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Evolution and Interactions of Solitary Waves (Solitons) in Nonlinear Dissipative Systems

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Abstract

A generalized wave equation containing production and dissipation of energy is derived in a heuristic fashion so as to have in the dissipationless limit the (two-way wave) Boussinesq equation, while for the slowly evolving in a moving frame (one-way) wave system, it reduces to the dissipation modified KDV equation (KDV–KSV) with the same energy-balance law. The new equation allows investigating the head-on collision of dissipative localized structures. A special difference scheme is devised which faithfully represents the balance law for energy. The numerical simulations show that if the production-dissipation rate is of order of a small parameter, the coherent structures upon collisions preserve their localized character and within a time interval proportional to the inverse of the small parameter they behave like (imperfect) solitons. The collisions are almost ideal without phase shift. The only difference from the strictly soliton collision is that during the time of interaction the dissipative structures are "aging" and changing their shapes.

1. Introduction

A variety of continuous systems are modelled by nonlinear PDE containing the first or second time derivative of the unknown function and spatial and mixed derivatives of different orders. With respect to the spatial variables they can be one-, two- or three-dimensional, respectively. The strict description of any physical process is three-dimensional but it is not always tangible for investigation and for this reason a plethora of simplified one-dimensional models have been derived and studied. The present work is concerned with the investigation of wave régimes for different nonlinear systems modelling the capillary flows in thin liquid layers. Here belong the inviscid models like Boussinesq and KDV equations on the one hand and the dissipation modified Korteweg-de Vries equation (KDV–KSV, for brevity) – on the other.

Under certain conditions the nonlinear and dispersive effects balance each other and localized, stationary propagating waves (solitary waves) are possible solutions. A dissipationless system – the KDV – was the first to undergo numerical investigation [23] unravelling the particle-like behaviour of the localized solutions upon overtaking interactions and the "soliton" concept was introduced. Since then a variety of conservation properties have been proved and a good deal of analytical techniques for the solitary waves is now available [8].

Completely different is the case with dissipative systems. The purpose of the present work is to show a way of extending the main tenets of soliton paradigm to such case. To this end a two-way-wave propagation model is proposed and investigated numerically hence discussing head-on collisions.

2. Models of thin-layer-flows with free surface

Boussinesq [2–4] derived in the framework of weakly-nonlinear long-wave approximation an equation for the shallow-water flows in which equation both nonlinearity and dispersion were present in appropriate balance. He found a solution of solitary-wave type. It goes beyond the scope of present work to discuss the different versions of Boussinesq equation that are currently in use. See, for instance [7] where a special effort was made and the coinage "Boussinesq Paradigm" was introduced to signify the main idea of Boussinesq-type derivations, rather than the particular equation obtained. As far as we are concerned here with the capillary flows then the Boussinesq equation is regularized by the presence of sufficiently strong surface tension (first considered in [15]), namely

$$u_{tt} = [\gamma^2 u + \alpha u^2 - \beta u_{xx}]_{xx}, \quad \beta = -\frac{1}{3} + \frac{\sigma}{\rho h_0^2} > 0, \tag{1}$$

where σ is the surface tension, ρ – the density, g – the gravity acceleration, h_0 – the thickness of the undisturbed fluid layer.

The Boussinesq equation (1) is formally a two-way wave equation since it contains the second time derivatives. However, the original version of Boussinesq was not applicable to the two-way waves, because of an assumption valid only in the right-going moving frame. Thus eq. (1) should be considered in a paradigmatic sense as a model problem allowing one to investigate head-on collisions of solitary waves.

The investigation of the role of viscosity on the capillary flow in thin layers was begun by Kapitza [14]. In other works of [1, 11, 17, 18, 20, 21] the long-wave weakly-nonlinear approximation was developed whose consistent application allows one after rescaling the variables to derive the following equation ([12]):

$$\phi_t + \phi\phi_x + \phi_{xx} + \phi_{xxxx} = 0. \tag{2}$$

or

$$\psi_t + \tfrac{1}{2}(\psi_x)^2 + \psi_{xx} + \psi_{xxxx} = 0 \quad \text{where} \quad \psi_x \equiv \phi. \tag{3}$$

The last form of the *NEE* was obtained in [16] for the evolution of reaction fronts. The eq. (2), or eq. (3) are usually referred to as Kuramoto-Sivashinsky equation (KS for brevity).

