Well-posed Boussinesq paradigm with purely spatial higher-order derivatives

C. I. Christov
Centre for Nonlinear Phenomena and Complex Systems, Université Libre de Bruxelles, Campus Plaine, Case Postale 231, Boulevard du Triomphe, Bruxelles 1050, Belgium

G. A. Maugin
Laboratoire de Modélisation en Mécanique, CNRS URA 229, Université Pierre et Marie Curie, Tour 66, Place Jussieu 4, 75252 Paris, France

M. G. Velarde
Instituto Pliridisciplinar, Universidad Complutense, Paseo Juan XXIII, No. 1, Madrid 28040, Spain
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The derivation of Boussinesq's type of equations is reexamined for the shallow fluid layers and nonlinear atomic chains. It is shown that the linearly stable equation with purely spatial derivatives representing dispersion must be of sixth order. The corresponding conservation and balance laws are derived. The shapes of solitary stationary waves are calculated numerically for different signs of the fourth-order dispersion. The head-on collisions among different solitary waves are investigated by means of a conservative difference scheme and their solitonic properties are established, although the inelasticity of collisions is always present. [S1063-651X(96)00209-7]

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I. INTRODUCTION

The permanent wave was observed by Russell [1,2] around ''Turning Point'' in Union Canal near Edinburgh and further results were obtained in laboratory systematic experimental investigations by Russell and Bazin. Boussinesq [3–5], and later on independently Lord Rayleigh [6], provided the pertinent theoretical description. The importance of this discovery went unnoticed at the beginning although Korteweg and de Vries [7] further developed its understanding. It was only after Zabusky and Kruskal [8] showed the particlelike (''solitonic'') behavior of the localized waves of the Korteweg–de Vries equation (KdV), that the individualized (permanent) wave captured for good the attention of investigators and the study of solitons became an important field of nonlinear physics. Nowadays, the Boussinesq idea that the permanent-wave shapes are the result of an appropriate (local) balance between dispersion and nonlinearity has already become a paradigm. The Boussinesq equations appear not only in the study of the dynamics of thin inviscid layers with free surface but also in the study of the propagation of waves in elastic rods and in the continuum limit of lattice dynamics or coupled electrical circuits. On the other hand the Korteweg–de Vries (KdV) equation served as the prime example of the integrability theory and various properties have since been established for KdV, Boussinesq, and related equations.

Yet, an exhaustive analytical description can be obtained only in certain rather special cases. It takes just the smallest step in the direction of making the model more realistic and the integrability (or at least the analytical form of the solutions) is lost. It is clear that a model or a paradigm can be of practical importance only if its properties are robust, i.e., structurally stable. Then it can be simulated numerically and predictions can be made for large intervals of the variation of the governing parameters. It happened not to be the case with the original equation derived by Boussinesq himself, since it was linearly unstable with respect to short-wave-length disturbances and can be called ''incorrect in the sense of Hadamard'' since the smallest disturbance in the initial conditions results in a significant change in the solution after a finite time. This spurred a significant activity for improving the Boussinesq equation (BE) and nowadays ''good,'' ''improved,'' ''proper,'' etc., Boussinesq equations are known which differ from Boussinesq's derivation. For the sake of clarity we call the equation derived by Boussinesq himself ''Boussinesq's Boussinesq equation'' (BBE). Thus a Boussinesq equation will be a wave equation to which a fourth-order dispersion term and certain nonlinearity are added. ''Boussinesq Paradigm'' refers to this in a broad sense.

A way to make BE mathematically correct is to change the improper sign of the dispersion term of Boussinesq's Boussinesq equation. In fluid dynamics it amounts to considering a very strong surface tension (which hardly corresponds to the case observed by Russell), while in lattices it means an overwhelming presence of long-range interactions (five-point differences) which is never true in reality. This means that the mathematically improper sign of the dispersion coefficient in BBE reflects a deeper physical nature and could not be simply changed without compromising the main assumptions of the model. In our view, the mathematical incorrectness of BBE is due to missing (or badly rearranged) terms, rather than to the physics it was attempting to reflect.

Another approach is to replace the fourth spatial derivative by a mixed spatiotemporal one of the same order. This keeps intact the physical assumptions but then makes the model less amenable to the analytical techniques since the ''improved'' equation is no more fully integrable [9]. It seems important to pursue further the research with purely spatial derivatives representing the dispersion. As we show here, this can be done mathematically correct if at least the sixth spatial derivative is retained when approximating the dispersion.

Here we point out that there exists a physical situation where an equation of Boussinesq's type naturally appears with the proper sign of the dispersion. This is the case of transverse vibrations of nonlinear rods [10–12].

II. LONGITUDINAL VIBRATIONS IN NONLINEAR CHAINS

A. Discrete dynamics

Consider a chain of points of equal masses, connected to each other through (nonlinear) springs. Let us denote by l the lattice constant (the equidistant spacing between the material points in the initial state or ''reference configuration''). We consider here a chain which is a straight line coinciding with the coordinate axis Ox. This conjecture gives a good approximation for any curved one-dimensional (1D) filament whose local radius of curvature is large enough in comparison with the distance l between points. In the nondeformed state the coordinates of points are nl. The longitudinal positions assumed in the deformed state are denoted by $x_0,\ldots,x_n,\ldots,x_N$. It is convenient to also introduce the relative displacements (loosely speaking ''strains'') and the rates of strains

$$u_{n+1}=x_{n+1}-x_n\equiv r_{n+1}, \quad \dot{u}_{n+1}=\dot{x}_{n+1}-\dot{x}_n\equiv\dot{r}_{n+1}, \tag{2.1}$$

where the dots over the variables denote time derivatives. Let us now denote by $\Psi(r_{n+1})$ and $\Psi'(r_{n+1})$, respectively, the potential and the elastic force of interaction between the masses at sites n and $n+1$. If one considers an exponentially nonlinear (Toda) lattice, these are expressed as follows:

$$\Psi(r_{n+1})=\frac{a}{b}\,[\exp(-br_{n+1})-1]+ar_{n+1}, \tag{2.2}$$

$$\Psi'(r_{n+1})=a[1-\exp(-br_{n+1})]. \tag{2.3}$$

If the characteristic length b^{-1} of nonlinearity of the problem is large enough, the springs can experience a large elongation before the nonlinear effects become important. Then in the limit $lb\ll 1$ one can reduce the exponential (Toda's) potential to the following cubic one

$$\Psi(r_{n+1})\approx ab\left(\frac{1}{2}\,r_{n+1}^2-\frac{b}{6}\,r_{n+1}^3\right), \qquad \Psi'(r_{n+1})\approx ab\left(r_{n+1}-\frac{b}{2}\,r_{n+1}^2\right). \tag{2.4}$$

