FOURIER-GALERKIN ALGORITHM FOR 2D LOCALIZED SOLUTIONS

(Dedicated to the memory of my young colleague K. L. Bekyarov)

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The paper presents the numerical implementation in 2D of a Fourier-Galerkin expansion with complete orthonormal basis system of localized functions. The bilinear Laplace equation is considered as a featuring example. Coordinate splitting is used to reduce the cost of inversion of the linear matrices for the coefficients. The axisymmetric soliton is calculated as a 2D problem and compared to a numerical solution, found by means of a difference scheme.

Keywords: Fourier-Galerkin method, localized solutions, bilinear Laplace equation.
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1. INTRODUCTION

Calculating the shapes of localized waves, e.g. solitons, is of importance for the modern theory of non-linear waves. The difficulties are connected with the unboundness of the integration domain. For example, in numerical treatment, when using finite-difference or finite-element schemes, one has to consider large enough domains in order to reduce the influence of the truncation (the so-called "actual infinity"). In 1D the problems of domain size and mesh resolution can still be tackled, although sometimes up to 20000 grid points (see, e.g. [12]) have to be used. Clearly, in 2D, when the mesh size is at least the square of the 1D mesh-size, it is a very hard problem.

One of the ways to circumvent the said difficulty is to employ a complete orthonormal (CON) system of functions on the infinite interval and to devise an

algorithm for implementation of one of the spectral techniques: Galerkin's, pseudospectral, *tau*-method (see [5, 3]). The successful application of the Galerkin method requires, however, that the product of two members of the system can be conveniently represented by means of the functions of the system. CON system with the required properties was introduced first in [6] and applied for finding a localized solution to the Burgers equation. Later on, the numerical Fourier-Galerkin technique was extended to Korteweg-de Vries (KdV) and Kuramoto-Sivashinsky (KS) equations [11] and the fifth order KdV [1]. Boyd [2, 4] showed that the new CON system can be obtained by an algebraic mapping of the Tchebishev polynomials on an infinite interval, see also [3]. In this way he derived a variety of properties of the expansion.

Employing a spectral expansion with a specialized CON basis system drastically reduces the required computational resources. They can be even further reduced if the resulting algebraic system is treated in the appropriate manner by means of a splitting method. The aim of the present paper is the creation of an algorithm for implementing the Fourier-Galerkin technique in 2D.

2. POSING THE PROBLEM

Consider the following generic equation (the non-linear Klein-Gordon equation)

$$\frac{\partial^2 u}{\partial t^2} = -u + 3u^2 + \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{2.1}$$

which, as is well-known, possesses localized solutions that propagate stationary. In the co-ordinate system connected with the center of the localized structure (the so-called "moving frame") one can introduce new independent variables $\xi x - c_1 t$, $\eta = y - c_2 t$, where c_1, c_2 are the components of the phase speed of the center of the localized structure. Then for the stationary localized solution one arrives at the equation

$$-u + 3u^2 + \left(\beta_1 \frac{\partial^2 u}{\partial \xi^2} + \beta_2 \frac{\partial^2 u}{\partial \eta^2}\right) = 0, \tag{2.2}$$

where $\beta_i = 1 - c_i^2$. Here we consider only "subsonic" solitons for which $\beta_i > 0$. The boundary conditions stem from the vanishing of the solution at infinity:

$$u \to 0 \quad \text{for} \quad \xi, \eta \to \pm\infty. \tag{2.3}$$

Clearly, the problem (2.2), (2.3) is a bifurcation one, since the trivial solution $u \equiv 0$ always persists. A similar problem was treated in [14] for the classical spectral method with harmonic functions in application to the sixth order Boussinesq equation. To avoid the trivial solution, one can impose a condition at the origin of the co-ordinate system, say,

$$u(0, 0) = \text{const.} \tag{2.4}$$

Strictly speaking, (2.4) will overpose the problem unless some additional degree of freedom is introduced, say, through an additional coefficient of the non-linear term

$$-u + 3\alpha u^2 + \left(\beta_1 \frac{\partial^2 u}{\partial \xi^2} + \beta_2 \frac{\partial^2 u}{\partial \eta^2} \right) = 0, \tag{2.5}$$

which is to be calculated so as to fit the imposed boundary conditions at the origin of the co-ordinate system. The definitive relation for the new unknown is the equation taken in the origin:

$$\alpha = \frac{1}{3u^2(0,0)} \left[u(0,0) - \left(\beta_1 \frac{\partial^2 u}{\partial \xi^2} + \beta_2 \frac{\partial^2 u}{\partial \eta^2} \right) \Big|_{x=0,\, y=0} \right]. \tag{2.6}$$

