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Inelasticity of Soliton Collisions in Systems of Coupled NLS Equations

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Abstract

Movatived by a problem of the dynamical elasticity of crystals with microstructure, a conservative numerical scheme is employed to study the very long time evolution and interaction of soliton-like solutions in systems of Coupled Nonlinear Schrödinger Equations. Head-on collision and overtaking receive special attention. The results obtained demonstrate the inelasticity (change of polarization) of the interaction even for initially circularly polarized components. Thus it may be said that the interactions in such systems break the symmetry of the input.

1. Introduction

Many problems of mathematical physics, condensed-matter physics, mechanics of solids or fluids, and biological structures lead to the consideration of systems of partial differential equations which comprise two nonlinearly coupled (cubic) Schrödinger (NLS) equations in the following normalized form

$$\mathrm{i}\,\frac{\partial \phi}{\partial t} + \beta\,\frac{\partial^2 \phi}{\partial x^2} + [\alpha_1 |\phi]^2 + (\alpha_1 + 2\alpha_2)|\psi|^2]\phi = 0,$$

$$\mathrm{i}\,\frac{\partial \psi}{\partial t} + \beta\,\frac{\partial^2 \psi}{\partial x^2} + [\alpha_1 |\psi]^2 + (\alpha_1 + 2\alpha_2)|\phi|^2]\psi = 0, \qquad (1)$$

where ϕ and ψ are complex-valued (x, t)-dependent scalars, i is the imaginary unit, and β, α_1 and α_2 are real scalars. The writing (1) exhibits a rather high symmetry between the two equations governing a two-degree of freedom system as the same coefficients for self-nonlinearities and mutual nonlinear couplings are considered. The same holds true of the dispersion coefficient β. In addition, as the same time derivative occurs in both components of the system (1), this means that the two corresponding linear modes were assumed to have the same characteristic velocities [i.e. the reduced time t in (1) was introduced in both equations using the same characteristic coordinate]. Such a situation will certainly prevail in optics and isotropic elasticity for two circularly polarized modes of two transverse shear modes, respectively. As a matter of fact, system (1) is canonical in many physical systems such as in nonlinear pulse propagation in optical fibers [1–4], in nonlinear wave interactions in discrete transmission lines [5–7], in antiferromagnetic chains [8], in plasma physics (ion-acoustic modes) [9, 10], in molecular structures of the DNA type in which a system of coupled sine-Gordon equations is shown to yield (1) after reduction to the case of small-amplitude modulated monochromatic motions [11], in shear-horizontal nonlinear wave propagation in superimposed layers [12], and more generally in nonlinear wave interactions involving two dispersive modes with equal group velocity [13, 14]. Closer to our usual interest in the mechanics of deformable solids and crystals is the appearance of systems such as (1) for coupled rotational modes in elastic Cosserat continua (also called micropolar continua, i.e., elastic crystals with a rigidly rotating microstructure) [15, 16] where, indeed, the two linear modes do have the same velocity. However, the application of (1) would be questionable in another case of great interest which concerns the nonlinear coupling between surface shear-horizontal and Rayleigh modes in elasticity with capillarity effects (i.e. dispersion) included [17] as, clearly, the correponding two linear modes have different group velocities.

What is to be retained from the wealth of applications of (1) is its canonical structure. From the mathematical point of view, the complete integrability of (1) has been established by Zakharov and Schulman [18] whenever $\alpha_2 = 0$. In each physical application this contraint materializes in a specific relationship to be satisfied jointly by the frequency and wave-number of the carrier wave whose two-component envelope is governed by (1), hence very peculiar points of the working regime for the carrier (along the linear dispersion relation of the physical system). In practice, however, one cannot expect to be in such a precise régime (if it exists at all) so that the constraint $\alpha_2 = 0$ must be relaxed, from which there follows the lack of complete integrability of the system in general. But it is still of interest in more realistic situations to look at the solitonic behaviour, in particular during collision or overtaking of waves as the system may not be too "bad", i.e., it may be somewhat nearly integrable