In a series of papers, Velarde and collaborators (see [9, 10, 22] and references therein) showed the consistent way of incorporating the Marangoni effect into the one-way long-wave assumption and in particular obtained the following equation

$$u_t + 2\alpha_1 uu_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0, \quad (4)$$

containing also the dispersion term u_{xxx} and additional nonlinearity $\alpha_5(uu_x)_x$. We call in what follows the eq. (4) dissipation modified Korteweg-de Vries equation (KDV–KSV – for brevity). Equation (4) explains at least to a qualitative level Bénard-Marangoni waves in thin liquid layers heated from the air side [19].

The influence on the dynamical behaviour of the additional nonlinear term (with coefficient α_5) as a destablizing factor (according to its sign in the equation) was elucidated in [13]. The role of the new nonlinearity played in the formation of the shapes of the solitary waves was treated in [6]. In the present paper we focus our attention mostly on the interplay between the dispersion and production-dissipation, and hence we consider the case $\alpha_5 = 0$.

Our recent simulations of the eq. (4) have shown that upon over taking interactions the localized solutions of KDV–KSV behave as quasi-solitons in the sense that they preserved the localization for long enough times. In the present paper we investigate the same property for head-on collisions and derive a new equation to this end.

3. Two-way wave generalization of KDV–KSV

In this Section we concern ourselves with the minimal heuristic generalization of the dissipative model called KDV–KSV in a manner to allow two-way waves. This means that it should contain second time derivative just in the same manner as the Boussinesq equation does. For this reason we start from the Boussinesq equation and incorporate in its the production-dissipation terms that are supposed to engender the respective terms of KDV–KSV equation. Here, however, we do not aim at generalizing the KDV–KSV equation to fully account for Bénard-Marangoni two-way-wave propagation as done in [22].

There are two equations that satisfy the outlined requirements. The first one is

$$u_{tt} = [\gamma^2 u + \alpha_1 \gamma u^2 + \alpha_3 \gamma u_{xx} - \alpha_4 u_{xxt} - (\alpha_2 + \alpha_5 u) u_t]_{xx}, \qquad \alpha_3 \equiv -\beta, \quad (5)$$

while the alternative version is as follows

$$u_{tt} = [\gamma^2 u + \alpha_1 \gamma u^2 + \alpha_3 \gamma u_{xx} + \gamma \alpha_4 u_{xxx} + \gamma(\alpha_2 + \alpha_5 u) u_x]_{xx}, \qquad \alpha_3 \equiv -\beta, \quad (6)$$

Here the characteristic γ speed appears in some places merely for the sake of further convenience.

Let us begin with the "right-going" moving frame. Introduce new independent coordinates and sought function

$$t_1 = \tfrac{1}{2} t, \quad x_1 = x - \gamma t, \quad u(t, x) = u_1(t_1, x_1).$$

The different derivatives are expressed as follows

$$\frac{\partial u}{\partial x} = \frac{\partial u_1}{\partial x_1}, \quad \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial u_1}{\partial t_1} - \gamma \frac{\partial u_1}{\partial x_1},$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4}\frac{\partial^2 u_1}{\partial t_1^2} - \gamma \frac{\partial^2 u_1}{\partial t_1 \partial x_1} + \gamma^2 \frac{\partial^2 u_1}{\partial x_1^2}. \quad (7)$$

If we consider only motions for which the evolution in the moving frame is very slow (and that is precisely the condition under which all kinds of KDV-type equations are being derived), then we can disregard the local-time derivatives with respect to the local spatial derivatives in the sense, that

$$\left|\frac{\partial^m u_1}{\partial t_1 \partial x_1^{m-1}}\right| \ll \gamma \left|\frac{\partial^m u_1}{\partial x_1^m}\right|, \quad \left|\frac{\partial^2 u_1}{\partial t_1^2}\right| \ll \gamma \left|\frac{\partial^2 u_1}{\partial t_1 \partial x_1}\right|.$$