Here the first term gives a harmonic potential with a spring constant $\kappa=ab$. For simplicity and with no lack of generality we constrict the considerations in what follows to the cubic potential (2.4). Yet, the cubic potential is qualitatively different from the exponential one and is inherently improper in the sense that the force which corresponds to it becomes unbounded for large relative displacements. At the time the exponential potential gives a saturation for the force [see (2.3)]. In fact the cubic approximation of the potential is the cause for the nonlinear blowup of the model. Thus the cubic-pentic approximation from [13] appears more appropriate. However, it goes beyond the frame of the present work to investigate the consequences of different approximations for the potential. For the behavior of solitons in the cubic-pentic model we refer the reader to [14].

Newton's law for the mass point of number n reads

$$m\ddot{x}_n=\Psi'(u_{n+1})-\Psi'(u_n), \tag{2.5}$$

or which is the same

$$m\ddot{x}_n=ab(x_{n+1}-2x_n+x_{n-1})+[F(x_{n+1}-x_n) - F(x_n-x_{n-1})], \tag{2.6}$$

where

$$F(x_{n+1}-x_n)\equiv-\frac{ab^2}{2}\,(x_{n+1}-x_n)^2$$

is the nonlinear part of the force.

In a similar fashion the Newton law with the potential of interactions that depends on the relative position of three particles can be derived. It can be noted here that a quadratic potential depending on the three-point difference yields in the equations a linear term proportional to the five-point difference $x_{n-2}-4x_{n-1}+6x_n-4x_{n+1}+x_{n+2}$.

In terms of relative displacements u_i of atoms in a lattice, the governing equation has the following form ([15,16]):

$$\ddot{u}_i=\chi(u_{i+1}-2u_i+u_{i+1})+[F(u_{i+1})-2F(u_i)+F(u_{i-1})], \tag{2.7}$$

where $\chi=ab/m$ is proportional to the square of the characteristic speed in the crystal.

Taking into considerations the triple interactions among the points of the chain (atoms in the lattice) one ends up with an equation also containing the five-point difference [13]

$$\ddot{u}_i=\chi(u_{i+1}-2u_i+u_{i+1})+[F(u_{i+1})-2F(u_i)+F(u_{i-1})] - \delta(u_{i+2}-4u_{i+1}+6u_i-4u_{i-1}+u_{i-2}). \tag{2.8}$$

Here δ controls the triple interactions and the linear stability demands that $\delta>0$, which is the proper sign for a discrete equation of the discussed type.

B. Continuum limit

The most natural way to predict the behavior of a chain seems to be making use of the ''difference'' equation (2.8) as the governing equation and simulating it numerically as it is correct in the sense of Hadamard. The problem is that it is a microscopic equation with l being of the order of intermolecular distances. Hence too many computational points (coinciding with the number of atoms of the chain) will be needed for direct numerical simulations if one is to model even the smallest system of macroscopic relevance. To overcome this difficulty the continuum limit is used, assuming that the relative displacement u is a continuous and smooth enough function whose values in the geometric points representing the material points of the chain are exactly u_n. Then a Taylor-series expansion for the strain in the vicinity of point x_i gives

$$(u_{i+1}-2u_i+u_{i-1})=l^2u_i''+\frac{l^4}{12}u_i^{(4)}+\frac{l^6}{360}u_i^{(6)}+\frac{l^8}{20\,160}u_i^{(8)}+\frac{l^{10}}{1\,814\,400}u_i^{(10)}, \tag{2.9}$$

$$(F_{i+1}-2F_i+F_{i-1})=l^2F_i''+\frac{l^4}{12}F_i^{(4)}+\frac{l^6}{360}F_i^{(6)}+\frac{l^8}{20\,160}F_i^{(8)}+\frac{l^{10}}{1\,814\,400}F_i^{(10)}, \tag{2.10}$$

$$(u_{i+2}-4u_i+6u_i-4u_{i-1}+u_{i-2})=l^4u_i^{(4)}+\frac{l^6}{6}u_i^{(6)}+\frac{l^8}{80}u_i^{(8)}+\frac{17l^{10}}{30\,240}u_i^{(10)} \tag{2.11}$$

and hence

$$u_{tt}=l^2\chi u_{x^2}+l^2\frac{\partial^2}{\partial x^2}F[u(x,t)]+l^4\left(\frac{\chi}{12}-\delta\right)u_{x^4}+l^6\left(\frac{\chi}{360}-\frac{\delta}{6}\right)u_{x^6}+\cdots . \tag{2.12}$$

Now higher-order spatial derivatives appear in the model reflecting more information about the interaction between the atoms. Equations of the type of (2.12) are ''generalized wave equations'' (GWE). The problem is that after the contribution of these new terms is acknowledged the truncation after the fourth derivative does not necessarily give a linearly stable model.

The fourth-order truncation of Eq. (2.12) would be proper only if $\delta>\chi/12$, which is hardly realizable since the multiple interactions are always ''screened'' by the lower-order ones, i.e., actually $\delta\ll\chi$. Then it is the sixth-order truncation which is of practical interest since it is well posed for $\delta<\chi/60$.

One can proceed even further by considering the eighth-order GWE but the condition for correctness of the latter appears to be qualitatively similar to the fourth-order equation with the only difference that the limitation now is not so restrictive, namely, $\delta>\chi/252$, but still well above the practical range of parameters. Then the tenth-order GWE can be considered and it is correct for very large δ but lesser than $17\chi/60$.