[14]. This is why the present work is devoted to the numerical simulation of collisions of solitary waves which are solutions of (1). In order to be closer to the aysmptotic boundary conditions we consider in our numerical study large enough spatial intervals $x \in [-L_1, L_2]$ at the borders of which the following trivial conditions are prescribed

$$\psi = \phi = 0 \quad \text{for} \quad x = -L_1, L_2.$$

2. Soliton solutions

The system under consideration is in a sense not strongly coupled and polarized solutions of type of $\psi \neq 0$, $\phi = 0$, or $\psi = 0$, $\phi \neq 0$ (which are in fact solutions to the decoupled system) do exist for the full system for arbitrary values of parameters. Yet the coupling present here is strong enough not to allow for the existence of envelopes with different carrier frequencies for the two unknown functions ϕ and ψ. In this instance, the envelope solitons should always be polarized (Parker *et al.* – for the coinage) in the sense that $\psi = C\phi$, where C is generally a complex constant. When $C = 0$ one has what Parker *et al.* call circular polarization and for $C = \exp 2i\nu$ – linear polarization. In both cases, the system reduces to a single NLS equation for which the one- and two-solition solutions are well known [19]. Our main purpose is to investigate collisions of initially polarized solutions and to understand the process of exciting of solutions with polarization different than the original. In fact a bifurcation takes place and alongside with the solution of original symmetry another solution occurs which is of different symmetry. The latter appears to be more stable and yields breaking of symmetry.

A "one-soliton" envelope solution of (1) is known in the form

$$\psi, \phi = a \exp(i\theta) \exp\{i[\tfrac{1}{2}c(x - ct) + nt]\} \times \operatorname{sech}\left[\frac{a(x - ct)}{\sqrt{2}}\right] \tag{2}$$

where

$$a^2 = 2(n - \tfrac{1}{4}c^2) > 0$$

and θ is the phase. Note that the coefficient a is the same for both functions ϕ and ψ by virtue of our choice $\alpha_1 = \alpha_2 = 0.25$. In the present work we also restrict ourselves to interactions of solitons of the same phase, and without loss in the generality we take $\theta_\phi = \theta_\psi = 0$.

We shall reasonably assume that the initial conditions of our simulations will indeed consist of two sufficiently set apart solitary-wave solutions (2). The difference scheme sketched out in Section 4 is especially devised for the purpose with due attention to the conservative features of this scheme for the coupled evolution system. The results are presented and discussed in Section 5 and illustrated by many graphs. The inelasticity of the interactions observed is not an artifact of the numerical scheme. This inelasticity is rather conspicuous for the selected values of α_1, α_2 and precludes any nice "nearly integrable" behaviour. This somewhat contrasts with the pictures obtained by Parker, Newboult and Faulkner [3, 4] using explicit computational schemes for a system in which ϕ and ψ are none other than the complex amplitudes of two circularly polarized modes, whereas our conclusion for general ϕ and ψ coincides with Hirota's analysis [19] – see Ref. [14].

3. Hamiltonian representation

First we introduce the notations

$$\phi = p(x, t) + ir(x, t), \qquad \psi = q(x, t) + is(x, t), \tag{3}$$

and then the original system is transformed to

$$\begin{aligned} r_t &= \beta p_{xx} + [\alpha_1(p^2 + r^2) + (\alpha_1 + 2\alpha_2)(q^2 + s^2)]p, \\ -p_t &= \beta r_{xx} + [\alpha_1(p^2 + r^2) + (\alpha_1 + 2\alpha_2)(q^2 + s^2)]r. \end{aligned} \tag{4}$$

and

$$\begin{aligned} s_t &= \beta q_{xx} + [\alpha_1(q^2 + s^2) + (\alpha_1 + 2\alpha_2)(p^2 + r^2)]q, \\ -q_t &= \beta s_{zz} + [\alpha_1(q^2 + s^2) + (\alpha_1 + 2\alpha_2)(p^2 + r^2)]s. \end{aligned} \tag{5}$$

It is easily shown that the quantity called mass or "total number of particles" is conserved for each of the modes ψ, ϕ, namely

$$\frac{d}{dt}\int_{-\infty}^{\infty} \frac{p^2 + r^2}{2}\, dx = 0, \qquad \frac{d}{dt}\int_{-\infty}^{\infty} \frac{q^2 + s^2}{2}\, dx = 0. \tag{6}$$

i.e. due to the "weak" coupling of the system, the functions ψ and ϕ are allowed to have separate conservation laws. It is not the case, however, with the energy of the system, which is a single property for both equations, namely

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} &[\alpha_1(q^2 + s^2)^2 + \alpha_1(p^2 + r^2)^2 \\ &+ 2(\alpha_1 + 2\alpha_2)(p^2 + r^2)(s^2 + q^2) \\ &- \beta(p_x^2 + q_x^2 + r_x^2 + s_x^2)]\, dx = 0, \end{aligned} \tag{7}$$

which is not a positive definite function (as usually for the NLS).