Introducing (7) into eq. (5) or eq. (6) and neglecting the terms according to the above scheme we arrive at the following approximate equation

$$-\gamma \frac{\partial^2 u_1}{\partial x_1 \partial t_1} = \frac{\partial^2}{\partial x_1^2}\left[\alpha_1 \gamma u_1^2 + (\alpha_2 + \alpha_5 u)\gamma \frac{\partial u_1}{\partial x_1} \times \delta + \alpha_3 \gamma \frac{\partial^2 u_1}{\partial x_1^2} + \alpha_4 \gamma \frac{\partial^3 u_1}{\partial x_1^3}\right], \quad (8)$$

After dividing both sides of the equation by γ (here becomes apparent the reason of incorporating γ in the definition of the equation), and disregarding the index "1" without fear of confusion we arrive at

$$u_t + 2\alpha_1 uu_x + \alpha_2 u_{xx} + \alpha_3 u_{xxx} + \alpha_4 u_{xxxx} + \alpha_5 (uu_x)_x = 0, \quad (9)$$

which is nothing else but the KDV–KSV eq. (4).

Let us examine now the case of left going moving frame. Then it is convenient to introduce the new variables in the form

$$t_1 = \tfrac{1}{2} t, \quad x_1 = -x - \gamma t, \quad u \equiv -u_1(t_1, x_1).$$

and the generalized Boussinesq equations (5), (6) reduce once again to (9).

4. Energy balance law

Henceforth, for the sake of simplifying the notations we drop out the multiplier γ of some of the coefficients of eq. (5). For the same reasons we return in what follows to the coefficient $\beta = -\alpha_3 > 0$.

Upon introducing an auxiliary function q into eq. (5) we render it into the following system

$$u_t = q_{xx},$$
$$q_t = \gamma^2 u + \alpha_1 u^2 - \beta u_{xx} - \alpha_4 u_{xxt} - (\alpha_2 + \alpha_5 u) u_t. \quad (10)$$

The conservative boundary conditions are as follows:

$$u = 0, \quad q_x = 0 \quad \text{for} \quad x = -L_1, L_2, \quad (11)$$

where $-L_1, L_2$ are the values of spatial coordinate at which we truncate the infinite interval (so-called "actual infinities"). The shape of a localized solution approaches at each infinity a constant and hence all its derivatives automatically decay to zero. Then, when treating the problem analytically one may also impose b.c. on the second, third, etc. derivatives that follows from the condition for decay (called asymptotic b.c.). However, in numerics and in physics as well, one never has a true infinity and the exact form of the b.c. in the finite interval is of crucial importance. In fact, any conservation or balance law is a property of the boundary value problem (b.v.p.) as a whole and not only of the equation itself.

We show now that (11) pose a proper set of b.c. Consider the quantities

$$M \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} u \, dx,$$

$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \tfrac{1}{2}[\gamma^2 u^2 + q_x^2 + \tfrac{1}{3}\alpha_1 u^3 + \beta u_x^2] \, dx. \quad (12)$$

Integrating the first of eqs (10) with respect to x and acknowledging the appropriate b.c. from (11) we get that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \tag{13}$$

which can be called "conservation law for the mass of wave".

Upon multiplying l.h.s. of the first of eqs (10) by the r.h.s. of the second one and r.h.s. of the first one by the l.h.s. of the second, adding them to each other and integrating with respect to x within the limits $[-L_1, L_2]$ is obtained