It is clear that the increased δ interval for correctness in the case of the tenth-order equation does not pay off the increased complexity added to the model. Thus we shall limit the consideration to the sixth-order GWE which is the *minimal* order that is linearly stable. After rescaling the variables, we arrive at the following equation for the transverse strain:

$$u_{tt}=\gamma^2u_{xx}+\frac{d^2F(u)}{dx^2}+\beta u_{xxxx}+u_{xxxxxx},\quad F(u)\equiv-\frac{dU(u)}{du}. \tag{2.13}$$

Here $U(u)$ is the nonlinear part of the potential. In the cubic case we have: $U(u)=(\alpha/6)u^3$, $\alpha=ab^2$. We call Eq. (2.13) the sixth-order generalized Boussinesq equation (6GBE).

C. Pseudomomentum formulation

Equation (2.13) is a corollary of the system

$$u_t=q_{xx},\quad q_t=\gamma^2u-\frac{dU}{du}+\beta w+w_{xx},\quad w=u_{xx}. \tag{2.14}$$

Different boundary conditions (B.C.) can be imposed. On a finite interval $[-L_1,L_2]$, however, the system (2.14) admits conservation laws, only for the following B.C.:

$$u=0,\quad u_x=0,\quad q_x=0\quad \text{for } x=-L_1,L_2. \tag{2.15}$$

Indeed, consider the quantities

$$M\overset{\text{def}}{=}\int_{-L_1}^{L_2}u\,dx,\quad P\overset{\text{def}}{=}\int_{-L_1}^{L_2}uq_x\,dx. \tag{2.16}$$

$$E\overset{\text{def}}{=}\int_{-L_1}^{L_2}\tfrac{1}{2}[\gamma^2u^2+q_x^2-2U(u)+\beta u_x^2+w^2]dx. \tag{2.17}$$

Upon an appropriate manipulation of (2.14), integrating with respect to x and using the B.C. (2.15), one obtains the following conservation and balance laws (for the fourth-order BE see a similar derivation in [17–19]):

$$\frac{dM}{dt}=0,\quad \frac{dP}{dt}=\frac{1}{2}[u_{xx}^2]\Big|_{-L_1}^{L_2}\equiv F,\quad \frac{dE}{dt}=0. \tag{2.18}$$

Here M can be interpreted as the mass of the wave and E as its energy. [Note that this energy is not a positive definite functional. Hence its conservation does not bound the solution which may well diverge (nonlinear blowup).] Following [20–22] we call P *pseudomomentum*, and F—*pseudoforce*.

The mechanical and field interpretation embodied in Eqs. (2.17), (2.18) which grants to a nonlinear wave process the essential attributes of a ''quasiparticle,'' is made more salient by remarking the following. If one introduces the potential $\overline{u}$ of u by $u=\overline{u}_x$ and assumes that for dynamical solutions of interest $\overline{u}_t(x=-L_1)=0$, it is verified, on account of (2.14), that $\overline{u}_t=q_x$ and thus P and E are none other than the canonical (wave) momentum and energy associated to the Lagrangian

$$L=\int_{-L_1}^{L_2}\mathcal{L}dx,\quad \mathcal{L}=\mathcal{K}-\mathcal{W}, \tag{2.19}$$

with

$$\mathcal{K}=\tfrac{1}{2}\overline{u}_t^2,\quad \mathcal{W}=\tfrac{1}{2}[\gamma^2\overline{u}_x^2-2U(\overline{u}_x)+\beta\overline{u}_{xx}^2+\overline{u}_{xxx}^2], \tag{2.20}$$

with the *canonical* field-theoretical definitions [20,21]

$$P=-\int_{-L_1}^{L_2}\overline{u_x}\frac{\delta\mathcal{L}}{\delta\overline{u_t}}dx=-\int_{-L_1}^{L_2}\overline{u_x u_t}dx, \tag{2.21}$$

$$E=\int_{-L_1}^{L_2}\mathcal{H}dx,\quad \mathcal{H}=\mathcal{P}\overline{u_t}-L,\quad \mathcal{P}=\frac{\delta\mathcal{L}}{\delta\overline{u_t}}, \tag{2.22}$$

where $\delta/\delta\overline{u_t}$ denotes the Euler-Lagrange variational derivative. In this mechanical reinterpretation, it is $\overline{u}$ that is the *displacement* (or basic field) and u that is the *strain* (field gradient). Thus we have a mathematical object pertaining to the class of solitonic systems.

III. INVISCID FLOW IN SHALLOW LAYER: BOUSSINESQ'S APPROACH

In this section we revisit Boussinesq's derivation with the purpose of obtaining a form that may be more useful in some instances, e.g., when showing conservativeness with higher-order derivatives. We derive the general case of two-dimensional (2D) motion in the plane of the layer but restrict ourselves to one spatial dimension in numerical calculations.

Consider the 2D inviscid flow in a thin layer with a free surface. We limit the derivations to the case when the shape function $h(x,y,t)$ of the free surface is single valued, i.e., there is no breaking of the waves. The motion in the bulk is governed by the Laplace equation for the potential Φ.

Let H be the scale for the vertical spatial coordinate and L (the yet undefined wave length) for the horizontal one. We introduce dimensionless variables according to the scheme

$$\Phi=UH\phi,\quad h=H\eta,\quad z=Hz',\quad x=Lx',\quad y=Ly',$$
$$t=HU^{-1}t',$$

where $U=\sqrt{gH}$ is the characteristic scale for the velocity. Henceforth, the primes will be omitted without fear of confusion.

Then the Laplace equation takes the form

$$\beta\Delta\phi+\frac{\partial^2\phi}{\partial z^2}=0, \tag{3.1}$$

where $\beta\equiv H/L$ is the *dispersion* parameter. This is a small parameter for long-length scales of the motion. The kinematic and dynamic conditions then become (the free surface in dimensionless form is $z=1+\eta$)

$$\frac{\partial\eta}{\partial t}+\nabla\phi\cdot\nabla\eta=\frac{1}{\beta}\frac{\partial\phi}{\partial z}, \tag{3.2}$$

and

$$\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2+\frac{1}{2\beta}\left(\frac{\partial\phi}{\partial z}\right)^2+\eta=0. \tag{3.3}$$

Here the unknown function of time that enters the dynamic condition is identified as gH, assuming that in the initial moment of time the system was at rest (i.e., $\Phi\equiv 0>$, $h\equiv 0$ at $t=0$).