4. Difference scheme

Consider a uniform mesh in the interval $x \in [-L_1, L_2]$

$$x_i = (i - 1)h, \qquad h = \frac{L_1 + L_2}{N - 1},$$

where N is the total number of grid points in the interval under consideration.

In order to treat the very-long-time evolution of the solitons, we construct here a conservative scheme, reflecting the first two conservation laws for the system under consideration (mass and energy).

$$\begin{aligned} \frac{s_i^{n+1} - s_i^n}{\tau} &= \frac{\beta}{2h^2}[q_{i-1}^{n+1} - 2q_i^{n+1} + q_{i+1}^{n+1} \\ &+ q_{i-1}^n - 2q_i^n + q_{i+1}^n] + \frac{q_i^{n+1} + q_i^n}{4} \\ &\times \{\alpha_1[(q_i^{n+1})^2 + (q_i^n)^2 + (s_i^{n+1})^2 + (s_i^n)^2] \\ &+ (\alpha_1 + 2\alpha_2) \\ &\times [(p_i^{n+1})^2 + (p_i^n)^2 + (r_i^{n+1})^2 + (r_i^n)^2]\}, \end{aligned} \tag{8}$$

$$-\frac{q_i^{n+1}-q_i^n}{\tau}=\frac{\beta}{2h^2}\left[s_{i-1}^{n+1}-2s_i^{n+1}+s_{i+1}^{n+1}+s_{i-1}^n-2s_i^n+s_{i+1}^n\right]+\frac{s_i^{n+1}+s_i^n}{4}\times\{\alpha_1[(q_i^{n+1})^2+(q_i^n)^2+(s_i^{n+1})^2+(s_i^n)^2]+(\alpha_1+2\alpha_2)\times[(p_i^{n+1})^2+(p_i^n)^2+(r_i^{n+1})^2+(r_i^n)^2]\}. \quad (9)$$

A similar pair of difference equations holds for the functions p, r.

4.1. *Conservative properties*

The conservation of the masses of modes (number of particles) are easily seen to be represented by our scheme. After some algebra is is shown that the difference approximation of energy

$$\mathscr{E}^{n+1}=-\frac{\beta}{4}\sum_{i=2}^{N}\left[\left(\frac{s_i^{n+1}-s_{i-1}^{n+1}}{h}\right)^2+\left(\frac{q_i^{n+1}-q_{i-1}^{n+1}}{h}\right)^2+\left(\frac{p_i^{n+1}-p_{i-1}^{n+1}}{h}\right)^2+\left(\frac{r_i^{n+1}-r_{i-1}^{n+1}}{h}\right)^2\right]-\frac{\beta}{4}\sum_{i=2}^{N}\left[\left(\frac{s_i^{n}-s_{i-1}^{n}}{h}\right)^2+\left(\frac{q_i^{n}-q_{i-1}^{n}}{h}\right)^2+\left(\frac{p_i^{n}-p_{i-1}^{n})}{h}\right)^2+\left(\frac{r_i^{n}-r_{i-1}^{n}}{h}\right)^2\right]+\frac{1}{4}\sum_{i=2}^{N-1}\{\alpha[(s_i^{n+1})^2+(q_i^{n+1})^2]^2+\alpha_1[(p_i^{n+1})^2+(r_i^{n+1})^2]^2+2(\alpha_1+2\alpha_2)[(s_i^{n+1})^2+(q_i^{n+1})^2]\times[(p_i^{n+1})^2+(r_i^{n+1})^2]\} \quad (10)$$

is conserved in the sense that $\mathscr{E}^{n+1}=\mathscr{E}^n$.

