MIXING IN GEOPHYSICAL FLOWS
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TRUNCATED STOCHASTIC FUNCTIONAL SERIES AS A TWO-POINT CLOSURE TECHNIQUE IN TURBULENCE

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1. Introduction

After Keller and Friedman [1] introduced the first cascade system, the problem of truncation and closure for hierarchies of moments (cumulants) has always been one of the central issues in modelling chaotic regimés of nonlinear dynamical systems and turbulence. As pointed out in [2], the simplest closure is the mere neglecting of the nonlinear terms, but then the only case to be treated this way remains the decaying turbulence.

The predominant part of theoretical works deal with the problem of coupling of the truncated version of the system for moments by means of expressing the higher-order moments (cumulants) as functions of the lower-order ones through additional (sometimes quite arbitrary) relations. These are known as hierarchy techniques and the most popular of them are, perhaps, the quasi-normal assumption called also *zero-fourth-cumulant-approach* [3] and the *local independence assumption* [4], [5], [6] in which the fourth moments are related to products of the second or first and third moments, respectively. It turns out, however, that the hierarchy techniques are intrinsically incorrect and often lead to physically inappropriate results, e.g., negative energy spectrum [7]. Another shortcoming is that they are, as a rule, *one-point* closures whose generalization is not feasible.

It was Wiener who proposed in the 1940's (see, [8]) to develop the solution into a Volterra functional series with non-random kernels and random basis function; the latter embodying the information about the random behavior of the solution. In [13] it is shown that each continuous stochastic functional can be expressed Volterra-Wiener series with a Gaussian expansion base and that the series are orthogonal in statistical sense if Hermitean polynomials of the basis function are used.

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Once again, an infinite cascade system was derived, but this time for the kernels. The truncation and closure had simple meaning: discarding higher-order terms in the functional expansion. Wiener employed the Gaussian white noise as a basis function, but the series (named afterwards Wiener-Hermite) lacked a clear physical meaning. As a result, the application to turbulence problems originated in [9] (see also [10] and cited there works) was faced with difficult convergence problems and required frequent renormalization [11] since it was calculated from a truncated system higher-order kernels (representing the deviation from normal distribution) increased swiftly with time. Yet the Wiener method was a considerable step ahead because it provided the rigorous basis for two-point (and multi-point) closure.

The Gaussian basis function was replaced in [12] by the Poisson one and it was shown that the theorem of Cameron & Martin [13] holds for the case, i.e. each continuous stochastic functional can be developed into Wiener series with Poisson basis function. The crucial importance of the choice of Poisson basis function for the practical performance of the functional-series method is demonstrated in works [14],[15],[16],[17] featuring Burgers turbulence. The main advantage of Poisson-Wiener method over the Wiener-Hermite method was that the system of kernels appeared to be nonlinearly closed, in the sense that at a given level of truncation, the kernel of the respective order appeared in the nonlinear term with the same kind of nonlinearity as in the governing system. Note that "nonlinear closure" is strictly impossible in Wiener-Hermite expansions because of the fact that the third moments of the basis function are identically equal to zero (Gaussian white noise).

The Poisson approximation was applied to the Lorenz system [18], [19]; Poiseuille flow [20], [21]; [22], [23]; Kuramoto-Sivashinsky equation [24], [25] (see, [26] for review) and closed system for the kernels were derived and their localized solutions (homoclinics) were found numerically. Thus a complete (though approximate) solution was found in each case and on its base the multipoint statistical characteristics were calculated and compared to the data from physical or numerical experiments. The first moments (the mean values) were, as a rule, in very good quantitative agreement. What is more important is that the predictions for the two-point statistical characteristics showed also satisfactory quantitative agreement (on occasions a very good one) with the experiments.

The good performance of the Poisson approximation is a consequence of the fact that the Poisson function is physically more adequate in its virtue of being a Random Point Function (RPF) representing random trains (random assemblages) of equal shapes. Thus, in the very basis function, more adequate information about the process is included and as a result even the lower-order terms of the expansion are instrumental in quantitative prediction of the two-point statistical characteristics. A way to further improve the practical performance of the functional-series method is to bring the basis function even closer to the physics. The natural choice for the basis function is the class of (generally marked) random point functions (RPF) which is the straightforward generalization of the Poisson function. A RPF is composed by structures of similar

deterministic shape that are randomly located in the region under consideration (time interval, spatial domain, etc.). This is strictly the case with the particulate multiphase media (composites, suspensions) and then the Wolterra-Wiener series provide the rigorous basis for estimating their overall properties (see, [27],[28],[29]).

For the chaotic dynamical systems, the Volterra-Wiener expansion with RPF basis appears to be rigorous only for systems which exhibit so-called homoclinic bifurcation (giving rise to localized structures of bump shape) and then near to the threshold of the instability where the structures do not influence each other in the course of interaction. For more general systems, for which the presence of homoclinic bifurcation has not been yet established our approach is in fact a heuristic assumption based on the observation that the instability gives rise to disturbances that develop and eventually decay leaving the system approximately in the same initial unstable state and only then can a new disturbance occur and the scenario is repeated. During its life span a structure is stable to disturbances of the same characteristic length and the secondary instabilities result in smaller-scale disturbances superimposed upon the main one. This notion fits very well into the picture of a turbulent flow with coherent structures, the latter forming random trains. The secondary disturbances are smoothed out by the truncation procedure and only the large-scale coherent structures contribute to the averaged characteristics.

Although strict, the functional-series may prove not very practical if higher-order kernels must be evaluated numerically as localized solutions of multidimensional PDE. Hence, the speed of convergence is of primary concern. The number of terms needed for adequate approximation of the random field under consideration can drastically be reduced if some more specific information about the RPF-basis is known, namely the two-point probability distribution functions for the generating system of random points – centers of structures. The most rigorous way is to derive Bogolyubov hierarchies for the multi-point distributions. It is complicated to follow this way without having analytical solutions for the shapes of coherent structures and for the "potentials" of interaction among them. The most feasible way is, perhaps, to employ some minimal (maximal) principles [30],[31],[32]. Before embarking on this non-trivial task any approximate closures based on certain approximations of the two-point probability densities are of interest.

The present paper discuss the RPA from the point of view of a multi-point closure. As a featuring example we present the 2D viscous flow in mixing layer. The typical results for the turbulent characteristics are compared the experimental data and show good agreement.

2. Random Point Functions

A Random Point Function (RPF) is a superposition of similarly-shaped structures occurring in random moments of time $\{\tau_\alpha\}$ or at random spatial positions $\{(x_\alpha^1,\dots,x_\alpha^m)\}$. For the sake of brevity we denote by $\mathbf{x}\equiv(x^1,\dots,x^m,t)\in\mathcal{R}^{(m+1)}$ the spatio-temporal coordinate and $m+1$ is the dimension of the configurational

space. Then the set of points $\mathbf{x}_\alpha$ (called "generating set of random points") represents the random spatio-temporal position at which structures appear. In general, the structures could be of variable shape, the latter parameterized by a μ dimensional vector $\mathbf{u}\in\mathcal{U}$ called "mark" and then one has a Marked Random Point Function MRPF. Each MRPF can be expressed as the following linear transformation

$$\kappa(\mathbf{x})=\sum_\alpha h(\mathbf{x}-\mathbf{x}_\alpha;\mathbf{u}_\alpha;\mathbf{x})\equiv\int_\Omega\int_{\mathcal{U}} h(\mathbf{x}-\mathbf{x}_l;\mathbf{u};\mathbf{x})\,\omega(\mathbf{x}_l;\mathbf{u})\,d^{m+1}\mathbf{x}_l d^\mu\mathbf{u} \tag{1.1}$$

where h is the shape of a coherent structure. One should be reminded here about the asymmetry between the configuration variables and the marks. In fact, the configuration variable $\mathbf{x}$ can appear also as a mark in the sense, that the shape of coherent structure may be different in different regions of the configuration space. Only for statistically homogeneous MRPF the explicit dependence of h on $\mathbf{x}$ can be suppressed.