Boussinesq expanded the solution of the Laplace equation (3.1) into a power series with respect to β. With the nonflux condition $\partial\phi/\partial z=0$ at the bottom of the layer the power series contains only the even powers of the coordinate z, namely,

$$\phi(x,y,z,t)=\sum_0^\infty(-\beta\Delta)^m f(x,y,t)\frac{z^{2m}}{(2m)!}, \tag{3.4}$$

where $f(x,y,t)=\phi(x,y,z=0,t)$ is the unknown function representing the value of potential at the bottom of the layer. Then for the derivatives entering the surface conditions (3.2), (3.3) one has

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=1+\eta}=\sum_1^\infty(-\beta\Delta)^m f(x,y,t)\frac{(1+\eta)^{2m-1}}{(2m-1)!}, \tag{3.5}$$

$$\left.\frac{\partial\phi}{\partial t}\right|_{z=1+\eta}=\sum_0^\infty(-\beta\Delta)^m \frac{\partial f(x,y,t)}{\partial t}\frac{(1+\eta)^{2m}}{(2m)!}, \tag{3.6}$$

$$\nabla\phi|_{z=1+\eta}=\sum_0^\infty(-\beta\Delta)^m\nabla f(x,y,t)\frac{(1+\eta)^{2m}}{(2m)!}. \tag{3.7}$$

Note that in our 2D case there is no dependence on y, hence $\Delta\equiv\partial^2/\partial x^2$ and $\nabla\equiv\partial/\partial x$.

Introducing these expressions into the system governing the surface motion and keeping within the order of approximation $O(\beta^2)$ one arrives at the following approximate system containing the 1D variables η, f, only:

$$\frac{\partial\eta}{\partial t}+\left[\nabla f-\frac{\beta}{2}\nabla[(1+\eta)^2 f_{xx}]\right]\nabla\eta$$
$$=-(1+\eta)\Delta f+\frac{\beta}{6}(1+\eta)^3\Delta^2 f, \tag{3.8}$$

$$\frac{\partial f}{\partial t}-\frac{\beta}{2}\frac{\partial}{\partial t}[(1+\eta)^2\Delta f]+\frac{1}{2}(\nabla f)^2+\eta$$
$$-\frac{\beta}{2}\nabla f\cdot\nabla[(1+\eta)^2\Delta f]+\frac{\beta}{2}[(1+\eta)\Delta f]^2=0, \tag{3.9}$$

which is the gist of Boussinesq's derivation.

The linearized version of the system for Boussinesq's functions is obtained from (3.8), (3.9) upon neglecting η in comparison with unity and ηf, f^2—in comparison with f. Then the function η is readily excluded to obtain a single equation

$$\frac{\partial^2 f}{\partial t^2}-\frac{\beta}{2}\frac{\partial^2\Delta f}{\partial t^2}=\Delta f-\frac{\beta}{6}\Delta^2 f, \tag{3.10}$$

which is well posed as an initial value problem. Naturally, its energy functional

$$E=\frac{1}{2}\int_{-\infty}^{\infty}\left[f_t^2+(\nabla f)^2+\frac{\beta}{2}(\nabla f_t)^2+\frac{\beta}{6}(\Delta f)^2\right]dx \tag{3.11}$$

is positive definite and it is a conserved quantity due to Eq. (3.10).

In the literature Eq. (3.10) is called the regularized long-wave equation (RLW) (see, [23–25]) suggesting that something had to be regularized in the thin-film equations. RLW is the natural equation that appears in Boussinesq's type of derivation (see. Sec. II C) and curiously enough some effort is needed to ''deregularize'' it making it incorrect.

If an approximation valid only in the moving frame is sought, then following Boussinesq [3–5] one can argue that the time derivatives can be approximated by the spatial ones for motions that evolve slowly in the coordinate frame moving to the right (with unit velocity). Then upon replacing the mixed fourth derivative in Eq. (3.10) by the fourth spatial derivative one obtains

$$\frac{\partial^2 f}{\partial t^2}=\frac{\partial^2 f}{\partial x^2}+\frac{\beta}{3}\frac{\partial^4 f}{\partial x^4} \tag{3.12}$$

which apparently has a mathematically more pleasant form lacking mixed derivatives. However, Eq. (3.12) is unstable to short-length disturbances, as linear stability analysis shows.

Physically speaking this deficiency seems to be of no relevance, because from the very beginning the equation was derived to only account for the long-wave motions. This is indeed the case when one can find an analytical solution (as Boussinesq did). Yet avoiding the short-wave-length instability is crucial when direct numerical simulations are attempted because it can be triggered by the inevitable errors (truncation, round-off, mismatch between analytical initial conditions and finite difference solution for evolution, etc.). Note that here the mixed-derivative expression naturally appears while the purely spatial dispersion is an approximation. It is opposite to the case of nonlinear chains where the mixed-derivative expression is used to regularize the equation.

IV. THE 6GBE

A. Reformulating Boussinesq's approach

Although Boussinesq arrived at an ill-posed problem when replacing the mixed spatiotemporal derivative by the purely spatial fourth derivative, getting rid of the mixed fourth derivative might prove useful in the end. This idea nowadays enjoys a revived actuality in the light of the quest for conservation laws and integrability of the models. So far, attempts to show integrability for models with mixed derivatives have failed [9]. For this reason we reformulate the Boussinesq derivation in a manner to have only spatial higher-order derivatives, while avoiding the trap of ill posedness.

Our approach requires inversion of infinite series and we carry it on in an asymptotic manner up to terms of order of β^3 included.