The last relation does not overpose the problem, since the equation in the origin is not used in the scheme for u, but rather it is replaced by the prescribed boundary condition (2.4). Thus we arrive to a boundary value problem (b.v.p.) which does not possess a trivial solution. In addition, for the unknowns (u, α) explicit relations are available. Then the construction of an iterative procedure is straightforward. In some cases, however, the convergence is achieved only when a relaxation for α is performed.

Note that the above procedure is valid only when the expected solution has non-zero amplitude in the origin of the co-ordinate system. When this is not the case (say, for solutions that are odd functions), one can impose a similar condition on one of the partial derivatives of u in the origin. In order not to overload the presentation, we skip the details of such a case and consider here only the case of even functions.

3. FOURIER-GALERKIN EXPANSION

3.1. THE BASIS SYSTEM OF FUNCTION IN $L^2[-\infty, \infty]$

The first CON system in $L^2(-\infty, \infty)$ suited for non-linear problems was proposed in [6]. The different formulas were compiled and verified in [7]. Here we cite the necessary formulas in order to make the paper self-content.

The products of members of series are expanded in series of the system

$$C_n C_k = \frac{1}{2\sqrt{2\pi}} [\ C_{n+k+1} - C_{n+k} - C_{n-k} + C_{n-k-1}] = \sum_{n=0}^{\infty} \beta_{nk,l} C_l, \tag{3.1}$$

$$S_n S_k = \frac{1}{2\sqrt{2\pi}} [\ C_{n+k+1} - C_{n+k} + C_{n-k} - C_{n-k-1}] = \sum_{n=0}^{\infty} \alpha_{nk,l} C_l, \tag{3.2}$$

$$S_n C_k = \frac{1}{2\sqrt{2\pi}} [-S_{n+k+1} + S_{n+k} + S_{n-k} - S_{n-k-1}] = \sum_{n=0}^{\infty} \gamma_{nk,l} S_l. \tag{3.3}$$

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The first derivatives of the functions of the system are expressed as

$$\frac{dS_n}{dx} = \frac{1}{2}\left[nC_{n-1} + (2n+1)C_n + (n+1)C_{n+1}\right],$$

$$\frac{dC_n}{dx} = -\frac{1}{2}\left[nS_{n-1} + (2n+1)S_n + (n+1)S_{n+1}\right].$$

Respectively, for the second derivatives one has

$$\frac{d^2C_n}{dx^2} = -\frac{1}{4}\left\{n(n-1)C_{n-2} - 4n^2C_{n-1} + \left[n^2 + (n+1)^2 + (2n+1)^2\right]C_n\right.$$

$$\left. -4(n+1)^2C_{n+1} + (n+1)(n+2)C_{n+2}\right\}, \tag{3.4}$$

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$$\left. -4(n+1)^2S_{n+1} + (n+1)(n+2)S_{n+2}\right\}. \tag{3.5}$$

3.2. THE GALERKIN EXPANSION

The simplest and oldest spectral technique is the Galerkin one in which the sets of test and trial functions coincide. The main purpose of the present work is to provide an efficient iterative algorithm for treating the linear part of the system. For this reason we select a system with a quadratic non-linearity, for which the Galerkin method is the most efficient. When a more complicated non-linearity is present, then one of the pseudo-spectral techniques should be used. In addition, our equation admits even solution. That is why, for the sake of simplicity, we consider the following series:

$$u = \sum_{n=0}^{n=N} a_{mn}C_m(x)C_n(y). \tag{3.6}$$

3.3. THE CONDITIONS FOR COUPLING THE SYSTEM

Introducing the expressions for the derivatives in the differential equation, one gets a five-diagonal system for each subsystem of coefficients C_n, S_n. The system has to be truncated at $n = 0$ (no terms of negative order show up, since they are expressed by the functions of positive order) and for certain sufficiently large $n = N$. Then the problem of coupling conditions arises. Here we resort to even functions only and the formulas are similar for the odd functions. The condition for coupling the system for $n = 0$ and $n = 1$ comes from the very formulae of the second derivatives (3.4)

$$\frac{d^2C_0}{dx^2} = -\frac{1}{2}C_0 + C_1 - \frac{1}{2}C_2, \tag{3.7}$$

$$\frac{d^2 C_1}{dx^2} = C_0 - \frac{7}{2}C_1 + 4C_2 - \frac{3}{2}C_1. \tag{3.8}$$