$$q_x q_t |_{-L_1}^{L_2} - \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L_1}^{L_2} \tfrac{1}{2}(q_x)^2\,\mathrm{d}x = \gamma^2 \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L_1}^{L_2} \tfrac{1}{2}u^2\,\mathrm{d}x + \alpha\frac{\mathrm{d}}{\mathrm{d}t}\int_{-L_1}^{L_2}\tfrac{1}{3}u^3\,\mathrm{d}x + \alpha_3 \frac{\mathrm{d}}{\mathrm{d}t}\int_{-L_1}^{L_2}\tfrac{1}{2}u_x^2\,\mathrm{d}x - \alpha_3 u_x u_t|_{-L_1}^{L_2} - \alpha_4 u_{xt}u_t|_{-L_1}^{L_2} - \int_{-L_1}^{L_2}(\alpha_2+\alpha_5 u)u_t^2\,\mathrm{d}x + \alpha_4\int_{-L_2}^{L_2} u_{xt}^2\,\mathrm{d}x.$$

Since taking $u = 0$ at the borders of the interval implies also $u_t = 0$, then the terms in the above formula that are evaluated at the borders of the interval vanish identically. Here is to be mentioned that these terms vanish as well if we impose the condition $q = 0$ at $x = -L_1, L_2$, instead of $q_x = 0$, but then the mass M is not conserved. This is the justification for the particular set of boundary conditions taken in the present work.

Rewriting the above equality and discarding the terms that are trivially equal to zero, one obtains

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-L_1}^{L_2}(\alpha_2+\alpha_5 u)u_t^2\,\mathrm{d}x - \alpha_4\int_{-L_1}^{L_2} u_{xt}^2\,\mathrm{d}x. \tag{14}$$

which is the balance law for the quantity E, which we call pseudoenergy. Note the difference between a balance law and a conservation law. The latter is a limiting case of the former when the terms responsible for production and dissipation of energy are trivially equal to zero. The role of the terms we added to the Boussinesq equation can now be identified from the balance law. The term $[(\alpha_2 + \alpha_5 u)u_t]_{xx}$ represents the production of energy, while u_{xxxxt} is the dissipation. In those parts of the region where $(\alpha_2 + \alpha_5 u) < 0$ the production is negative, i.e. the Marangoni term damps the motion. In the regions where $u > 0$ it boosts the production of energy.

At this point we can justify our choice of eq. (5) over (6). The latter could have yielded to a balance law of the following type

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_{-L_1}^{L_2}(\alpha_2+\alpha_5 u)u_t u_x\,\mathrm{d}x - \alpha_4\int_{-L_1}^{L_2} u_{xt}u_{xx}\,\mathrm{d}x,$$

whose r.h.s. could not be interpreted in physically satisfactory way.

5. Difference scheme

Let us introduce a regular mesh in the interval $[-L_1, L_2]$ with spacing h, i.e.,

$$x_i = -L_1 + (i-1)h, \quad h = \frac{L_1+L_2}{N-1}, \tag{15}$$

where N is the total number of grid points in the said interval. Respectively, we define the set function $u_i \stackrel{\text{def}}{=} u(x_i)$.

It is clear that when the system under consideration satisfies balance laws, one's desire is to have them reproduced by the numerical scheme. Bearing upon our earlier work [5, 7] we construct here an implicit difference scheme which complies with the balance laws for the mass and pseudoenergy.

$$\frac{q_i^{n,k} - q_i^n}{\tau} = \gamma^2\frac{u_i^{n,k}+u_i^n}{2} - \alpha\frac{u_i^{n,k}u_i^{n,k-1} + u_i^{n,k}u_i^n + (u_i^n)^2}{3} - \frac{\alpha_3}{2}\times\left[\frac{u_{i+1}^{n,k} - 2u_i^{n,k} + u_{i-1}^{n,k}}{2h^2} + \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{2h^2}\right] - \alpha_4 \times\left[\frac{u_{i+1}^{n,k} - 2u_i^{n,k} + u_{i-1}^{n,k}}{\tau h^2} - \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\tau h^2}\right] + \left(\alpha_2 + \alpha_5\frac{u_i^{n,k-1}+u_i^n}{2}\right)\frac{u_i^{n,k}-u_i^n}{\tau}, \tag{16}$$

$$\frac{u_i^{n,k}-u_i^n}{\tau} = \frac{1}{2}\left[\frac{q_{i+1}^{n,k} - 2q_i^{n,k} + q_{i-1}^{n,k}}{h^2} + \frac{q_{i+1}^n - 2q_i^n + q_{i-1}^n}{h^2}\right]. \tag{17}$$

with b.c.