The simplest way to avoid mixed derivatives is to use the value of the original potential function at the surface (denote it by $\psi(x,y,t)\equiv\phi[x,y,1+\eta(x,t),t]$) rather than the Boussinesq function $f\equiv\phi(x,y,0,t)$ which is the restriction of ϕ to the bottom boundary. We invert the Boussinesq series Eq. (3.4) to express f in terms ψ. To the order $O(\beta^4)$ (the fourth order here secures the third order of the overall procedure) it gives

$$\begin{aligned}f=\psi&+\frac{(1+\eta)^2}{2}\beta\Delta\psi+\beta^2\left[\frac{(1+\eta)^2}{2}\Delta\frac{(1+\eta)^2}{2}\Delta\psi\right.\\&\left.-\frac{(1+\eta)^4}{24}\Delta^2\psi\right]\\&+\beta^3\left[\frac{(1+\eta)^2}{2}\Delta\frac{(1+\eta)^2}{2}\Delta\frac{(1+\eta)^2}{2}\Delta\psi\right.\\&-\frac{(1+\eta)^2}{2}\Delta\frac{(1+\eta)^4}{24}\Delta^2\psi-\frac{(1+\eta)^4}{24}\Delta^2\frac{(1+\eta)^2}{2}\Delta\psi\\&\left.+\frac{(1+\eta)^6}{720}\Delta^3\psi\right]+O(\beta^4).\end{aligned} \tag{4.1}$$

And upon introducing the last formula into expression (3.5) we get

$$\begin{aligned}\frac{1}{\beta}\left.\frac{\partial\phi}{\partial z}\right|_{z=1+\eta}=&-(1+\eta)\Delta\psi-\beta\left\{(1+\eta)\Delta\left[\frac{(1+\eta)^2}{2}\Delta\psi\right]\right.\\&\left.-\frac{(1+\eta)^3}{6}\Delta^2\psi\right\}\\&-\beta\left\{(1+\eta)\Delta\frac{(1+\eta)^2}{2}\Delta\frac{(1+\eta)^2}{2}\Delta\psi\right.\\&-(1+\eta)\Delta\left[\frac{(1+\eta)^4}{24}\Delta^2\psi\right]\\&-\frac{(1+\eta)^3}{6}\Delta^2\left[\frac{(1+\eta)^2}{2}\Delta\psi\right]\\&\left.+\frac{(1+\eta)^5}{120}\Delta^3\psi\right\}+O(\beta^3).\end{aligned} \tag{4.2}$$

$$\begin{aligned}\frac{1}{2\beta}\left[\left.\frac{\partial\phi}{\partial z}\right|_{z=1+\eta}\right]^2=&\frac{\beta}{2}[(1+\eta)\Delta\psi]^2\\&+\beta^2(1+\eta)^2\Delta\psi\left\{\Delta\left[\frac{(1+\eta)^2}{2}\Delta\psi\right]\right.\\&\left.-\frac{(1+\eta)^2}{6}\Delta^2\psi\right\}+O(\beta^3).\end{aligned} \tag{4.3}$$

Introducing (4.2), (4.3) into Eqs. (3.2), (3.3) we arrive at a system asymptotically correct to order $O(\beta^3)$, namely,

$$\begin{aligned}\frac{\partial\eta}{\partial t}+\nabla\psi\cdot\nabla\eta=&-(1+\eta)\Delta\psi-\beta\left\{(1+\eta)\Delta\left[\frac{(1+\eta)^2}{2}\Delta\psi\right]\right.\\&\left.-\frac{(1+\eta)^3}{6}\Delta^2\psi\right\}\\&-\beta^2\left\{(1+\eta)\Delta\frac{(1+\eta)^2}{2}\Delta\frac{(1+\eta)^2}{2}\Delta\psi\right.\end{aligned}$$

$$-(1+\eta)\Delta\left[\frac{(1+\eta)^4}{24}\Delta^2\psi\right]$$
$$-\frac{(1+\eta)^3}{6}\Delta^2\left[\frac{(1+\eta)^2}{2}\Delta\psi\right]$$
$$+\frac{(1+\eta)^5}{120}\Delta^3\psi\Bigg\}+O(\beta^3), \qquad (4.4)$$

$$\frac{\partial\psi}{\partial t}+\frac{1}{2}(\nabla\psi)^2+\frac{\beta}{2}[(1+\eta)\Delta\psi]^2$$
$$+\beta^2(1+\eta)^2\Delta\psi\left\{\Delta\left[\frac{(1+\eta)^2}{2}\Delta\psi\right]-\frac{(1+\eta)^2}{6}\Delta^2\psi\right\}$$
$$+O(\beta^3)=-\eta, \qquad (4.5)$$

which is complicated enough while being an approximate model due to the very fact of employing the Boussinesq series (3.4). It seems reasonable to simplify (although asymptotically inconsistently) the system and to retain only the terms responsible for introducing the qualitatively new effects like the linear stability. For instance, the leading nonlinear terms could not be neglected, as well as the leading dispersion terms, while their modifications of relative order $O(\beta)$ can be either neglected or reduced to simpler terms. This kind of heuristic but not so arbitrary reduction is called ''paradigmatic reduction'' to distinguish it from other asymptotically inconsistent reductions. Note that a true long-wave-length solution can exist for the Boussinesq system only if it is also weakly nonlinear ([26,36]). As far as Boussinesq seches are concerned, this is the case when the celerities are very close to the characteristic velocity of the system [36] (unity in the particular dimensionless form considered here). Thus in the process of reduction we envisage quantitative applications to shallow-layer flows only for the case $\psi,\eta\sim O(\beta)$. Yet, we obtain a system for investigating the ''quasiparticle'' behavior of the localized nonlinear waves.

Accordingly, we set

$$\frac{1}{\beta}\frac{\partial\phi}{\partial z}\bigg|_{z=1+\eta}=-(1+\eta)\Delta\psi-\frac{\beta}{3}\Delta^2\psi-\frac{2\beta^2}{15}\Delta^3\psi,$$

$$\frac{1}{2\beta}\left[\frac{\partial\phi}{\partial z}\bigg|_{z=1+\eta}\right]^2=0+O(\beta),$$

where the sixth spatial derivative of ψ is kept although it contributes to the order β^2 at the time when all other terms of the same order were neglected. As argued in Sec. III, the only way to have a linearly stable system is to keep this term. It is even more inconsistent with the nonlinear term where, in fact, all the terms have been neglected. However, this allows to derive a conservation law for energy, i.e., the simplification is ''paradigmaticly consistent.'' Finally, we obtain

$$\frac{\partial\eta}{\partial t}+\frac{\partial}{\partial x}\left(\eta\frac{\partial\psi}{\partial x}\right)=-\frac{\partial^2\psi}{\partial x^2}-\frac{\beta}{3}\frac{\partial^4\psi}{\partial x^4}-\frac{2\beta^2}{15}\frac{\partial^6\psi}{\partial x^6}, \quad (4.6)$$

$$\frac{\partial\psi}{\partial t}+\frac{1}{2}\left(\frac{\partial\psi}{\partial x}\right)^2=-\eta. \qquad (4.7)$$

Hereafter we neglect the dependence on the variable y and consider only 1D case. Thus we arrive at a system which we shall call ''sixth-order classical Boussinesq system'' (6CBS). A similar coinage was used in [28] for a system to which (4.6) is reduced if the sixth derivative is neglected, which was linearly unstable.