Here

$$\omega(\mathbf{x};\mathbf{u})=\sum_\alpha\delta(\mathbf{x}-\mathbf{x}_\alpha)\delta(\mathbf{u}-\mathbf{u}_\alpha)\equiv\sum_\alpha\delta(\mathbf{r}-\mathbf{r}_\alpha) \tag{1.2}$$

is called Random Density Function (RDF) and the notation

$$\mathbf{r}\equiv(x^1,\dots,x^m,t;u^1,\dots,u^\mu)\in\mathcal{R}\otimes\mathcal{U} \tag{1.3}$$

is introduced for the sake of convenience.

It is clear that the randomness of a MRPF is exhaustively described by its RDF. In its turn, the statistics of a RDF is defined by the respective properties of the generating set of random points, namely by the multipoint probability densities F_n, which define the elementary probability

$$dP=F_n(\mathbf{r}_1,\dots,\mathbf{r}_n;\mathbf{x})d^{m+1}\mathbf{x}_1\dots d^{m+1}\mathbf{x}_n\,d^\mu\mathbf{u}_1\dots d^\mu\mathbf{u}_n \tag{1.4}$$

to simultaneously find n random points in the vicinities of the n "points" $\mathbf{r}_n$. Once again, the explicit dependence on the configuration variable $\mathbf{x}$ is possible when the distribution is not homogeneous. In what follows it will be only displayed in those formulas for which a confusion may arise.

It is convenient to introduce also the "correlation functions of distribution" G_n (see [33] for definition) that are related to F_n in the same manner as cumulants to the moments. For the case of MRPF the said relations were generalized in [26] as follows

$$\begin{aligned}
F_1(\mathbf{r}_1)&=G_1(\mathbf{r}_1)\\
F_2(\mathbf{r}_1,\mathbf{r}_2)&=G_2(\mathbf{r}_1,\mathbf{r}_2)+G_1(\mathbf{r}_2)G_1(\mathbf{r}_2)\\
F_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)&=G_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)+3\{G_1(\mathbf{r}_1)G_2(\mathbf{r}_2,\mathbf{r}_3)\}_{sym}\\
&\quad+G_1(\mathbf{r}_1)G_1(\mathbf{r}_2)G_1(\mathbf{r}_3)\\
&\quad\dots
\end{aligned} \tag{1.5}$$

or

$$\begin{aligned} G_1(\mathbf{r}_1) &= F_1(\mathbf{r}_1) \\ G_2(\mathbf{r}_1,\mathbf{r}_2) &= F_2(\mathbf{r}_1,\mathbf{r}_2) - F_1(\mathbf{r}_2)F_1(\mathbf{r}_2) \\ G_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) &= F_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) - 3\{F_1(\mathbf{r}_1)F_2(\mathbf{r}_2,\mathbf{r}_3)\}_{sym(1,2,3)} \\ &\quad + 2F_1(\mathbf{r}_1)F_1(\mathbf{r}_2)F_1(\mathbf{r}_3) \\ &\quad \dots \end{aligned} \tag{1.6}$$

where $\{\cdot\}_{sym}$ stands for the symmetrization procedure (see, e.g., [33]). Functions G_n are called by some authors "product densities".

The statistics of RPF is decidedly simpler than the one of a general random function. In the groundlaying works [34] and [35], the way has been shown of how to express the multiple moments and cumulants of a random point function without marks (e.g., RDF) through the statistical characteristics of the generating set of random points. Compilation of the necessary technique and the generalization to marked random point functions MRPF is done in [19],[27],[28],[29],[26]. Here we review the main equations which will be of use henceforth.

The multiple moments are given by

$$\begin{aligned} \langle\omega_1(\mathbf{r}_1)\rangle &= F_1(\mathbf{r}_1) = \gamma(\mathbf{x})P(\mathbf{u}_1)\,, \\ \langle\omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\rangle &= F_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2) + F_2(\mathbf{r}_1,\mathbf{r}_2)\,, \\ \langle\omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\omega(\mathbf{r}_3)\rangle &= F_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3) \\ &\quad + 3\{\delta(\mathbf{r}_1-\mathbf{r}_2)F_2(\mathbf{r}_1,\mathbf{r}_3)\}_{sym} + F_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) \\ \langle\omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\omega(\mathbf{r}_3)\omega(\mathbf{r}_4)\rangle &= F_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}_4) \\ &\quad + 4\{\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)F_2(\mathbf{r}_3,\mathbf{r}_4)\}_{sym} \\ &\quad + 3\{\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_4)F_2(\mathbf{r}_1,\mathbf{r}_3)\}_{sym} \\ &\quad + 3\{\delta(\mathbf{r}_3-\mathbf{r}_4)F_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\}_{sym} + F_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\,, \\ &\quad \dots\,, \end{aligned} \tag{1.8}$$

Here γ is called " number density" and is in fact the number of non-random shapes (coherent structures) per unit volume in the configuration space. This quantity is a constant in case of homogeneous RPF (no dependence on $\mathbf{x}$). Respectively $P(\mathbf{u})$ and $P(\mathbf{u}_1,\dots,\mathbf{u}_n)$ are the one-dimensional and multi-dimensional probability distributions of the mark. If the RPF under consideration is not marked, then the formulas presented above are valid if the functions $P(\mathbf{u})$ are omitted and the variable $\mathbf{r}$ is reduced simply to $\mathbf{x}$.

When the multiple distributions of generating set adhere to the rule

$$P_n(\mathbf{u}_1,\dots,\mathbf{u}_n) \equiv P(\mathbf{u}_1)\dots P(\mathbf{u}_n)\,, \quad F_n = \gamma^n P(\mathbf{u}_1)\dots P(\mathbf{u}_n)\,, \quad G_n = 0\,, \tag{1.9}$$

then one arrives at the compound Poisson random point function [36] for which the points and marks of the generating system are statistically independent (functions G_n vanish identically). This could happen if the interaction and/or overlapping of the structures do not affect significantly the dynamics of their centers. It is clear that this is an idealization, because even in the most simplified models (e.g., KdV or Boussinesq equations) the structures (solitons) do acquire phase shift in the course of interaction, although passing through each other without changing the shapes. This means that the occurrence of nontrivial product densities could not be ruled out.