$$u_N^{n,k} = u_{N-1}^{n,k} = 0, \quad q_N^{n,k} - q_{N-1}^{n,k} = q_2^{n,k} - q_1^{n,k} = 0.$$

The "inner" iterations are conducted starting from the initial conditions $u_i^{n,0} = u_i^n$ and $q_i^{n,0} = q_i^n$, and are terminated at certain $k = K$ after the following criterion is satisfied

$$\frac{\max|u_i^{n,K} - u_i^{n,K-1}|}{\max|u_i^{n,K}|} \leqslant 10^{-11}$$

After the inner iterations are converged one obtains, in fact, the solution for the "new" time stage $n+1$ of the nonlinear difference scheme, namely

$$u_i^{n+1} \stackrel{\text{def}}{=} u_i^{n,K}.$$

Following [5], we prove here that the difference approximations of mass M and pseudoenergy E

$$\mathcal{M} = \sum_{i=2}^{N-1} u_i h$$

and

$$\mathcal{E} = \sum_{i=2}^{N-1}\left(\frac{\gamma^2 u_i^2}{2} - \frac{\alpha u_i^3}{3}\right)h + \sum_{i=1}^{N-1}\left[\alpha_3\left(\frac{u_{i+1}-u_i}{h}\right)^2 + \left(\frac{q_{i+1}-q_i}{h}\right)^2\right]h,$$

satisfy the difference approximations of the balance laws (13)–(14), namely

$$\frac{\mathcal{M}^{n+1} - \mathcal{M}^n}{\tau} = 0$$

and

$$\frac{\mathscr{E}^{n+1}-\mathscr{E}^{n}}{\tau}=\sum_{i=1}^{N-1}\left(\alpha_2+\alpha_5\frac{u_i^{n+1}+u_i^n}{2}\right)\left(\frac{u_i^{n+1}-u_i^n}{\tau}\right)^2 - \alpha_4\sum_{i=1}^{N-1}\left(\frac{u_{i+1}^{n+1}-u_i^{n+1}-u_{i+1}^n+u_i^n}{\tau h}\right)^2. \quad (18)$$

In this instance our scheme represents exactly on difference level (within the adopted second order of approximation) the properties of the original differential system. The practical organization of calculations is the same as described in [5].

6. Results and discussion

First we begin with the case of small production-dissipation, i.e. for small α_2 and α_4. For definiteness we take $\alpha_1 = 3$, $\beta = 1$ and $\gamma = 10$, the latter being large enough in order to have the right to compare with the results for KDV–KSV (the assumption of slow evolution in the moving frame is justified when the frame is moving fast enough). We reduce the degree of freedom taking $\varepsilon = \alpha_2 = \alpha_4$. Our calculations show that for the selected parameters, an appreciable deviation from the classical dissipationless Boussinesq solitons is observed for $\varepsilon > 0.1$. In terms of KDV–KSV equation this corresponds to $\alpha_2 = \alpha_4 > 0.01$ for $\alpha_1 = 3$ and $\beta = 1$. The upper part of Fig. 1 presents the head-on collision of the dissipative solitons of *sech* initial shape and celerities $c_l = 9.5$ and $c_r = -9.5$. The lower part gives an impression about the phase shift. The interesting thing here is that the phase shift is virtually absent, i.e. much smaller than the corresponding dissipationless case (see, [7]). In a sense, the incorporation of dissipation "tamed" the adverse effects of excessive elasticity of the system. One should not be deceived, however, that the system becomes more elastic if a phase shift is absent. On the other hand we should keep clear that eq. (5) or eq. (6) are the minimal heuristic generalizations of eq. (4). A more general equation could indeed account for the phase shift as noted in Ref. [22]. Figure 2 shows the initial and final shapes of the wave system whose evolution is shown in Fig. 1. The bumps have experienced considerable "aging" in the course of interaction diminishing in amplitude and increasing their support. Obviously this is a dissipation dominated case. The localization, however, is preserved, so one can consider the bumps after the collision as the same original entities ("particles"), but somewhat deformed by the interaction and the aging. Then it makes sense to extend the notion of "soliton" to this non-conservative case. The most appropriate description is "aging solitons", since during the collision they behave in a very similar fashion as the solitons of conservative systems, but because their shapes do not provide for conservation of energy, the total wave profile is evolving (aging). Yet the balance of the energy is strictly observed.