4.2. *The linearization*

The above described conservative scheme is nonlinear and requires linearization. However, any change in the scheme would spoil the conservative properties. Then the only option is to introduce an iterative procedure (see [20]) and to conduct it until convergence, i.e., we substitute the sought values at the $(n+1)$-st stage by their approximations according to Newton's quasi-linearization

$$(p^{n+1,k})^3\approx 3(p^{n+1,k-1})^2p^{n+1,k}-2(p^{n+1,k+1})^3, \text{ etc.}$$

$$(p^{n+1,k})^2\approx 2p^{n+1,k-1}p^{n+1,k}-2(p^{n+1,k-1})^2, \text{ etc.}$$

$$(p^{n+1,k})^2q^{n+1,k}\approx(p^{n+1,k-1})^2q^{n+1,k}+2p^{n+1,k}p^{n+1,k-1}q^{n+1,k-1}-2(p^{n+1,k-1})^2q^{n+1,k-1}, \text{ etc.}$$

$$p^{n+1,k}q^{n+1,k}\approx p^{n+1,k-1}q^{n+1,k}+p^{n+1,k}q^{n+1,k-1}-p^{n+1,k-1}q^{n+1,k-1}, \text{ etc.}$$

Beginning from the initial conditions

$$p^{n+1,0}\equiv p^n, \text{ etc.}$$

the iterations are conducted until convergence, in the sense that

$$\max(N_p,N_q,N_r,N_s)\leqslant\varepsilon$$

where

$$N_p=\max_i\left[\frac{p^{n+1,k}-p^{n+1,k-1}}{p^{n+1,k}}\right], \text{ etc.}$$

For the calculations of double precision we require $\varepsilon=10^{-12}$. After the iterations converge, one obtains in fact a time step of the nonlinear scheme which has the desired property of conservativeness.

5. Results and discussion

In our computations we have taken different values of α_1, α_2. The significant interaction occurs for small α_1 and large α_2 but this case has little physical significance. For the applications envisaged (see [15, 16]), at most, the said coefficients can be equal. As a rule α_1 dominates over α_2. So we only present here the results obtained with equal coefficients. The obvious renormalization is $\alpha_1+\alpha_2=0.5$ which would lead to unit value of the coefficient of the nonlinear term if one takes $\phi\equiv\psi$. Hence $\alpha_1=\alpha_2=0.25$ is the case considered in the present paper.

As mentioned above, we concern ourselves with the solitary wave solutions. In order to investigate the interaction properties we compose the initial condition by means of superimposing two solitons of type (2). It is clear that this allows a vast room for maneuvering and it is impossible to track the evolution of all the interesting initial configurations of solitons. In order to simplify the task we consider only the "pure case" when the initial state is circularly polarized, in the sense that one of the solitons (the left one in the figures attached) has nontrivial value for function ψ and trivial zero for ϕ. It is shown in the upper box of the figures presented. The right soliton is $\phi=0$ and $\psi\neq 0$ and is depicted in the lower box of the respective figure. If there is to be inelasticity it would be much more conspicuous here that in the case treated by Parker *et al.*

The first to undergo investigation was the head-on collision. The important observation here is that for moderate and large phase velocities of the solitons, the time of interaction ("the cross section") is too short to allow excitation of coupled waves of considerable amplitude. The interaction is observable for sufficiently low phase velocities. Due to the nature of the envelope solitons, small phase speeds produce relatively short length scales of the soliton, in the sense that the support of the localized solution encompasses only fractions of a period of the carrier frequency. Then one is to consider also low carrier frequencies in order to span at least one period. This is exactly the case of head-on collision presented in the sequence of pictures in Fig. 1(a)–(f). Two polarized waves of equal carrier frequencies $n_{\text{left}}=n_{\text{right}}=0.05$ and equal celerities $c_{\text{left}}=c_{\text{right}}$ are considered. It follows automatically that their amplitudes are also equal. We clearly see the scenario of symmetry breaking – how the coupled wave is gradually excited during the interaction and carried away with its mate. The interaction, however, has further inelastic features and some additional vibrations are excited apart from the purely localized waves. This is not an artifact of the numerical scheme, because the latter is strictly conservative. It is not an artifact of the approximation