Among the RPF with nontrivial product densities we discern the class for which

$$\gamma/L < 1\,, \quad G_n \approx O(\gamma^n)\,, \quad F_n \approx O(\gamma^n)\,, \tag{1.10}$$

where L is the characteristic scale of a structure. These can be called "well mixed" systems of random points in the sense that there is no significant clustering among the generating points, i.e., there are no components of type of delta-functions (or other kind of steep functions) in the expressions for G_n/γ^n. The conjecture (1.10) rules completely out the case with stationary (or long-lasting) bound states among the structures.

3. Functional Series

Wiener extended the idea of Volterra functional series to the case of random functions introducing for an arbitrary random function $\mathbf{w}(\mathbf{x})$ the "power series"

$$\begin{aligned} \mathbf{w}(\mathbf{x}) =& \mathbf{K}_0(\mathbf{x}) + \sum_1^\infty \int_{\mathcal{R}}\dots\int_{\mathcal{R}}\int_{\mathcal{U}}\dots\int_{\mathcal{U}} \mathbf{K}^{(n)}(\mathbf{x}-\mathbf{x}_1,\dots,\mathbf{x}-\mathbf{x}_n;\mathbf{u}_1,\dots,\mathbf{u}_n;\mathbf{x}) \\ & \chi(\mathbf{x}_1;\mathbf{u}_1),\dots,\chi(\mathbf{x}_n;\mathbf{u}_n)\, d^{m+1}\mathbf{x}_1\dots d^{m+1}\mathbf{x}_n\, d^\mu\mathbf{u}_1\dots d^\mu\mathbf{u}_n \end{aligned} \tag{2.1}$$

where $> \mathbf{K}^{(n)}$ are non-random kernels and χ is a basis function responsible for the randomness of the field under consideration. Wiener used Gaussian white noise as a basis function. Cameron & Martin [13] proved that each random function can be developed in Volterra-Wiener series with basis function of type of Gaussian white noise. H. Ogura [12] went further to show that the same holds true for Poisson white noise. It could be shown that the provisions of the Cameron-Martin-Ogura theorem apply for the case of more general MRPF basis function. In this sense any random function can be expressed in terms of functional series with respect to MRPF basis function. The problem is the number of terms which will be necessary in order to obtain a reasonable approximation. Hence, it is of crucial importance to chose a basis function which is as closer as possible to the behaviour of the system under consideration and to achieve a satisfactory approximation with the lowest-order terms. It is not the case with the original Wiener choice – the Gaussian white noise since the latter is connected with the asymptotic case of large number of infinitesimally small disturbances.

It seems much more plausible that near the threshold of the homoclinic bifurcation, one may be faced with a random train of weakly interacting homoclinics rather

than with a Gaussian-like random function. A random train is just another name for the random point functions.

It is important to render the series (at least approximately) orthogonal in order to separate the effects from the different kernels and reduce the complexity of the resulting hierarchical systems. In [12] is shown that the Wiener series with Poisson basis function are rendered orthogonal in statistical sense if the Charlier Polynomials of the Poisson function are employed in the place of the multinomes of the latter. The generalization of the Charlier Polynomials to the case of Compound RPF (Functions of Perfect Disorder Type) was proposed in [27,28,29] but it led to tedious expression whose use were justified in the statistical theory of suspension. It suffices here to present just the first several of Charlier polynomials and to use them to make the series orthogonal in virial sense (with respect to the powers of the small parameter γ). So that, we do not go to the full scale generalization of Charlier polynomials, but rather replace the number density γ of [12] with the more general expression $F_1(\mathbf{r})$, namely

$$
\begin{aligned}
&C_\omega^{(0)} = 1\,, \qquad C_\omega^{(1)} = \omega(\mathbf{r}_1) - F(\mathbf{r}_1)\,,\\
&C_\omega^{(2)} = \omega(\mathbf{r}_1)\omega(\mathbf{r}_2) - \delta(\mathbf{r}_1-\mathbf{r}_2)\omega(\mathbf{r}_1)\\
&\quad - [\omega(\mathbf{r}_2)F_1(\mathbf{r}_1) + \omega(\mathbf{r}_1)F_1(\mathbf{r}_2)] + F_1(\mathbf{r}_1)F_1\mathbf{r}_2)\,,\\
&C_\omega^{(3)} = \omega(\mathbf{r}_1)\omega(\mathbf{r}_2)\omega(\mathbf{r}_3) + 2\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\omega(\mathbf{r}_1)\\
&\quad - [\omega(\mathbf{r}_1)F_2(\mathbf{r}_2,\mathbf{r}_3) + \omega(\mathbf{r}_2)F_2(\mathbf{r}_3,\mathbf{r}_1) + \omega(\mathbf{r}_3)F_2(\mathbf{r}_1,\mathbf{r}_2)]\\
&\quad - \omega(\mathbf{r}_1)[\omega(\mathbf{r}_2) - \delta(\mathbf{r}_1-\mathbf{r}_2)] - \omega(\mathbf{r}_2)[\omega(\mathbf{r}_3) - \delta(\mathbf{r}_2-\mathbf{r}_2)]\\
&\quad - \omega(\mathbf{r}_3)[\omega(\mathbf{r}_1) - \delta(\mathbf{r}_3-\mathbf{r}_1)] + F_1(\mathbf{r}_1)F_1(\mathbf{r}_2)F_1(\mathbf{r}_3)\,,
\end{aligned}
\tag{2.2}
$$

The above defined Generalized Charlier polynomials are not centered random variables, namely

$$
\langle C_\omega^{(n)}(\mathbf{r}_1,\ldots,\mathbf{r}_n)\rangle = G_n(\mathbf{r}_1,\ldots,\mathbf{r}_n)\,, \tag{2.3}
$$

but this difficulty is not essential for what follows.

Respectively, for the first couple of moments of Charlier Polynomials we have

$$
\begin{aligned}
\langle C_\omega^{(1)}(\mathbf{r}_1)C_\omega^{(1)}(\mathbf{r}_2)\rangle &= F_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2) + F_2(\mathbf{r}_1,\mathbf{r}_2) - F_1(\mathbf{r}_1)F_1(\mathbf{r}_2)\,,\\
&\equiv G_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2) + G_2(\mathbf{r}_1,\mathbf{r}_2)\,,
\end{aligned}
$$

$$
\begin{aligned}
\langle C_\omega^{(1)}(\mathbf{r}_1)C_\omega^{(1)}(\mathbf{r}_2)C_\omega^{(1)}(\mathbf{r}_3)\rangle &= G_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\\
&\quad + 3\{\delta(\mathbf{r}_1-\mathbf{r}_2)G_2(\mathbf{r}_1,\mathbf{r}_3)\}_{sym} + G_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\,,\\
&= G_1(\mathbf{r}_1)\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\\
&\quad + 3\{\delta(\mathbf{r}_1-\mathbf{r}_2)G_2(\mathbf{r}_1,\mathbf{r}_3)\}_{sym} + O(\gamma^3) >,
\end{aligned}
$$

$$
\begin{aligned}
\langle C_\omega^{(2)}(\mathbf{r}_1,\mathbf{r}_2)C_\omega^{(1)}(\mathbf{r}_3)\rangle &= G_2(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_2-\mathbf{r}_3)\\
&\quad + G_2(\mathbf{r}_3,\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_1) + G_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\,.\\
&= G_2(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_2-\mathbf{r}_3) + G_2(\mathbf{r}_3,\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_1) + O(\gamma^3)
\end{aligned}
\tag{2.4}
$$

$$
\begin{aligned}
\langle C_\omega^{(2)}(\mathbf{r}_1,\mathbf{r}_2)C_\omega^{(2)}(\mathbf{r}_3,\mathbf{r}_4)\rangle &= G_2(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r}_4)\\
&\quad + G_2(\mathbf{r}_1,\mathbf{r}_2)\delta(\mathbf{r}_1-\mathbf{r}_4)\delta(\mathbf{r}_2-\mathbf{r}_3) + O(\gamma^3)\,.\\
&\ldots\\
\left\langle C^{(n_1)}\ldots C^{(n_k)}\right\rangle &= O(\gamma^{max(n_1,\ldots n_k)})\,.
\end{aligned}
$$

It is easily seen that (2.4) reduce to the Charlier polynomials introduced by H.Ogura [12] for the Poisson basis function.