The same observation is valid also for larger γ. The results are not presented here, because they have the same qualitative bearing. Rather we give an example for the evolution of the wave system in the case when our generalized equation is not supposed to be very close to KDV–KSV equation in the moving frame. This is the case of moderate and small γ, when the convective speed of the moving frame is not dominating the local evolution in the moving frame.

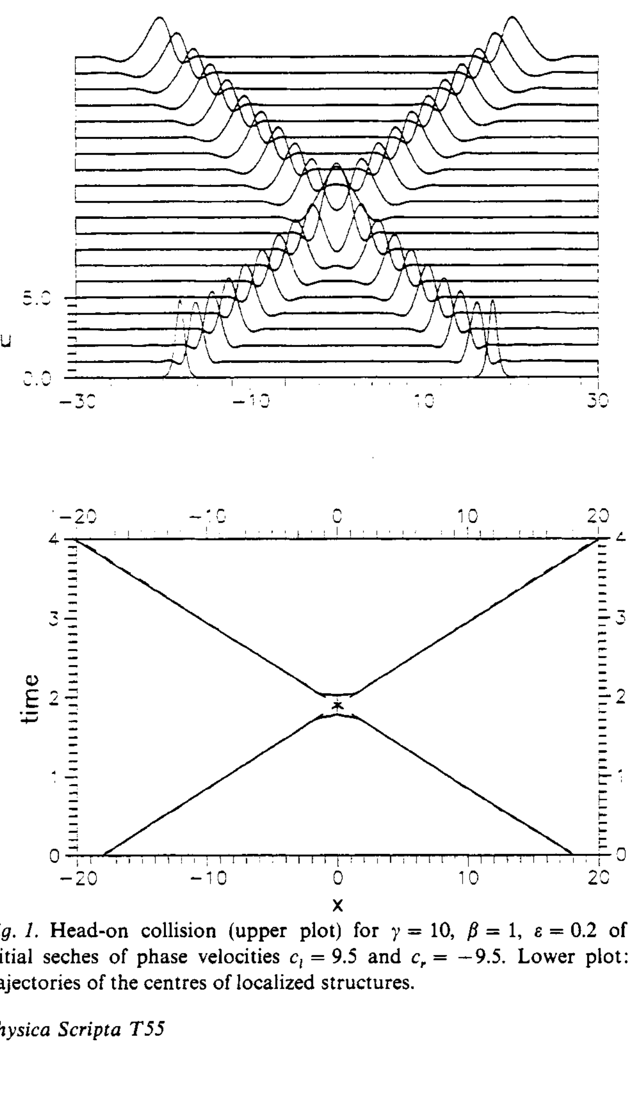

Fig. 1. Head-on collision (upper plot) for $\gamma = 10$, $\beta = 1$, $\varepsilon = 0.2$ of the initial seches of phase velocities $c_l = 9.5$ and $c_r = -9.5$. Lower plot: the trajectories of the centres of localized structures.