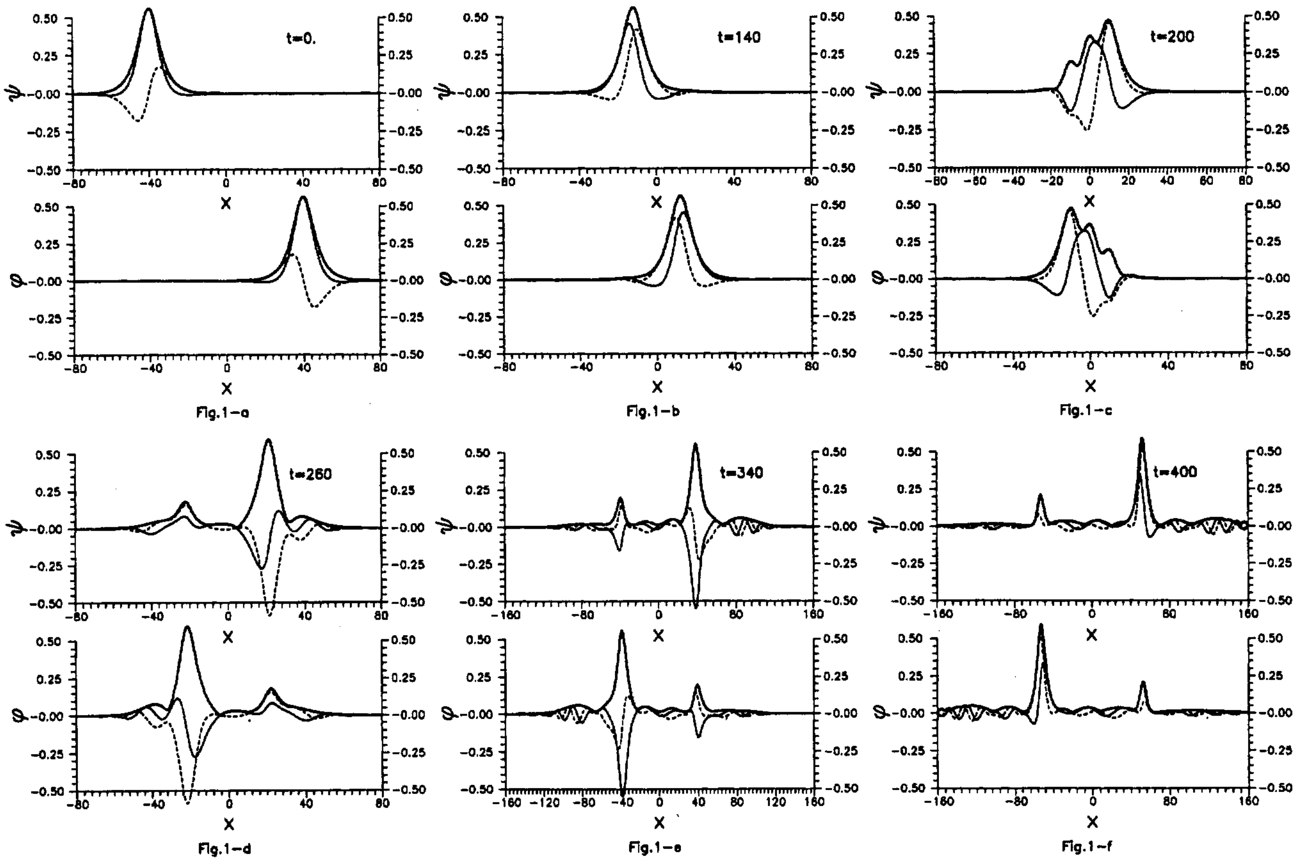

Fig. 1. Head-on collision for equal frequencies and amplitudes: $n_{\text{left}} = n_{\text{right}} = 0.05$; $c_{\text{left}} = c_{\text{right}} = 0.2$. The thick line represents the modulus; the thin line – the real part; the dashed thin line – the imaginary part. (a) $t = 0$, (b) $t = 140$, (c) $t = 200$, (d) $t = 260$, (e) $t = 340$, (f) $t = 400$.