In the framework of the Galerkin method, the truncation of the system at $n = N$ requires to assume that $C_n \equiv 0$ and $C_{n+1} \equiv 0$. Then, for the last two members of the series one gets the following expressions for their second derivatives:

$$\frac{d^2 C_{n-1}}{dx^2} = -\frac{(n-2)(n-1)}{4}C_{n-3} + (n-1)^2 C_{n-2}$$

$$-\frac{3n^2 - 3n + 1}{2}C_{n-1} + n^2 C_n, \tag{3.9}$$

$$\frac{d^2 C_n}{dx^2} = -\frac{n(n-1)}{4}C_{n-2} + n^2 C_{n-1} - \frac{3n^2 + 3n + 1}{2}C_n. \tag{3.10}$$

Thus the second x-derivative in the governing equation (2.2) is approximated by 5-point finite difference in the system. Denote by Λ_{xx} and Λ_{yy} the respective five-diagonal matrices which are obtained after half of the identity operator, $\frac{1}{2}$, is subtracted from each of the second-derivative operators. Then the original equation is approximated by the algebraic system

$$3\alpha \sum_{n_1=0}^{n_1=N} \sum_{m_1=0}^{m_2=M} \sum_{n_2=0}^{n_2=N} \sum_{m_2=0}^{m_2=M} \beta^x_{n_1 m_1, n} \beta^y_{n_2 m_2, m} a_{m_1 n_1} a_{m_2 n_2}$$

$$+ (\Lambda_{xx} + \Lambda_{yy})a_{mn} = 0, \tag{3.11}$$

where β^x and β^y are the matrices of coefficients from (3.1) for x and y, respectively. The system (3.11) is taken for all $n \neq 0$ and $m \neq 0$. In the origin the boundary condition

$$a_{00} = 1$$

is imposed. Respectively, the system (3.11), taken at $n = 0, m = 0$, gives the definitive relation for α, namely

$$\hat{\alpha} = \frac{-\beta_1 \left(-\frac{1}{2}a_{00} + a_{10} - \frac{1}{2}a_{20}\right) - \beta_2 \left(-\frac{1}{2}a_{00} + a_{01} - \frac{1}{2}a_{02}\right)}{3 \sum_{n_1=0}^{n_1=N} \sum_{m_1=0}^{m_1=M} \sum_{n_2=0}^{n_2=N} \sum_{m_2=0}^{m_2=M} \beta^x_{n_1 m_1, 0} \beta^y_{n_2 m_2, 0} a_{m_1 n_1} a_{m_2 n_2}}, \tag{3.12}$$

where the unknowns a_{mn} are from the new iteration (fictitious-time stage) $k + 1$. The relaxation for α is performed as follows:

$$\alpha^{k+1} \stackrel{\text{def}}{=} \alpha^k (1 - \omega) + \hat{\alpha}\omega.$$

4. THE SPLITTING SCHEME

In previous works on 1D problems ([11, 1]) we used the Brent's routine for solving the non-linear system for the coefficients. Despite of the rather simple expressions for the products of members of system into series in the system (see

(3.1) — (3.3)), using a pseudo-Newton algorithm like the Brent's one becomes too expensive in 2D because of the large size of the Jacobian. This justifies the search for alternative algorithms. Here we use a simple iteration for the non-linear term. The appropriate series representation of the products of the terms in the system is rather "sparse," so a lot of iterations can be easily performed. It is desirable, however, to have the linear part approximated implicitly. We split it to reduce the calculations. Thus we use the following scheme corresponding to the so-called (see [15]) scheme of stabilizing correction:

$$\frac{\tilde{a}_{ij} - a_{ij}^n}{\tau} = \Lambda_{xx}\tilde{a}_{ij} + \Lambda_{yy}a_{ij}^k + F[a_{ij}^k] \tag{4.1}$$

$$\frac{a_{ij}^{k+1} - \tilde{a}_{ij}}{\tau} = \Lambda_{yy}[a_{ij}^{k+1} - a_{ij}^k]. \tag{4.2}$$