The system (4.6), (4.7) can be reformulated by introducing the auxiliary variables $q_x\equiv-\eta$ and $u\equiv\psi_x$, the latter being simply the x component of the velocity at the surface. Then

$$\frac{\partial q}{\partial t}+u\frac{\partial q}{\partial x}=u+\frac{\beta}{3}\frac{\partial^2 u}{\partial x^2}+\frac{2\beta^2}{15}\frac{\partial^4 u}{\partial x^4}, \qquad (4.8)$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}=\frac{\partial^2 q}{\partial x^2}, \qquad (4.9)$$

where it is already integrated twice with respect to x using $q_x=\eta=0$ for $x=-L_1$.

B. Conserved quantities for 6CBS

Before turning to the integral characteristics we specify the asymptotic conditions at both infinities. Since we are concerned here with solitary waves, we set

$$\eta\equiv-q_x\to\eta_\pm \quad \text{and} \quad u\equiv\psi_x\to 0 \quad \text{for} \quad x\to\pm\infty. \qquad (4.10)$$

These are asymptotic conditions that also imply decay of all derivatives, namely [2],

$$\eta_x,\eta_{xx},\ldots\to 0 \quad \text{and} \quad \psi_{xx},\psi_{xxx},\psi_{xxxx},\psi_{xxxxx},\ldots$$
$$\to 0 \quad \text{for} \quad x\to\pm\infty. \qquad (4.11)$$

(In fact B.C. for η are not needed since no spatial derivatives of η are present in the system. Due to the system one has $\eta\to\psi_t$ for $x\to\pm\infty$.)

When a finite interval is considered, it does matter which of the B.C. (4.11) are imposed. For some B.C. one can have conservation of mass and energy in a finite interval. The conserving B.C. that are compatible with the physical B.C. (4.10) are the following:

$$\psi_x=\psi_{3x}=\psi_{5x}=0 \quad \text{for} \quad x=-L_1,L_2, \qquad (4.12)$$

where $-L_1$ and L_2 are the boundaries of the spatial interval under consideration.

The energy functional of system (4.6), (4.7) reads

$$E=\frac{1}{2}\int_{-L_1}^{L_2}\left[\eta^2+\eta\psi_x^2+\psi_x^2-\frac{\beta}{3}\psi_{xx}^2+\frac{2\beta^2}{15}\psi_{xxx}^2\right]dx. \qquad (4.13)$$

Note that the linear stability can be inferred either directly from (4.6) or from (4.13) since for any wave number k, the form $k^2-\beta k^4/3+2\beta^2k^6/15$ is positive definite. However, the possibility of nonlinear blowup (see, e.g., [29,30]) remains due to the presence of the cubic term $\eta\psi_x^2$ which may happen not to be positive for certain transients.

Here one can see the difference between the fluid layer and the nonlinear chain. In the lattice the possibility of nonlinear blowup was introduced by the inadequate cubic approximation of the potential. In the fluid layer, the nondefiniteness of the energy functional is inherent, because of the presence of free surface. The lack of positive definiteness reflects the fact that Eulerian coordinates are used in which the well known phenomenon of steepening of the surface waves due to nonlinearity cannot be followed beyond the instant in which the surface shape function η becomes double valued.

Now the *pseudomomentum* can be defined as

$$P \equiv \int_{-L_1}^{L_2} \eta \psi_x dx, \tag{4.14}$$

and the balance law for it can be derived as follows. Equation (4.6) is multiplied by ψ_x; the x derivative of Eq. (4.7) is multiplied by η; the results are added to each other and integrated in the interval $[-L_1, L_2]$. Then we get

$$\begin{aligned}\int_{-L_1}^{L_2} (\eta_t \psi_x + \psi_{xt}\eta) dx &= \int_{-L_1}^{L_2} [-\psi_x(\eta\psi_x)_x - \eta\eta_x \\ &\quad + \psi_x \psi_{xx}\eta] dx \\ &\quad - \int_{-L_1}^{L_2} \left[\psi_{xx} + \frac{\beta}{3}\psi_{4x} \right. \\ &\quad \left. + \frac{2\beta}{15}\psi_{6x} \right] \psi_x dx \\ &= -\frac{1}{2}\int_{-L_1}^{L_2} \frac{d}{dx}\left[2\eta\psi_x^2 + \eta^2 + \psi_x^2 \right. \\ &\quad \left. - \frac{\beta}{3}\psi_{xx}^2 + \frac{2\beta}{15}\psi_{3x}^2 \right] dx\end{aligned}$$

or which is the same

$$\begin{aligned}\frac{dP}{dt} = F &\equiv -\frac{1}{2}\left[\eta^2\psi_x^2 + 2\eta\psi_x^2 - \frac{\beta}{3}\psi_{xx}^2 + \frac{2\beta^2}{15}\psi_{3x}^2 + \eta^2 \right]_{-L_1}^{L_2} \\ &= \frac{1}{2}\left[\frac{\beta}{3}\psi_{xx}^2 - \eta^2 \right]_{-L_1}^{L_2},\end{aligned} \tag{4.15}$$

where F is called *pseudoforce* and the last equality acknowledges the conserving B.C. (4.12).

In terms of functions u,q, similar expressions are valid

$$P \equiv -\int_{-L_1}^{L_2} u q_x dx, \quad \frac{dP}{dt} = F \equiv \left[\frac{\beta}{3} u_x^2 - \frac{1}{2} q_x^2 \right]_{-L_1}^{L_2}, \tag{4.16}$$

$$E = \frac{1}{2}\int_{-L_1}^{L_2} \left[q_x^2 - q_x u^2 + u^2 - \frac{\beta}{3} u_x^2 + \frac{2\beta^2}{15} u_{xx}^2 \right] dx. \tag{4.17}$$

Due to B.C. (4.12) the only source for *pseudoforce* could be the differences $\eta_-^2 - \eta_+^2$ of the fluid levels which drives the unsteady waves. Only when this difference vanishes, are the stationary propagating waves possible.