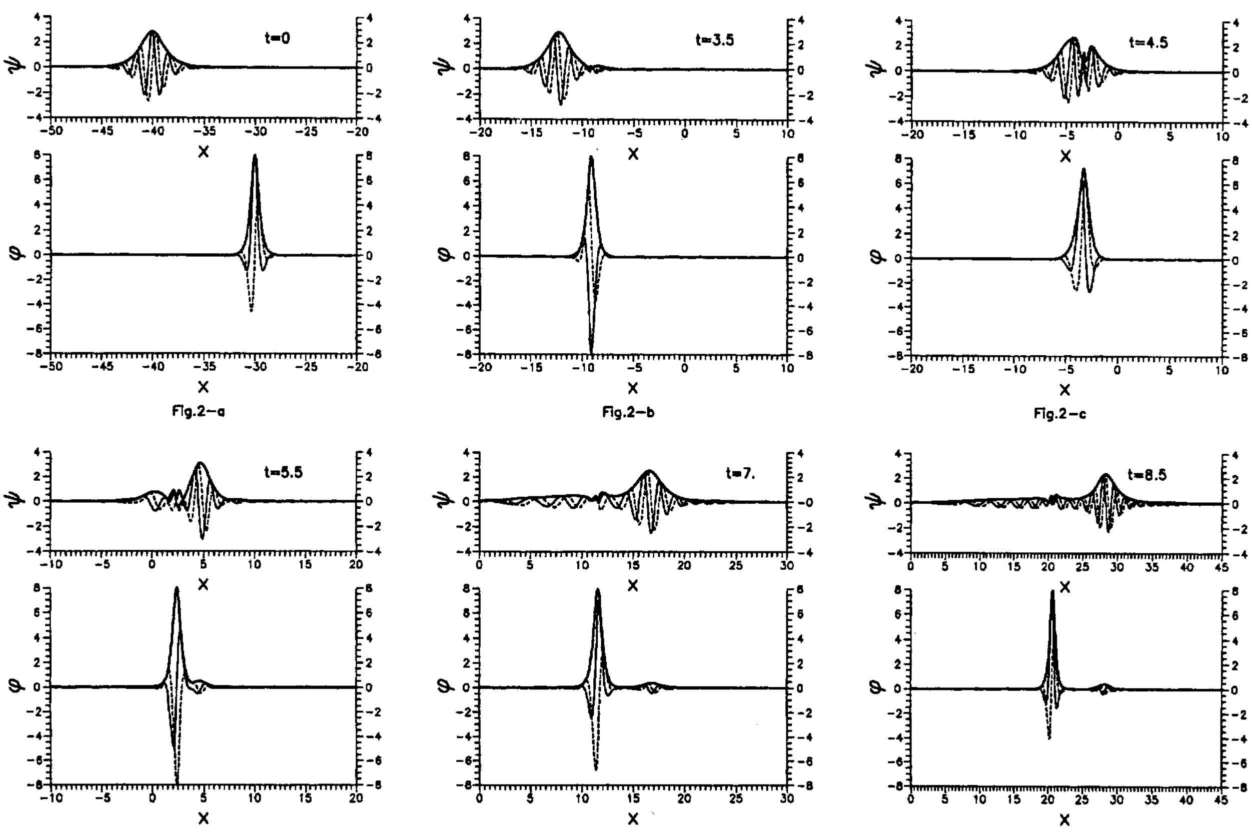

Fig. 2. Overtaking interaction for equal frequencies and different amplitudes: $n_{\text{left}} = n_{\text{right}} = 17$; $c_{\text{left}} = 8$, $a_{\text{left}} = 2$; $c_{\text{right}} = 6$, $a_{\text{right}} = 16$. The thick line represents the modulus; the thin line – the real part; the dashed thin line – the imaginary part. (a) $t = 0$, (b) $t = 3.5$, (c) $t = 4.5$, (d) $t = 5.5$, (e) $t = 7$, (f) $t = 8.5$.