Here τ is the time increment with respect to the fictitious time and it plays the role of an iteration parameter. Respectively, $F[a^n]$ is the expression for the non-linear term when evaluated with the values for a_{ij} from the "old" iteration

$$F[a^k] = \sum_{n=0}^{n=N} \sum_{m=0}^{m=M} \beta_{n_1 m_1, n}^x \beta_{n_2 m_2, m}^y a_{mn}^k \, .$$

After excluding the half-time-step variable $\tilde{a}$, one gets

$$\left(E + \tau^2 \Lambda_{xx}\lambda_{yy} \right) \frac{a^{k+1} - a^k}{\tau} = (\lambda_{xx} + \Lambda_{yy})a^{k+1} + F[a^k] \tag{4.3}$$

which converges to (3.11) in the limit $k \to \infty$, when $a^{k+1} \to a^k$. The important feature of the system (4.1), (4.2) is that it requires inversion of five-diagonal matrices for which special very fast elimination algorithms are available. We make use here of the algorithm from [9].

The iterations are terminated when the following criterion is satisfied

$$\left| a^{k+1} - a^k \right| < 10^{-10}.$$

5. RESULTS AND DISCUSSION

5.1. THE AXISYMMETRIC LOCALIZED SOLUTION

2D calculations of solitons are rarely found. That is why there are no available cases for comparison. However, for $\beta_1 = \beta_2$ one can compare a cross-section of the solution obtained by the 2D algorithm of the present work to 1D solution of the equation when the axial symmetry is acknowledged. Hence we consider the equation

$$u - 3u^2 - \frac{\beta_1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} \right) = 0, \tag{5.1}$$
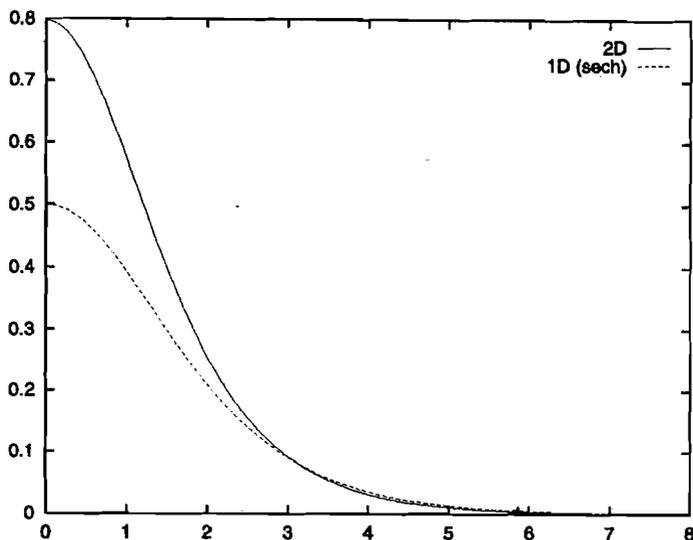
Fig. 1. The axisymmetric soliton for $\beta_1 = \beta_2 = 1$ as obtained with $N + 1 = 20$ functions in the spectral expansion

for which a localized solution is sought in $-\infty, \infty$. To this end we employ the so-called Method of Variational Imbedding (MVI), proposed in [8] for the homoclinic solution of the Lorenz system. To an equation of the type of (5.1), but with a cubic non-linearity, the MVI was applied in [10]. The algorithmic problems of application of MVI are elucidated in detail in [13] in application to the solitary-wave solution of the Kuramoto-Sivashinsky equation. For this reason we present here only the result for the axisymmetric soliton. Fig. 1 shows the shape of this solution alongside with the well-known *sech*-solution of the 1D case. It is seen that the axisymmetric soliton is taller (maximum height equal to 0.79735, while in 1D the maximum equals exactly 0.5) and of slightly smaller support. The solution presented in the figure is taken as a reference when assessing the approximation of the spectral scheme in the next subsection.