Now we are equipped to introduce the Volterra-Wiener series with MPRF basis:

$$
\begin{aligned}
\mathbf{w}(\mathbf{x}) = \mathbf{K}^{(0)}(\mathbf{x}) + \sum_{n=1}^{\infty} \int_{\mathcal{R}}\!\!\ldots\!\int_{\mathcal{R}} \int_{\mathcal{U}}\!\!\ldots\!\int_{\mathcal{U}} &\mathbf{K}^{(n)}(\mathbf{x}-\mathbf{x}_1,\ldots,\mathbf{x}-\mathbf{x}_n;\mathbf{u}_1,\ldots,\mathbf{u}_n;\mathbf{x})\\
&\left[C_\omega^{(n)}(\mathbf{x}_1,\ldots,\mathbf{x}_n;\mathbf{u}_1,\ldots,\mathbf{u}_n;\mathbf{x}) - G_n(\mathbf{r}_1,\ldots,\mathbf{r}_n;\mathbf{x})\right]\\
&d^{m+1}\mathbf{x}_1\ldots d^{m+1}\mathbf{x}_n\, d^{\mu}\mathbf{u}_1\ldots d^{\mu}\mathbf{u}_n\,.
\end{aligned}
\tag{2.5}
$$

The last of equalities (2.3) allows us to call the series "virial" in the sense that each term contributes to the average characteristics quantity proportional to the respective power of the number density.

The interpretation of the Volterra-Wiener series with RPF basis has a clear physical meaning. A kernel of certain order represents the interaction of respective order. The first kernel gives the shape of a single coherent structure. The second kernel is the contribution to the field due to the pure binary interaction of two structures (the main contribution is already included in the first-order kernel). The third kernel is the addition to the field due to the pure triple interaction. It is shown in [28] that for particulate media, the kernel of respective order vanishes exactly where the correlation function G_n is not equal to zero. It is not exactly the case with the nonlinear systems, but something similar is also observed. For instance, the two-point product density for the Lorenz system (see [18]) is strictly equal to zero in the interval [0,0.63] where the predominant "mass" of the structure is concentrated (homoclinics [19].

4. Nonlinear Stochastic Problems via RPF

In this section we show the way of application of RPF for approximate treatment of nonlinear stochastic problems. Consider the system,

$$\frac{\partial \mathbf{w}}{\partial t} + \nabla \mathbf{w}\mathbf{w} = \Delta \mathbf{w}, \tag{3.1}$$

where $\mathbf{w}$ is the vector of unknowns, which closely resembles the Navier-Stokes systems save the equation of continuity. There are no principle difficulties to consider higher-order nonlinearity or higher-order spatial or temporal derivatives and it is not done here only for the sake of not obscuring the main ideas.

For the moments one obtains a cascade system [1] which is not closed, namely

$$\begin{aligned} \langle \mathbf{w}\frac{\partial}{\partial t}\mathbf{w}\rangle + \langle \mathbf{w}\nabla \mathbf{w}\mathbf{w}\rangle &= \langle \mathbf{w}\Delta \mathbf{w}\rangle\,, \\ \langle \mathbf{w}\mathbf{w}\frac{\partial}{\partial t}\mathbf{w}\rangle + \langle \mathbf{w}\mathbf{w}\nabla \mathbf{w}\mathbf{w}\rangle &= \langle \mathbf{w}\mathbf{w}\Delta \mathbf{w}\rangle\,, \\ &\cdots \end{aligned} \tag{3.2}$$

The inherent deficiency of this approach is clearly seen, the system for moments is never closed in case of nonlinear systems. That is why the truncation and closure of the infinite cascade systems for the hierarchies of moments (cumulants) is a central issue in modelling chaotic regimés of nonlinear dynamical systems and turbulence.

Let us confine ourselves to the case of purely spatial chaos when we do not have time, t, among the independent variables comprising the configurational space. If we would have time among these, it would mean that we are assuming a process of death and birth. This kind of dependence was elucidated in [19] in connection with Lorenz system. Here we consider the other limiting case, when the time only plays the role of parameter connected with the evolution of fixed number of structures that are randomly distributed throughout the space. Then the functional series for function $\mathbf{w}$ has the form:

$$\begin{aligned} \mathbf{w}(\mathbf{x},t) = \mathbf{K}^{(0)}(\mathbf{x},t) + \int_{-\infty}^{\infty} \mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_l;\mathbf{x},t)C_\omega^{(1)}(\mathbf{x}_l)d^m\mathbf{x}_l\,, \\ + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_1,\mathbf{x}-\mathbf{x}_2;\mathbf{x},t)[C_\omega^{(2)}(\mathbf{x}_1,\mathbf{x}_2) - G_2(\mathbf{x}_1,\mathbf{x}_2)d^m\mathbf{x}_1 d^m\mathbf{x}_2 + \dots\,. \end{aligned} \tag{3.3}$$

Here $\mathbf{x}$ encompasses only the spatial variables. A sub-index denotes the local (dummy) variable and the semicolon serves to separate the dependences on the local and global coordinates. The meaning of the local coordinate is that it defines the shape of a structure. The presence also of the global variables $\mathbf{x}, t$ stand to remind that the structure may change from one spatial position to another or evolve in time. In other words, for different spatial or temporal positions the functional dependence on the local coordinates may be different. Note that we consider the case without marks, in order not to complicate much the exposition. The marks are easily incorporated if the respective formulas of the previous sections are employed.