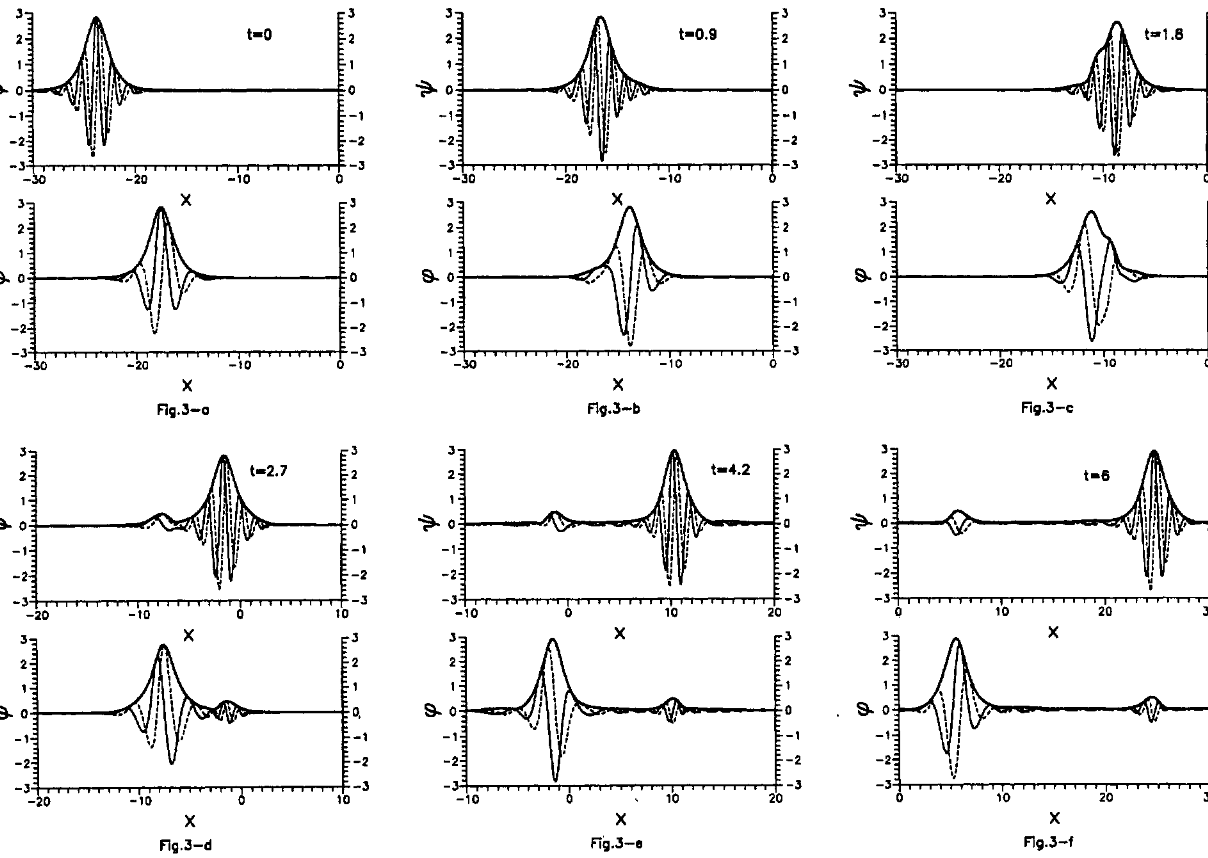

Fig. 3. Overtaking interaction for different frequencies and equal amplitudes: $n_{\mathrm{left}} = 17$, $n_{\mathrm{right}} = 5$; $c_{\mathrm{left}} = 8$, $c_{\mathrm{right}} = 4$; $a_{\mathrm{left}} = a_{\mathrm{right}} = \sqrt{2}$. The thick line represents the modulus; the thin line – the real part; the dashed thin line – the imaginary part. (a) $t = 0$, (b) $t = 0.9$, (c) $t = 1.8$, (d) $t = 2.7$, (e) $t = 4.2$, (f) $t = 6$.