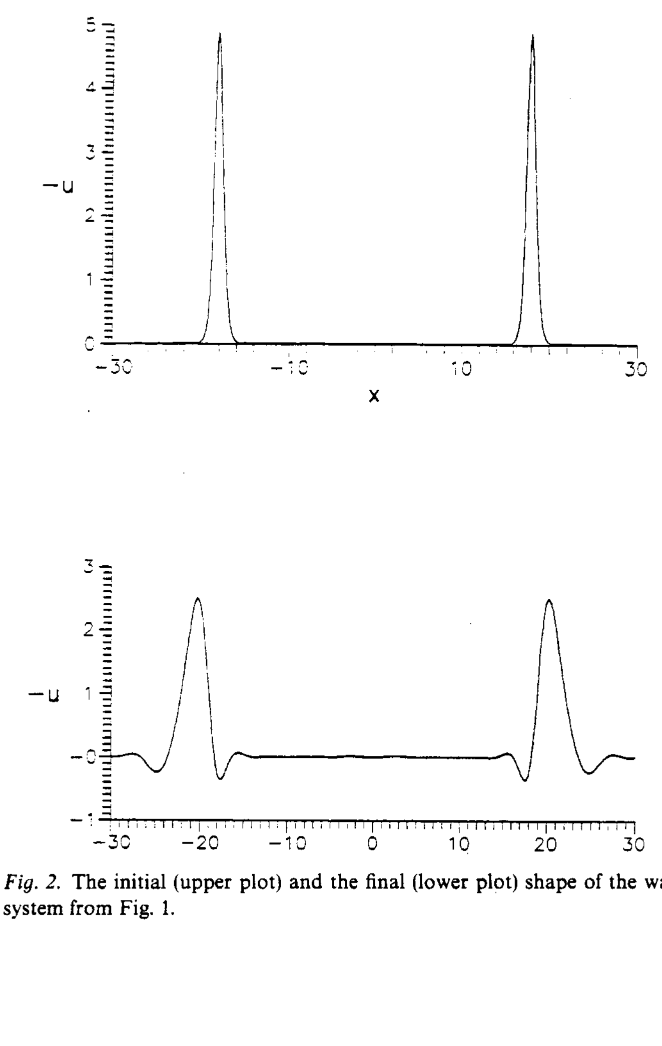

Fig. 2. The initial (upper plot) and the final (lower plot) shape of the wave system from Fig. 1.

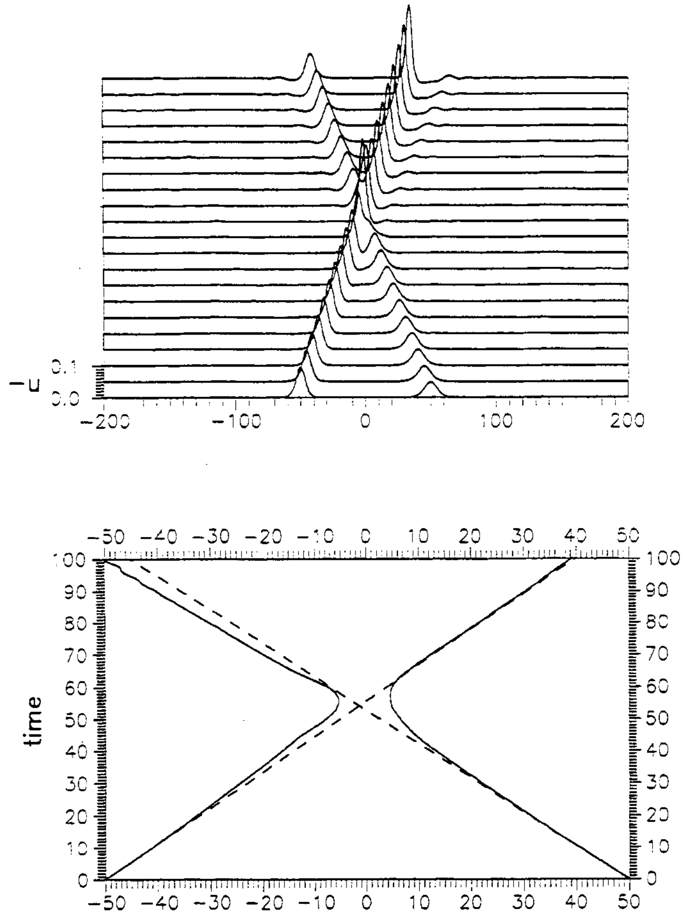

Fig. 3. Head-on collision (upper plot) for $\gamma = 1$, $\beta = 1$, $\varepsilon = 0.2$ of the initial seches of phase velocities $c_l = 0.9$ and $c_r = -0.95$. Lower plot: the trajectories of the centres of localized structures.