To the order $O(\gamma^3)$ we have for the nonlinear term

$$\begin{aligned} \mathbf{w}(\mathbf{x};t)\cdot\nabla\mathbf{w}(\mathbf{x};t) &= \mathbf{K}^{(0)}(\mathbf{x};t)\cdot\nabla\mathbf{K}^{(0)}(\mathbf{x};t) \\ &+\mathbf{K}^{(0)}(\mathbf{x};t)\cdot\nabla\int \mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_l;\mathbf{x};t)C^{(1)}(\mathbf{x}_l)d^m\mathbf{x}_l \\ &+\int \mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_l;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(0)}(\mathbf{x};t)C^{(1)}(\mathbf{x}_l)d^m\mathbf{x}_l \\ &+\int \mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_1;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_2;\mathbf{x};t)C^{(1)}(\mathbf{x}_1)C^{(1)}(\mathbf{x}_2)d^m\mathbf{x}_1 d^m\mathbf{x}_2 \\ &+\mathbf{K}^{(0)}(\mathbf{x};t)\cdot\nabla\int\int \mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_1;\mathbf{x}-\mathbf{x}_2;\mathbf{x};t) \\ &\times\left[C^{(2)}(\mathbf{x}_1;\mathbf{x}_2) - G_2(\mathbf{x}_1;\mathbf{x}_2;t)\right]d^m\mathbf{x}_1 d^m\mathbf{x}_2 \\ &+\int\int \mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_1;\mathbf{x}-\mathbf{x}_2;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(0)}(\mathbf{x};t) \\ &\times\left[C^{(2)}(\mathbf{x}_1;\mathbf{x}_2) - G_2(\mathbf{x}_1;\mathbf{x}_2;t)\right]d^m\mathbf{x}_1 d^m\mathbf{x}_2 \\ &+\int\int\int \mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_3;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_1;\mathbf{x}-\mathbf{x}_2;\mathbf{x};t) \\ &\times C^{(1)}(\mathbf{x}_3)\left[C^{(2)}(\mathbf{x}_1;\mathbf{x}_2) - G_2(\mathbf{x}_1;\mathbf{x}_2;t)\right]d^m\mathbf{x}_1 d^m\mathbf{x}_2 d^m\mathbf{x}_3 \\ &+\int\int\int \mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_1;\mathbf{x}-\mathbf{x}_2;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(1)}(\mathbf{x}-\mathbf{x}_3;\mathbf{x};t) \\ &\times C^{(1)}(\mathbf{x}_3)\left[C^{(2)}(\mathbf{x}_1;\mathbf{x}_2) - G_2(\mathbf{x}_1;\mathbf{x}_2;t)\right]d^m\mathbf{x}_1 d^m\mathbf{x}_2 d^m\mathbf{x}_3 \\ &+\int\int\int\int \mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_1;\mathbf{x}-\mathbf{x}_2;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(2)}(\mathbf{x}-\mathbf{x}_3;\mathbf{x}-\mathbf{x}_4;\mathbf{x};t) \\ &\times\left[C^{(2)}(\mathbf{x}_3,\mathbf{x}_4) - G_2(\mathbf{x}_3;\mathbf{x}_4;t)\right]\left[C^{(2)}(\mathbf{x}_1;\mathbf{x}_2) - G_2(\mathbf{x}_1;\mathbf{x}_2;t)\right] \\ &\times d^m\mathbf{x}_1 d^m\mathbf{x}_2 d^m\mathbf{x}_3 d^m\mathbf{x}_4 \\ &+\mathcal{O}(\gamma^3) \end{aligned} \tag{3.4}$$

Now we are equipped to take the averages of the linear and nonlinear terms. in order not to complicate the derivation we keep to order $O(\gamma^2)$. The main point in the present approach is that the dimensionless number γ is expected to be always lesser than unity if scaled by the effective measure of the support of a structure. The structures do no overlap significantly, or if they do, it is during a short period of time (the time for collision).

We start with the means

$$\begin{aligned}\langle \mathbf{w}(\mathbf{x};t)\rangle &= \mathbf{K}^{(0)}(\mathbf{x};t)\,,\\ \langle \mathbf{w}(\mathbf{x};t)\cdot\nabla\mathbf{w}(\mathbf{x};t)\rangle &= \mathbf{K}^{(0)}(\mathbf{x};t)\cdot\nabla\mathbf{K}^{(0)}(\mathbf{x};t)\\ +\gamma\int \mathbf{K}^{(1)}(\mathbf{x}_1;\mathbf{x};t)\cdot\nabla\mathbf{K}^{(1)}(\mathbf{x}_1;\mathbf{x};t)d^m\mathbf{x}_1 &+ O(\gamma^2)\,,\end{aligned} \tag{3.5}$$

according to the respective properties of Charlier polynomials.

Respectively, the first kernels are the leading terms in the following averages:

$$\begin{aligned}\langle C^{(1)}(\mathbf{0})\mathbf{w}(\mathbf{x};t)\rangle &= \mathbf{K}^{(1)}(\mathbf{x};t)+\mathcal{O}(\gamma^3)\\ \langle C^{(1)}(\mathbf{0})\mathbf{w}(\mathbf{x};t)\cdot\nabla\mathbf{w}(\mathbf{x};t)\rangle &= \gamma\left[\mathbf{K}^{(0)}\cdot\mathbf{K}^{(1)}\mathbf{K}^{(1)}\cdot\mathbf{K}^{(0)}\right]\end{aligned} \tag{3.6}$$

Employing these formulas we get the following hierarchical system to the second order with respect to the number density γ.

$$\begin{aligned}&\frac{\partial\mathbf{K}^{(0)}(\mathbf{x},t)}{\partial t}+\mathbf{K}^{(0)}(\mathbf{x},t)\cdot\nabla\mathbf{K}^{(0)}(\mathbf{x},t)-\Delta\mathbf{K}^{(0)}(\mathbf{x},t)\\ &\qquad=\gamma\int_{-\infty}^{\infty}\mathbf{K}^{(1)}(\mathbf{x}_l,t)\cdot\nabla\mathbf{K}^{(1)}(\mathbf{x}_l,t)d^m\mathbf{x}_l\\ &\frac{\partial\mathbf{K}^{(1)}(\mathbf{x},t)}{\partial t}+\mathbf{K}^{(0)}(\mathbf{x},t)\cdot\nabla\mathbf{K}^{(1)}(\mathbf{x},t)+\mathbf{K}^{(1)}(\mathbf{x},t)\cdot\nabla\mathbf{K}^{(0)}(\mathbf{x},t)\\ &\qquad+\mathbf{K}^{(1)}(\mathbf{x},t)\cdot\nabla\mathbf{K}^{(1)}(\mathbf{x},t)-\Delta\mathbf{K}^{(1)}(\mathbf{x},t)+O(\gamma^2)\end{aligned} \tag{3.7}$$

There are no difficulties in obtaining the equation for the second kernel, for which we would need

$$\langle C^{(2)}(\mathbf{0},\mathbf{z})\mathbf{w}(\mathbf{x};t)\rangle \quad\text{and}\quad \langle C^{(2)}(\mathbf{0},\mathbf{z})\mathbf{w}(\mathbf{x};t)\cdot\nabla\mathbf{w}(\mathbf{x};t)\rangle\,, \tag{3.8}$$

the algebra is tedious and not relevant in the present paper. The equation for the first order kernel is sufficient for the present discussion.

Near the onset of the homoclinic bifurcation the number density is indeed a small value and we do not have to know the exact shape of $G_2, G_3, \ldots, G_n$ if we target a $o(\gamma)$ solution. Neglecting the higher-order kernels far from the threshold is a herustic assumption which we call "closure". In fact keeping just the first order kernels we arrive at the lowest-level model in which the shapes of structures are approximately defined by the homoclinic solution and then their contribution to the averaged field serves to define the mean field. One should be aware that this kind of closure does not mean neglecting in full the interactions. In fact, the interaction of the structures through the mean changes the system and hence the condition for homoclinic bifurcation and the shape of the homoclinic itself. This is the reason for the surprisingly good quantitative predictions (see the references mentioned in the above) with the

lowest-order model. It turns out that the neglected terms containing the higher-order kernel could be effectively emulated by the un-exact form of the homoclinics which is now a solution of an approximate system. In this instance one can consider the eqs.(3.7) as living a life on its own. We refer to it as "two-point closure".