Like system (2.14), systems (4.6), (4.7) or (4.8), (4.9) may be reinterpreted in a field-theoretic framework by noting that total energy (4.13), on account of (4.10), is associated with the following Lagrangian:

$$L = \int_{-L_1}^{L_2} \mathcal{L}\, dx, \quad \mathcal{L} = \mathcal{K}(\psi_t, \psi_x) - \mathcal{W}(\psi_x, \psi_{xx}, \psi_{3x}), \tag{4.18}$$

where

$$\begin{aligned}\mathcal{K} = \frac{1}{2}(\psi_t^2 + \psi_x^2 \psi_t), \quad \mathcal{W} = \frac{1}{2}\left[\psi_x^2 - \frac{1}{4}\psi_x^4 - \frac{\beta}{3}\psi_{xx}^2 \right. \\ \left. + \frac{2\beta}{15}\psi_{3x}^2 \right].\end{aligned} \tag{4.19}$$

The general definition (2.21), and appropriate boundary conditions then yield the balance of *wave momentum* [21]

$$\begin{aligned}P^w &= -\int_{-L_1}^{L_2} \psi_x \frac{\delta\mathcal{L}}{\delta\psi_t} dx = -\int_{-L_1}^{L_2} \psi_x\left(\psi_t + \frac{1}{2}\psi_x^2 \right) dx \\ &= \int_{-L_1}^{L_2} \psi_x \eta\, dx.\end{aligned} \tag{4.20}$$

While the Euler-Lagrange equation derived from (4.18) by straightforward variation just yields the variant of (4.6) obtained by taking (4.7) into account, the canonical quantity (4.20) is the first of (4.14) or (4.16). Furthermore, while the Lagrangian (4.18) contains a term linear in ψ_t, this is not the case of the associated Hamiltonian density $\mathcal{H} = \psi_t(\partial\mathcal{L}/\partial\psi_t) - \mathcal{L}$ [this is the integrand in expression (4.17)]. Thus we have succeeded in reinterpreting our fluid-mechanics problem as a *field-theoretical construct*, namely, the *one-dimensional elastic crystal* endowed not only with *nonlinearity* and *dispersion*, but also with a Lagrangian contribution of the so-called ‘‘*gyroscopic*’’ type, that does not contribute to the total energy while altering the final expression of canonical momentum (this happens in spin systems such as in ferromagnetics).

C. Sixth-order corrections to Boussinesq's Boussinesq equation

It is instructive to add here the equation in the form obtained by Boussinesq himself. For this reason, following Boussinesq we neglect the nonlinear term in (4.9) as introducing ‘‘too much nonlinearity’’ (see, also, [28]). Boussinesq's conjecture was indeed physically sound and asymptotically correct since the long-wave assumption goes together with the weakly nonlinear assumption. The second Boussinesq conjecture was that in the nonlinear term of (4.8) one can replace q_x by $-u$ as in the right-moving frame, the derivative u_t can be replaced approximately by $-u_x$ in (4.9) and to integrate the latter once with respect to x (this integration is needed because we use here the auxiliary function q).

Apart from rendering the model linearly unstable, the Boussinesq manipulations as a ‘‘byproduct’’ destroy also the Galilean invariance. Thus the difference between the Eulerian and Lagrangian descriptions disappears and then the longitudinal coordinate can be thought of as a material coordinate in the reference configuration [21]. We have already demonstrated in a previous section that the pseudomomentum formulation of BBE coincides with the field-theoretical approach.

Finally, the sixth-order Boussinesq's Boussinesq Equation (6BBE) adopts the form

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left[u+u^2+\frac{\beta}{3}\frac{\partial^2 u}{\partial x^2}+\frac{2\beta^2}{15}\frac{\partial^4 u}{\partial x^4}\right]. \quad (4.21)$$

If we disregard the sixth derivative, we have exactly the equation derived by Boussinesq himself (with the coefficient of the nonlinear term rescaled)

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2}{\partial x^2}\left[u+u^2+\frac{\beta}{3}\frac{\partial^2 u}{\partial x^2}\right]. \quad (4.22)$$

This is an equation that contains both the nonlinearity and approximate dispersion (fourth-order derivative). This was the main contribution of Boussinesq. He also found an analytical solution of his equation which in terms of the variables used here reads

$$\eta=-u=\frac{3}{2}(c^2-1)\operatorname{sech}^2\left[\frac{x-ct}{2}\left(3\frac{c^2-1}{\beta}\right)^{1/2}\right], \quad (4.23)$$

and represents a *hump* over the surface propagating with phase velocity c. Thus Boussinesq put a full stop on the discussion of whether a wave of permanent form is possible. Explicitly obtaining the shape of the permanent form he vindicated John Scott Russell whose discovery of the solitary wave [1,2] was rejected by Airy [31] who did not consider the appropriate contribution of dispersion. Independently of Boussinesq, Lord Rayleigh [6] also found the permanent wave. This point was further strengthened by Korteweg and de Vries [7] who besides the Boussinesq sech found another permanent wave—the cnoidal one—consisting of a periodic train of crests. In the linear limit it gives the harmonic wave, while for significant nonlinearity it is a train of shapes similar to sech, but we had to wait 50 years before it was understood [32] (see, also [33] for an illuminating discussion on the cnoidal wave as ''imbrication'' of sech solitons). A historical account can be found in [34–36], among others.

The dispersion is weak (of order of the small parameter β) and hence the coefficient of the nonlinear term must also be small (i.e., $\alpha\sim\beta$) in order to have a balance between dispersion and nonlinearity while both being weak. At the time, the famous Boussinesq sech solution (4.23) is formally valid for all values of the parameters, which is a typical feature of a paradigmatic derivation. The weakly nonlinear long-length-scale solution is recovered only for phase velocities (celerities) c very close to the characteristic speed (unity in our notation). Then, indeed, the solution evolves slowly in the moving frame, at least for overtaking interactions of seches. This means that the physical validity of Boussinesq's Boussinesq equation is not wider than the validity of KdV equation, which is much simpler mathematically being merely an evolution equation in the moving frame. Formally one can solve one of the ''improved'' versions of Boussinesq equation for head-on interactions of seches but the result must be appreciated mostly qualitatively rather than quantitatively. The results of numerical simulations of head-on collisions in RLW exhibit considerable inelasticity (see [37,38]). The inelastic behavior was confirmed also by the calculations with the conservative scheme [36] which makes us believe that it is an innate property of the RLW rather than an artifact of the numerics.