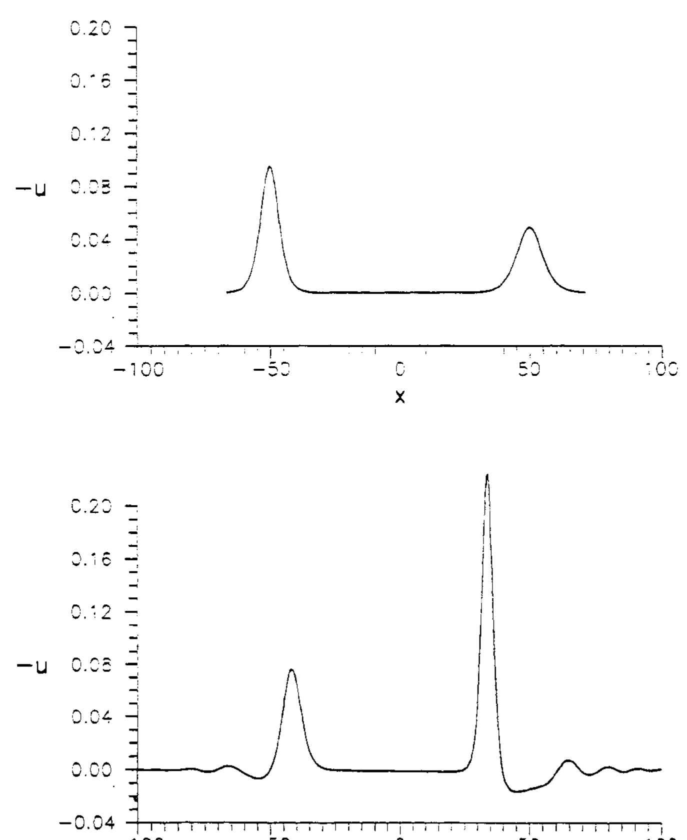

Fig. 4. The inital (upper plot) and the final (lower plot) shape of the wave system from Fig. 3.

Figure 3 presents the same kind of head-on collision, but for $\gamma = 1$. One sees, that the production-dissipation part is now much more influential for the same $\varepsilon = 0.2$. In fact the case depicted in Fig. 3 happens to be production dominated one. In Fig. 4 are shown the initial and final solution in the same time-interval as in Fig. 3. It is clearly seen that the bumps (initially being seches) have increased during the interaction and changed their shape (shortened their support). So the solitons are now "growing", but the main idea is still intact, namely that they preserve the localization and in the cross-section time of the collision behave as genuine localized structures, i.e. as particles.

7. Concluding remarks

The dissipation-modified KDV equation is generalized in a minimal, yet nontrivial heuristic manner to a wave equation of Boussinesq type with terms accounting for the energy production-dissipation mechanism [eqs (5)]. It is shown that if one is concerned with motions that evolve slowly in a moving frame with the characteristic velocity, then equation (5) reduces to the KDV–KSV equation (4). The generalized two-way wave equation derived here [eq. (5)], possesses the same balance law for energy as the KDV–KSV equation. The advantage of eq. (5) is that it allows consideration of two-way waves. This way we were able to track the head-on collisions of dissipative structures ("dissipative solitons").

An implicit scheme is developed representing "exactly" (within *a-priori* selected error bar, but regardless of the truncation error) the conservation law for the mass of localized structure, as well as the balance law for the pseudoenergy. The practical stability of the scheme allowed us to go after the very long time evolution of the solution.

Results are obtained for values of the production-dissipation ranging from very small to order of hundreds. For the small values, when the production-dissipation part is a small perturbation to the Boussinesq part, the present calculation confirm that for long enough times (of order of the inverse of the small parameter, and even beyond) the behaviour of the solitary waves is solitonic. The head-on collisions of the dissipative solitons in this case are qualitatively the same as for the pure Boussinesq seches, save some apparent "aging" of shapes of the structures.

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