The lowest-order closure modell omitts a whole universe of interactions. For this reason we can not claim this model as a complete model of turbulence. What is really done is to get a new kind of closure that is more robust and nonlinearly coupled.

Going beyond the lowest-order closure poses additional difficulties. The multi-point product densities enter the hierarchical system. Then one must derive kinetic equations for them. This is not an easy task, being reminded that the "potential of interactions" among the structures could be defined only if analytical expressions for the shapes are available. The most feasible way to proceed is to use a variational principle for approximate closure. It seems attractive at this stage to try to implement the principle of maximal entropy [30],[43]. The way is to keep the two-point product densities in the governing set (3.7) and to maximize a functional under the restrictions on the functions as provided by the very equations (3.7). The the latter would be two of the necessary conditions for minimization and the rest of them would provide the missing equations for the product densities.

5. A 2D Example of Application. Featuring the Plane Mixing layer

The Random Point Approximation RPA proves useful in the cases of Lorenz attractor [19], for Kuramoto-Sivashinsky equation [24],[25], and the initial-value Burgelence problem [15],[16],[19],[17] (see, also [26]). It is therefore tempting to face the challenge of a stochastic problem for Navier-Stokes equations in order to approach the turbulence problem. Due to the limited computational facilitites, it has to be a 2D problem. Unfortunatly, it is not clear that a homoclinic bifurcation does indeed exisit in every 2D shear flows. Our attempts to find a homoclinic bifurcation of the N-S equations for the 2D Poiseuille flows was not succesful. It turned out, however, that for a slightly modified version of the equations (in fact the Orr [32] equations), obtained from the so-called "principle of least dissipation" [31], we did find a localized solution [38],[20],[21]. It appeared that the coherent structure calculated in those works compared quantitatively surprisingly well with the Variable Interval Time Averaging VITA measurments in the wall region reported in [39].

An interesting example of structutural turbulence is provided by the plane mixing layer which is formed between two layers of different velocities. Due to the presense of an inflexion point in the mean-velocity profile, 2D turbulence is possible and that is the reason for considering the 2D mixing layer when displaying the RPA (see, [26],[22],[23]). Unfortunately, the price we are to pay is the spatial non-homegeneity of the flow. In this instance we are faced with a situation already described in the above – dependence on global and local variables.

The 2D Navier-Stokes equations in terms of stream-function adopt the form

$$\frac{\partial \Delta\psi}{\partial t} + \frac{\partial \psi}{\partial y}\frac{\partial \Delta\psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \Delta\psi}{\partial y} = \nu \Delta\Delta\psi \tag{4.1}$$

Let $\langle\psi\rangle$ be the ensemble average, and ψ' – the fluctuations. Then (4.1) yields

$$\frac{\partial \Delta\langle\psi\rangle}{\partial t} + \frac{\partial\langle\psi\rangle}{\partial y}\frac{\partial \Delta\langle\psi\rangle}{\partial x} - \frac{\partial\langle\psi\rangle}{\partial x}\frac{\partial \Delta\langle\psi\rangle}{\partial y} = \nu\Delta\Delta\langle\psi\rangle - \left\langle \frac{\partial\psi'}{\partial y}\frac{\partial\Delta\psi'}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\Delta\psi'}{\partial y}\right\rangle , \tag{4.2}$$

$$\begin{aligned} &\frac{\partial \Delta\psi'}{\partial t} + \left(\frac{\partial\langle\psi\rangle}{\partial y} + \frac{\partial\psi'}{\partial y}\right)\frac{\partial\Delta\psi'}{\partial x} - \left(\frac{\partial\langle\psi\rangle}{\partial x} + \frac{\partial\psi'}{\partial x}\right)\frac{\partial\Delta\psi'}{\partial y} \\ &+ \frac{\partial\psi'}{\partial y}\frac{\partial\Delta\langle\psi\rangle}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\Delta\langle\psi\rangle}{\partial y} = \nu\Delta\Delta\psi' - \left\langle \frac{\partial\psi'}{\partial y}\frac{\partial\Delta\psi'}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\Delta\psi'}{\partial y}\right\rangle , \end{aligned} \tag{4.3}$$

where $\langle\cdot\rangle$ stands for the operation of ensemble averaging.

One can approximately homogenize the flow by means of the von Kármán hypothesis about the different scales of the fluctuations and the mean flow and the similarity of the fluctuations. We denote the coordinate of the centre of a structure by $x_0(t)$ (so-called "slow" coordinate) and the coordinates in the frame moving with a structure (the "fast" variables) – by ξ, η, i.e.,

$$x = x_0 + \xi f(x_0)\,, \quad y = \eta f(x_0)\,. \tag{4.4}$$

Then, in the spirit of von Kármán's hypothesis we assume for the mean stream function and for the fluctuations the following

$$\langle\psi\rangle = U_1 f(x_0) F(\eta)\,, \quad \psi'(x,y,t) = U_1 h(x_0) Q(\xi,\eta,\tau)\,. \tag{4.5}$$

Then for the "Reynolds stress" we have

$$-\left\langle \frac{\partial\psi'}{\partial y}\frac{\partial\Delta\psi'}{\partial x} - \frac{\partial\psi'}{\partial x}\frac{\partial\Delta\psi'}{\partial y}\right\rangle = \frac{U_1 h^2(x_0)}{f^4(x_0)}\frac{\partial S}{\partial\eta}\,, \tag{4.6}$$

and the Reynolds equation adopts the form

$$FF'' + S = F'''\,, \tag{4.7}$$

provided that the scales are defined as follows

$$f = \left(\frac{2\nu x_0}{U_1}\right)^{\frac{1}{2}}\,, \quad h = \left(\frac{2\nu^3 x_0}{U_1}\right)^{\frac{1}{4}}\,, \quad c = \frac{1}{U_1}\frac{dx_0}{dt} = \frac{\dot{x}_0}{U_1}\,. \tag{4.8}$$

In terms of self-similar variables the equation for the fluctuations adopts the form

$$\begin{aligned} &\frac{U_1 f^2}{\nu h}\frac{\partial\Delta Q}{\partial\tau} + c\frac{ff'}{h}\left[\left(\frac{h'f}{hf'} - 2\right)\Delta Q - \xi\frac{\partial\Delta Q}{\partial\xi} - \eta\frac{\partial\Delta Q}{\partial\eta}\right] - c\frac{f}{h}\frac{\partial\Delta Q}{\partial\xi} \\ &+ \frac{\partial Q}{\partial\eta}\frac{\partial\Delta Q}{\partial\xi} - \frac{\partial Q}{\partial\xi}\frac{\partial\Delta Q}{\partial\eta} + \frac{f}{h}\left[\frac{dF}{d\eta}\frac{\partial\Delta Q}{\partial\xi} - \frac{d^3F}{d\eta^3}\frac{\partial\Delta Q}{\partial\eta}\right] = \frac{\nu}{U_1 h}\Delta\Delta Q + \frac{\partial S}{\partial\eta}\,, \end{aligned} \tag{4.9}$$

where c is the phase velocity of a coherent structure (the speed of the center of the moving and deforming coordinate frame).

$$c = \frac{1}{U_1}\frac{dx_0}{dt} \equiv \frac{\dot{x}_0}{U_1}\,. \tag{4.10}$$

It is convenient to introduce the following Reynolds number

$$Re = \frac{U_1 h}{\nu} = \left(\frac{2U_1 x_0}{\nu}\right)^{\frac{1}{4}} \tag{4.11}$$

The scaling allows us to treat in an unified manner the laminar and the RPF regimés. One sees that the solution is not perfectly self-similar, because the Reynolds number includes dependence on longitudinal coordinate x. Because of that we have some longitudinalwise evolution of the structure ("aging"). Anyway, this non-uniformity is slight and does not require changes in the numerical algorithm. The inhomogeneity is proportional to $Re^{\frac{1}{4}}$ which is of order of unity and this explains the fact that the shape of the coherent structure did not depend significantly on the particular value of Reynolds number [22], [23].