5.2. VERIFICATION OF ALGORITHM

The practical convergence of the method can be assessed if a cross section of the 2D solution is taken as function of the radial co-ordinate. Fig. 2 shows the result for different number of terms in the spectral series. Being reminded that the maximum of solution is approximately 0.8, one sees that even 8 functions are able to provide approximation closer to the solution than 0.3%, and 20 terms in the series give approximation better than 0.006%. It is to be mentioned here that no special care for optimization of the method has been taken in the present work. As shown in [11, 1], one can further improve the approximation with fewer number of terms by means of scaling the independent variable(s) in order to bring it closer to the characteristic measures (length of support) of the basis functions C_n, S_n.
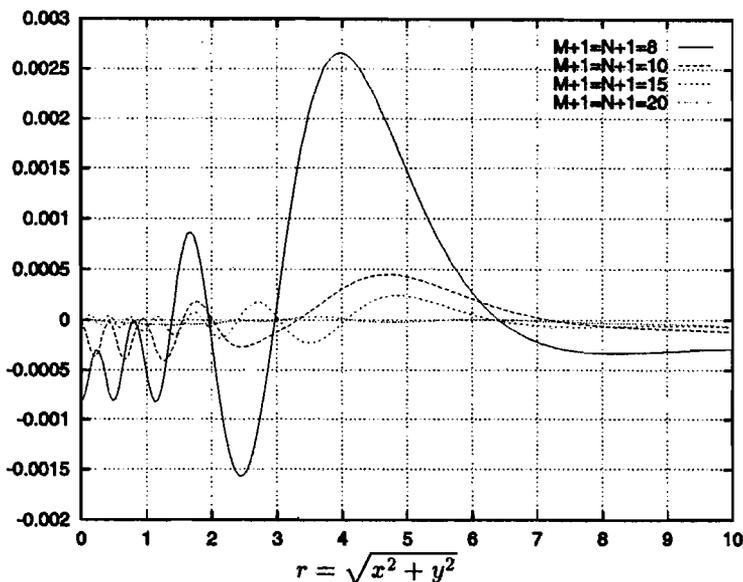
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Fig. 2. The difference between the spectral solution with different number of functions and the finite difference solution with 401 points in the interval [0, 9.9875]

In two dimensions the shape of soliton is presented in Fig. 3 as obtained by the 2D algorithm developed here. Note that a cross-section of this solution is compared in Fig. 2 to the solution with radial symmetry from Fig. 1.

5.3. THE NON-AXISYMMETRIC SOLUTION

As mentioned in the precedence, the convergence of the spectral series can be improved ([11]) if one succeeds to select the optimal scaling for the independent variable. This is especially important when in two dimensions the coefficients before the different highest-order derivatives differ significantly. In our case these are the coefficients β_1 and β_2. The optimization needs a special attention together with an extensive set of numerical experiments and goes beyond the framework of the present paper. Here we have only demonstrated the effectiveness of the splitting scheme for solving the algebraic system for the coefficients. For this reason we do not scale the independent variables even for the case shown in Fig. 4, where there is a considerable difference between the two coefficients $\beta_2 = 1 = 10\beta_1$, $\beta_1 = 0.1$.

