because of conservation of energy we can claim that our results prove the fact that the RLWE solitons behave inelastically during their collisions.

An asymmetric evolution is presented in Fig. 11 for two different phase velocities $c_1 = 1.6$ and $c_2 = 1.2$, which are bellow the threshold of nonlinear blow-up.

The stability domain of our scheme significantly exceeds the schemes of Bogolubsky [1977], Iskander & Jain [1980] and we were able to proceed in a region of rather strongly pronounced inelasticity. The inelastic portion of the signal that appears after the interaction increases rapidly with the increase of the supersonic phase speeds of the solitons.

Table 2. Comparison for the Phase Shift for the RLWE ($\beta > 0$).

c_1	s_1 (9)	s_1c_2 (9)	s_1 (num.)	c_2	s_2 (9)	s_2c_1 (9)	s_2 (num.)
1.2	0.9808	1.177	1.09	−1.2	0.9808	1.177	1.09
1.5	0.8773	1.316	1.18	−1.5	0.8773	1.316	1.18
2.	0.6805	1.361	1.24	−2.	0.6805	1.361	1.24
5.	0.2768	1.384	1.273	−5.	0.2768	1.384	1.273
1.6	0.6020	0.723	0.6	−1.2	1.133	1.813	1.84
4.	0.3245	0.9735	0.87	−3.	0.4444	1.778	1.66
5.	0.1689	0.3378	0.31	−2.	0.4777	2.389	2.44

Figure 12 provides an illustration of this statement for $c_1 = c_2 = 2$. The phase shift is now conspicuous although it is not clear whether it could be safely interpreted physically, since the calculations are blown-up promptly after the maximum of the residual signal exceeds the amplitude of the original *seches*. Anyway, within the time interval of the existence of the solution we calculated the phase shift (see the lower figure in Fig. 12). It is interesting to look at the dependence of the phase shift on the phase velocities. For this reason we investigated also the case $c_1 = c_2 = 5$ as presented in Fig. 13. As it should have been expected, the value of the phase shift increases with the increase of the phase velocities of the solitons (qualitatively in agreement with formula (9).

First we discuss the result in Fig. 11 where the phase shift can be defined for each time stage since the calculations are stable and long-time evolution is possible.

In order to examine more closely the phase shift, two more cases with different phase speeds (Fig. 14 and Fig. 15) are considered. Results are presented in Table 2. It is seen that there is no quantitative agreement with the formula of Toda & Wadati. It is hardly surprising since RLWE corresponds to OBE with a dispersion coefficient that is decreased c times. There is a conspicuous relation, however, between the phase shifts for OBE (9) and RLWE. To this testify the third and seventh columns in Table 2. If one scales the result for s_1 with c_2 and s_2 with c_1 then there is consistent, good agreement with the numerical results of our study. The above "experimental correlation" fits very well all the cases of different phase velocities considered here. We deem the qualitative agreement for the phase shifts, as well as the very existence of the mentioned relationship, as a significant sign of the universality for the soliton dynamics within the Boussinesq Paradigm.

Note that the results presented in Figs. 14 and 15 are valid only for finite times and were presented here mostly for completeness and for the sake of examination of the phase shift.

6. Concluding Remarks

In the present work we have studied the mathematical objects called Boussinesq Equations (Boussinesq Paradigm). Two improved versions of the Boussinesq equation [called the Proper Boussinesq Equation (PBE) and Regularized Long Wave Equation (RLWE)] have been considered which are well-posed (correct in the sense of Hadamard) as an initial-value problem. Fully implicit difference schemes have been developed representing, strictly on difference level, the conservation laws for the mass and the pseudoenergy of the wave and the balance law for the pseudomomentum. The head-on collision of *seches* (Boussinesq solitons) has been thoroughly investigated. In PBE they are *subsonic* and of negative amplitude, while in the RLWE they are *supersonic* and positive.

The negative *seches* (the depressions) of PBE are subject to positive phase shift (phase lag) after they re-emerge from the collision but they retain their shape perfectly and no residual signals are detected. The numerically obtained phase lag is in very good quantitative agreement with the analytical prediction based on the two-soliton solution.

Our simulations reveal that the subsonic humps of positive amplitude are not stable and gradually transform into oscillatory pulses whose supports increase with time and whose amplitudes decrease at the time when the total pseudoenergy is conserved. Despite of their "aging" (which is felt on a time interval that is several times larger than the time-scale of the collision) our calculations suffice to claim these pulses also as solitons because of the conservation of energy during the collision.

For *supersonic* phase velocities the collision of Boussinesq solitons has been investigated on the basis of RLWE. Our numerical experiments have shown that the apparently elastic system governed by RLWE exhibits inelastic properties of collisions of solitary waves of the *sech* type (Boussinesq solitons). In the weakly nonlinear limit, when the *seches* move with almost the characteristic speed, the interaction is perfectly elastic and, as it should have been expected, no appreciable phase shift is detected within the order $O(1)$. For larger phase velocities the collision becomes conspicuously *inelastic*, and not only significant negative phase shift is experienced by the solitons, but also a post-collision residual signal arises at the collision site. The residual signal is of sizable amplitude, but of negligible pseudoenergy. Its evolution is tracked numerically for very large times and it appears that a breather is formed at the post-collision site. The breather constantly feeds two ever-expanding pulses of "Big-Bang" type. The phase shift obtained numerically for RLWE is in qualitative agreement with the analytical prediction for the original (improper) Boussinesq equation.

The thresholds for a nonlinear blow-up are identified numerically for both PBE and RLWE.

Finally, let us point out that some of our results fit rather well with the experimental findings recently reported in the literature [Linde *et al.*, 1993a, 1993b]. Velarde *et al.* [1994a, 1994b]. It has been found that although solitary waves can be created by an instability and they are dissipative traveling localized structures, yet for the time scale of the experiment (what we may call a very long *transient* time interval) the solitary waves and the crests of (periodic) wave trains do interact like dissipationless Boussinesq *seches* (see also Christov & Velarde [1994]).

Acknowledgments

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