Eq. (4.9) adopts the following dimensionless form

$$\begin{aligned} &Re^3\frac{\partial\Delta Q}{\partial\tau} + \frac{c}{Re}\left[\left(Re^2 - 2\right)\Delta Q - \xi\frac{\partial\Delta Q}{\partial\xi} - \eta\frac{\partial\Delta Q}{\partial\eta}\right] - c\frac{f}{h}\frac{\partial\Delta Q}{\partial\xi} \\ &+ \frac{\partial Q}{\partial\eta}\frac{\partial\Delta Q}{\partial\xi} - \frac{\partial Q}{\partial\xi}\frac{\partial\Delta Q}{\partial\eta} + \frac{f}{h}\left[\frac{dF}{d\eta}\frac{\partial\Delta Q}{\partial\xi} - \frac{d^3F}{d\eta^3}\frac{\partial Q}{\partial\eta}\right] = \frac{1}{Re}\Delta\Delta Q + \frac{\partial S}{\partial\eta}\,, \end{aligned} \tag{4.12}$$

coupled by the dimensionless version of the "Reynolds equation" (4.6).

Now we can consider in the moving and deforming coordinate frame a homogeneous random train of structures. We content ourselves with the first order truncation when there is no difference between general RPF and plain Poisson. So, we seek for the solution in the form

$$Q(\xi,\eta) = \int_{-\infty}^{\infty} K(\xi-\zeta,\eta)[\omega(\zeta)-\gamma]d\zeta\,, \tag{4.13}$$

and then the Reynolds equation is simply

$$FF'' - \gamma \int_{-\infty}^{\infty} \frac{\partial K}{\partial \xi}\Delta K\, d\xi \tag{4.14}$$

with b.c.

$$u \equiv F'\,,\quad F'(-\infty)=1\,,\quad F'(\infty)=\frac{U_2}{U_1}\,,\quad F(0)=0\,. \tag{4.15}$$

Respectively, the "equation for fluctuations" is nothing else but the equation for the first kernel in which all the terms of order $O(\gamma^2)$ are neglected, namely

$$\left(Re\frac{dF}{d\eta} - cRe + \frac{\partial K}{\partial \eta} - \frac{c\xi}{Re}\right)\frac{\partial \Delta K}{\partial \xi} - \left(\frac{\partial K}{\partial \xi} + \frac{c\eta}{Re}\right)\frac{\partial \Delta K}{\partial \eta} - Re\frac{d^3F}{d\eta^3}\frac{\partial K}{\partial \xi} = \frac{1}{Re}\Delta\Delta K\,. \tag{4.16}$$

The last equation is not much different from the original N-S equation. The crucial difference is that it is not solved as initital-value problem which is unstable and whose solution goes eventually chaotic. Now we are looking for a smooth localized solution, when the following condition holds

$$\int_{-\infty}^{\infty}\left[\left(\frac{\partial K}{\partial \xi}\right)^2 + \left(\frac{\partial K}{\partial \eta}\right)^2\right] d\xi < +\infty\,, \tag{4.17}$$

which is the same as

$$\frac{\partial K}{\partial \xi}\,,\quad \frac{\partial K}{\partial \xi} \longrightarrow 0 \quad \text{for} \quad \xi\,,\eta \to \pm\infty\,. \tag{4.18}$$

We do not present details on the way the solution is obtained numerically. They can be found in [26], [22], [23] where the shape of the localized structure was first obtained. It is important to note that it should not be confused with the inviscid co-

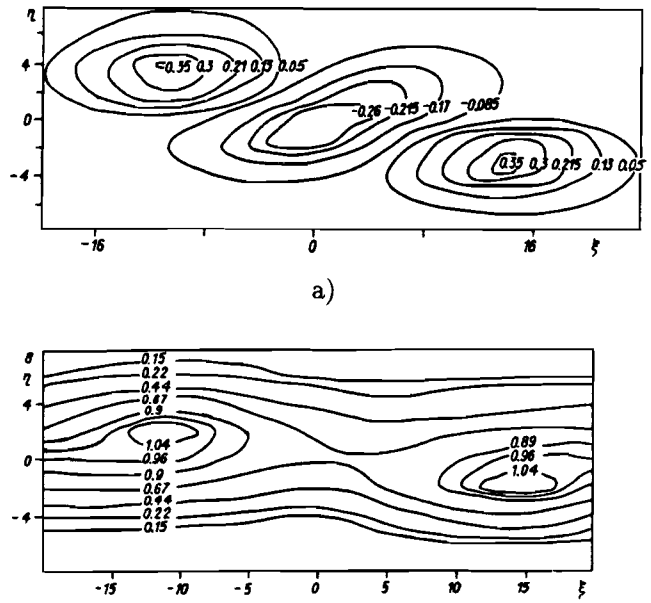

Fig.1 A typical shape of coherents structure for $\gamma = 0.07$ in terms of isolines of vorticity function: a) pulsational part; b)total vorticity.

herent structures obtained by the method of discrete vortices [40],[41]. In *Fig.* 1-*a* is presented the shape of the structure for the "fluctuating" part of the vorticity which, in our opinion, is the physically relevant variable. One sees that a single structure consists of three distinct vortices. In *Fig.* 1-*b* is presented the "full" vorticity and the structure consists of two "vortices". It is peculiar to note that for larger Reynolds numbers the structures draw closer to each other (in terms of scaled variables, of course) and the second vortex of a structure is closer to the center of the first vortex of the next structure, rather than to the center of the first vortex of the structure to which it formally belongs. This may be interpreeted as "pairing" of the vortices.