In this case a solution obtained by an independent numerical technique is not available and the convergence test is performed by the standard increase of the number of terms in the expansion and by assessing the contribution of the last term. Once again, employing 15 terms gives accuracy of 0.1% and 20 terms bring the difference down to 0.01%. This means that even for one order of magnitude difference between the coefficients of the second derivatives, 20 terms in the expansion is fully enough for securing a very good accuracy. When the ratio between the
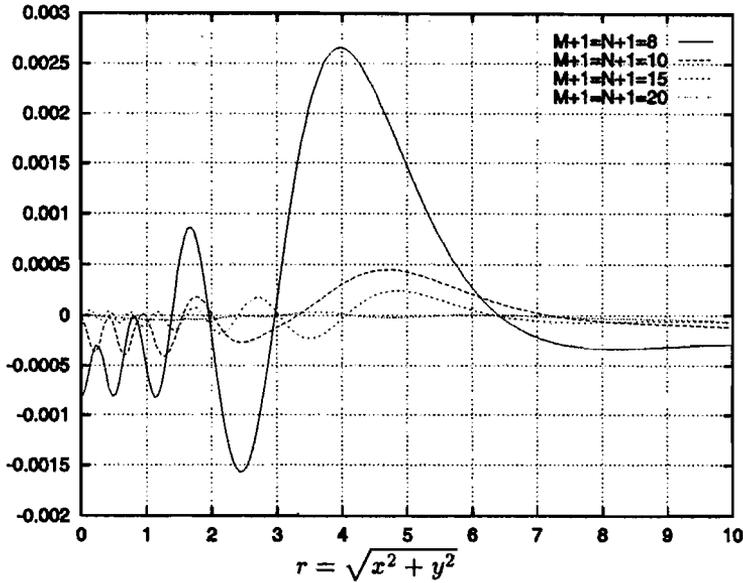
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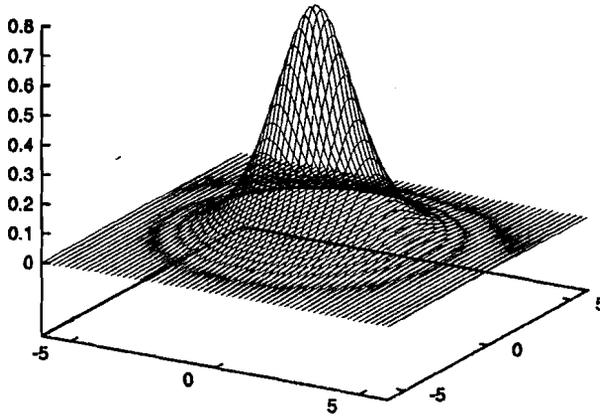
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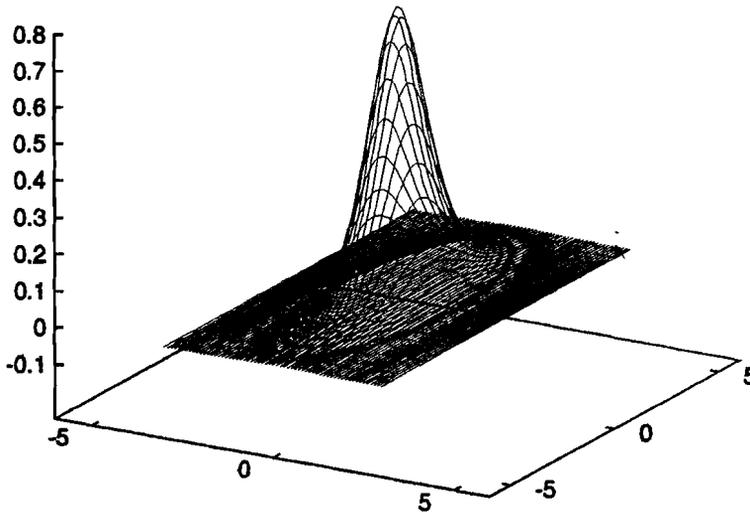

Fig. 4. The soliton for $\beta_1 = 0.1$, $\beta_2 = 1$ as obtained with $M + 1 = N + 1 = 20$ functions in the spectral expansion

coefficients β_i is still larger, one can attempt optimization of the algorithm through different scaling of the independent variables (see [11] for the details in 1D).

6. CONCLUSION

In the present paper a Fourier-Galerkin algorithm for numerical treatment of the bifurcation problem for localized solutions of 2D non-linear PDE is developed.

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To avoid the always present trivial solution, an additional boundary condition is imposed in the origin of the co-ordinate system and a coefficient is added before the non-linear term. The equation itself taken in the origin serves as an explicit definitive relation for the new coefficient. The iterative procedure involves artificial time and co-ordinate splitting of the linear operator corresponding to the partial derivatives. The convergence is secured through selecting the values of the artificial-time increment and the relaxation parameter for the sought coefficient of the non-linear term. In 2D the splitting-type procedure has a significant advantage over the direct Newton-type quasi-linearization algorithms for solving the algebraic system for the coefficients of the Galerkin expansion.

Results are obtained for a generic equation of Klein-Gordon's type with a quadratic non-linearity. The 1D an axisymmetric soliton of the equation in the moving frame is obtained by means of two different techniques and the comparisons give the quantitative assessment of the truncation errors of the spectral expansion.

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