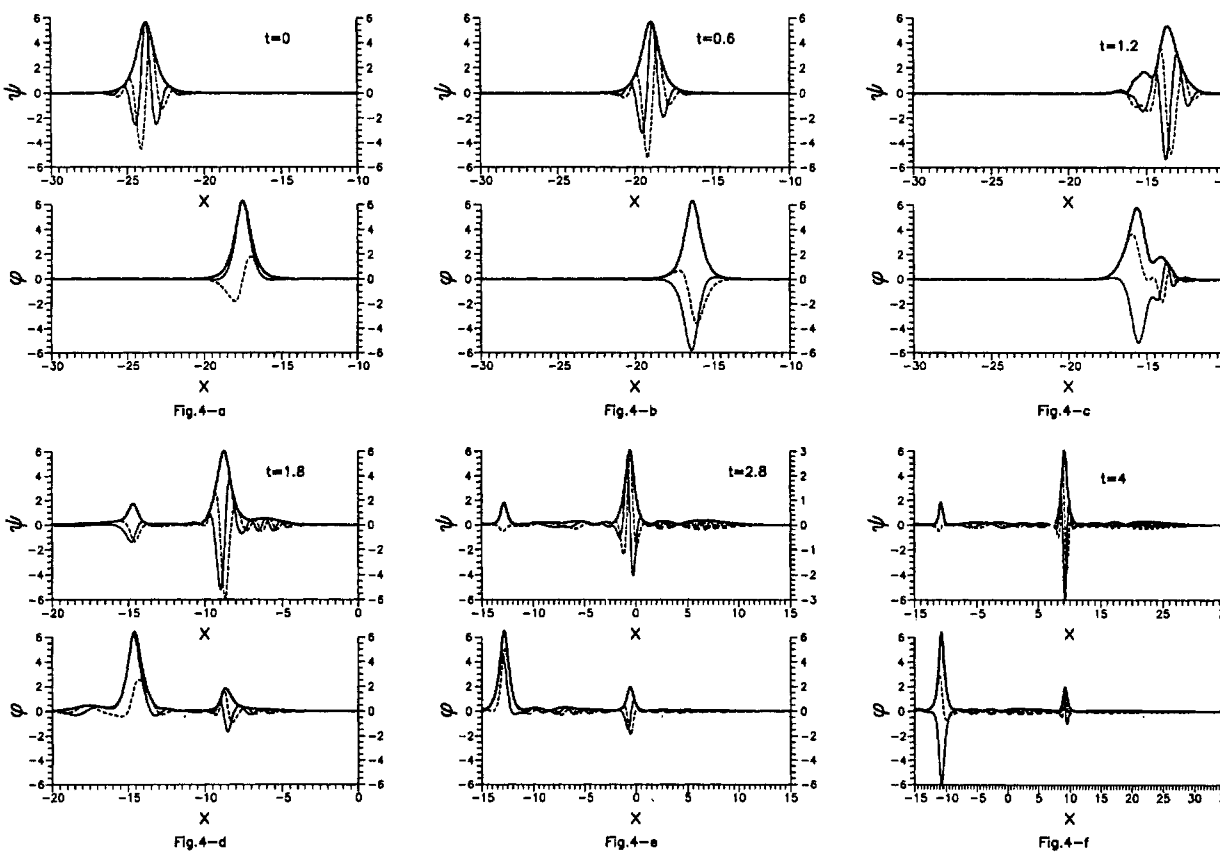

Fig. 4. Overtaking interaction for different frequencies and different amplitudes: $n_{\mathrm{left}} = 20$, $n_{\mathrm{right}} = 6$; $c_{\mathrm{left}} = 8$, $c_{\mathrm{right}} = 2$; $a_{\mathrm{left}} = 8$, $a_{\mathrm{right}} = 10$. The thick line represents the modulus; the thin line – the real part; the dashed thin line – the imaginary part. (a) $t = 0$, (b) $t = 0.6$, (c) $t = 1.2$, (d) $t = 1.8$, (e) $t = 2.8$, (f) $t = 4$.

either, because we have checked the results for different mesh sizes (different resolutions). Fig. 1(f) also shows the interaction of the waves with the boundaries. Note that in the last two figures, 1(e), (f), the horizontal scale has been changed in order to exhibit the whole pattern of the solution in the same box.

The over-taking collisions are more interesting, because they allow sufficient time (cross section) of interaction for the coupled waves to develop before the main waves finally separate from each other. It is clear that the phase velocities must be different in the overtaking collisions.

First we investigate the case when the waves have the same carrier frequency [see Figs 2(a)–(f)] but different amplitudes. Naturally, the faster wave has smaller amplitude. As it should have been expected, the wave of smaller amplitude suffers more from the interaction and larger part of its mass goes into the excited conjugated pattern.

It is interesting to trace in Figs 3(a)–(f) the case of equal amplitudes when the faster wave has larger carrier frequency. It turns out that the frequency does not really matter and the excited waves are of equal amplitudes, i.e. both the original waves are equally affected by the collision.

Figure 4 presents the overtaking collision of two waves of different frequencies and slightly different amplitudes. The solition of smaller amplitude actually becomes larger after the collision and slightly slower. This is the same effect as the one mentioned in [3, 4]. So, here the interaction is even more inelastic not only exhibiting some residual signals but also transforming the original signals. Once again, we remind the reader that the masses of the two waves and the total energy remain strictly constant in our calculations.

So far, we have shown that there is an inelastic behaviour (change of polarization) even for the case of circular polarization. In fact, Parker *et al.* did not actually treat this case numerically. They just mentioned that it is reduced to a single NLS. That is true, but the solution of the coupled system which also happens to be a solution to the single NLS is unstable and could not persist. After the collision it yields a solution with more complicated polarization (as shown in our figures).

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