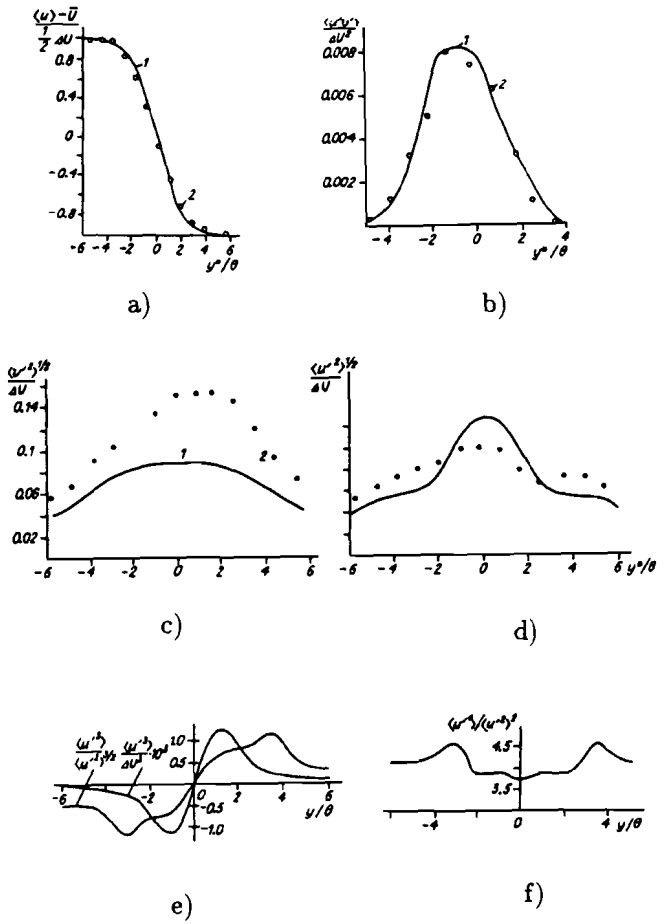

Fig. 2 The averaged characteristics for $\gamma = 0.07$ presented with transverse coordinate scaled by the thickness of displacement of the layer. Solid lines – present calculations; circles - experiments of Browand & Weidman, 1976. a) mean-velocity profile; b) Reynolds stress; c) kinetic energy of the transverse component of velocity; d) kinetic energy of the longitudinal component of velocity; e) asymmetry; f) excess.

The phase velocity (or celerity) is also an unknown of the problem. Our calculations turned out to give solution for various celerities. The solution presented here is for $c = (U_1 + U_2)/2$. It is to be mentioned that a better comparison with experiments can be achieved for celerities somewhat larger than the said value. The effect, however, could be attributed as well to the small scale turbulence through some kind of model as the one of [22]. It is not our purpose here to expound on that point, moreover that there is no decisive experimental data concerning the phase velocity (celerity) of a structure. What seems more important here in the context of Random Point Approximation is to stress the main bifurcation properties of the problem and the consequences and the very fact that a localized solution does exist to it, i.e., the presense of a homoclinic bifurcation is verified numerically.

Although the bifurcation appears even for $\gamma = 0$ (no contribution of the structures to the mean field), it is interesting to compare the shape of the solution with experiment for a given value of γ. We obtained localized solutions for $1 \leq Re \leq 15.37$, which corresponds to the experiments [42] in the interval $0.0005 \leq x \leq 60cm$. In *Fig.*1 is presented the case $\gamma = 0.07$ which roughly corrseponds to

$$Re = 11.1\,,\ x = 15\text{cm}\,,\ f = 0.2425\,,\ \frac{U_0 x_0}{\nu} \approx 7600\,. \tag{4.19}$$

We are able to substantiate the claims concerning the performance of the method proposed. In *Fig.*2 we present the turbulence characteristics. One sees that the agreement for the mean velocity profile (a) and Reynolds stress (b) is quantitatively very good. After selecting the value of γ the result is obtained without fudging any parameters, because there are no other parameters present in the model. The agreement for the fluctuations of the transverse (c) and longitudinal (d) components of velocity is not so good, but one should note the 2D nature of the present model, while in the experiments small-scale 3D fluctuations are also present. Finally, we present also the assymetry (e) and excess (f) of the velocity distributions just to show the type of quantities that can be predicted. For the time being we have not found experimental data concerning these quantities in order to validate our predictions.

6. Concluding remarks

The application of random-point approximation to nonlinear stochastic problems is outlined in the present paper. Basing upon the Volterra-Wiener expansion with Random Point Function as a basis, a new type of hierarchy is derived. The first-order kernel is interpreted as the shape function of the coherent structures, while the higher-order ones represent the double, tripple, etc. interactions among the localized structures. Truncation of the series provides a way for closure which has sound physical meaning – neglecting the interactions of order higher than a prescribed one. Nonlinear systems of equations are derived from the original governing systems for the kernels and solution is sought in the class of localized functions (generalized homoclinics). As a featuring example is considereed the plane mixing layer, where a 2D localized structure can be found numerically. Although approximate, the solution is complete

in statistical sense, since it gives the multi-point characteristics (moments,spectra, cumulants) which is the principal advantage of the proposed technique. The quantitative comparison with the experimental data is very good for the mean profile and Reynolds stress and fair for the "components" of kinetic energy of turbulence.

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MIXING IN GEOPHYSICAL FLOWS
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INTERNAL WAVE GENERATION BY TURBULENT WAKES

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At high Froude and Reynolds numbers, lee waves generated by a body moving horizontally in a stratified fluid give way to 'random' internal waves generated by the turbulent wake. This paper considers the second type of waves, whose presence is attributed to the collapse of vortex loops or turbulent bursts, associated with the coherent vortex structures of the wake. A simple analytical model of the collapse is given, leading to two distinct analyses of the waves. At moderate cross-stream distances the individual effect of each loop or burst is observed; waves have the same structure as for an isolated impulsive point source. At large cross-stream distances the collective effect of the loops or bursts is observed; waves reduce to those generated by a source of mass moving with the body, and emitting fluid at the frequency of vortex shedding and at some of its harmonics. Here, only the fundamental frequency is taken into account, in which case the waves decompose into several systems and are confined within conical caustics. All of this agrees with available experimental results. More attention is finally paid to the shape of surfaces of constant phase.

1. Introduction

Stratified flows of geophysical interest are often characterised by complicated combination of. and strong interaction between, turbulence and internal gravity waves. They are governed by two dimensionless parameters, the Reynolds number Re representing the ratio of inertial forces to viscous forces. and the internal Froude number Fr representing the ratio of inertial forces to buoyancy forces. For instance. for sufficiently low Fr, homogeneous stratified turbulence decreases more rapidly than its unstratified counterpart, as a result of both the buoyant collapse of the large-scale eddies and the subsequent radiation of internal waves (see, e.g., [1]). Similarly, for sufficiently high Re, lee waves, i.e., internal waves generated by horizontal stratified flow past a fixed obstacle or by horizontal motion of a body in a stratified fluid at rest, are superseded by 'random' waves generated by the turbulent wake of this obstacle or body. This paper considers the second aspect, namely internal wave radiation by a three-dimensional turbulent wake.

The first mention of this phenomenon dates back to Lin & Pao [2], who noticed that "[when] turbulence has decayed appreciably, random internal waves appeared near the wake boundary". The existence of these waves was then confirmed, and its origin clarified, by experiments carried out by Gilreath & Brandt [3] for a self-propelled slender spheroid and by Hopfinger *et al.* [4] and Bonneton *et al.* [5] for a towed sphere. In fact, the internal wave field accompanying a horizontally moving body comprises several distinct components: deterministic lee waves generated by the body itself; deterministic waves observed only in the self-propelled case, and generated by the collapse of the wake due to initial mixing and by the propeller swirl; random internal waves generated by the turbulent wake, at Reynolds numbers $Re = 2aU/\nu$ such that this wake is turbulent. Here ν is the kinematic viscosity, N the buoyancy frequency, a the transverse radius