Finally, upon rescaling the variables of Eq. (4.21), the latter is recast in the form (2.13) which will be henceforth referred to as the sixth-order Boussinesq equation (6GBE). In what follows we turn to the numerical investigation of 6GBE.

V. THE STATIONARY SHAPES

First we begin with the equation for the shapes that are stationary in the moving frame $\xi=x-ct$. Denoting by primes the derivatives with respect to the variable ξ we arrive at the following ordinary differential equation (ODE) for the stationary shapes

$$0=\lambda u+\alpha u^2+\beta u''+u'''', \quad \lambda=(\gamma^2-c^2), \quad (5.1)$$

which is the same ODE to which the fifth-order KdV (FKdV) is reduced for solutions in the moving frame (see, [39,49,58,35,60]. Apparently, the first numerical study of (5.1) is due to the Kawahara [39], which is the reason why some authors call the oscillatory solutions of (5.1) ''Kawahara solitons.''

Before embarking on numerical investigations we recall here the results of the linear analysis of the tails of the solitary waves. The linearized equation possesses harmonic solutions of the type $e^{k\xi}$. The dispersion relation for these waves reads

$$k^4+\beta k^2+\lambda=0\Rightarrow k=\pm\left(\frac{-\beta\pm\sqrt{\beta^2-4\lambda}}{2}\right)^{1/2}. \quad (5.2)$$

For the sake of definiteness let us set $|\beta|=1$, $\gamma=1$. The other cases can be obtained by rescaling the variables.

A. Monotone shapes

For negative dispersion $\beta=-1$ and subsonic celerities $\lambda\equiv 1-c^2>0$ one has

$$k_{1,2}=\pm\left(\frac{-\beta+\sqrt{\beta^2-4|\lambda|}}{2}\right)^{1/2},$$

$$k_{3,4}=\pm i\left(\frac{\beta+\sqrt{\beta^2-4|\lambda|}}{2}\right)^{1/2}, \quad (5.3)$$

and two cases can be distinguished. In the first case $c>\sqrt{1-0.25\beta}\approx 0.866$ and we get two pairs of real roots. In this case an analytical solution of (5.1) can also be found [40] in the ubiquitous sech shape

$$u=\frac{105}{169}\frac{\beta^2}{2\alpha}\operatorname{sech}^4\left(\frac{x}{2}\left[\frac{-\beta}{13}\right]^{1/2}\right), \quad |c|=\left(\gamma^2-\frac{36}{169}\beta^2\right)^{1/2}, \quad (5.4)$$

where c is the phase velocity or *celerity* of the wave. Note that this is the subsonic case, hence $c<\gamma=1$. The difference

TABLE I. Checking the algorithm for stationary shapes and comparison with the analytical solution: $c=0.887\,12$, $x\in[-80,80]$.

	analytical	$h=0.1$	$h=0.2$	$h=0.4$
amplitude	0.310 651	0.310 661	0.310 689	0.310 801
difference	0	0.000 010	0.000 038	0.000 150

with the fourth-order Boussinesq and the classical KdV equations is that the analytical solution of sech type (5.4) exists only for a single value of celerity. For the selected parameters $\gamma=1$ and $\beta=-1$, the celerity of the analytical solution is $c\approx0.887\,12$ which falls in the range $c>0.866$.

We start the numerical experiments with this case. The difference scheme is given in the Appendix. The difference solution we obtain for this particular value of celerity virtually coincides with the analytical solution. Table I shows the maximal amplitude of the solutions obtained as a function of the spacing h of the scheme. It is seen that the deviation from the analytical solution is indeed of order $O(h^2)$, which is the accuracy of the scheme. It is no surprise that even the roughest mesh $h=0.4$ gives very good accuracy of 0.015%, since the solution is very smooth.

It is hard to believe that (5.4) presents the only value of celerity for which a solution of monotone shape is possible, i.e., that the spectrum of the nonlinear eigenvalue problem under considerations is discrete, and consists only of a single value. What can indeed be unique is the analytical representation of the solution. So we treated numerically the whole range of subsonic celerities $c<\gamma=1$ and obtained solutions to (5.1) for a continuous spectrum of celerities c. Similarly to the case of the fourth-order proper Boussinesq equation (see, [36] for details), the subsonic humps have larger amplitude when they are slower. In Fig. 1 the numerically obtained shapes of the sechlike solutions for different celerities are presented. The amplitude and the independent coordinate are scaled by

$$\frac{105\alpha(1-c^2)}{72} \quad \text{and} \quad \left(\frac{1-c^2}{24}\right)^{1/2},$$

respectively. Thus the normalized analytic solution has unit amplitude and approximately unit space support. The figure shows that there is a deviation from the sech shape.

For $c\leq\sqrt{0.75}\approx0.866$ a complex conjugate pair of roots appears and the localized waves do have oscillatory damped tails but of extremely low amplitude so that the fact that the shapes are not strictly monotone cannot be discerned on the graphs with normal scales for the variables. In order to show that we present in Fig. 2 two successive zooms to illustrate this statement. This means that the real pair of roots dominates the behavior of the solution, as far as the steady propagating waves are concerned. Note that for smaller β one can find more intense oscillations of the outskirts of the ''monotone'' shapes.

B. Damped oscillatory shapes (Kawahara solitons)

The algorithm developed is applied next to the case of positive fourth-order dispersion $\beta=1$. The difference here is that the oscillatory shapes become stable in the iterative process while the humplike shapes disappear. The upper graph in Fig. 3 shows the result for different celerities where the amplitudes are scaled by the Lorentzian factor $\sqrt{0.75-c^2}$ and the abscissa is not scaled. We have obtained the shapes found by Kawahara for the continuous spectrum $0<c<c_0=\sqrt{0.75}$.

An important feature of the case with positive fourth-order dispersion is that the stationary shapes form bound states. Depending on the amount of initial energy put into the initial condition, the algorithm goes to different solutions. The solutions containing more than one hump can be considered as bound states of solitons, i.e., wave trains of humps separated by different distances between the main peaks. The lower graph in Fig. 3 depicts the bound state of two solitons

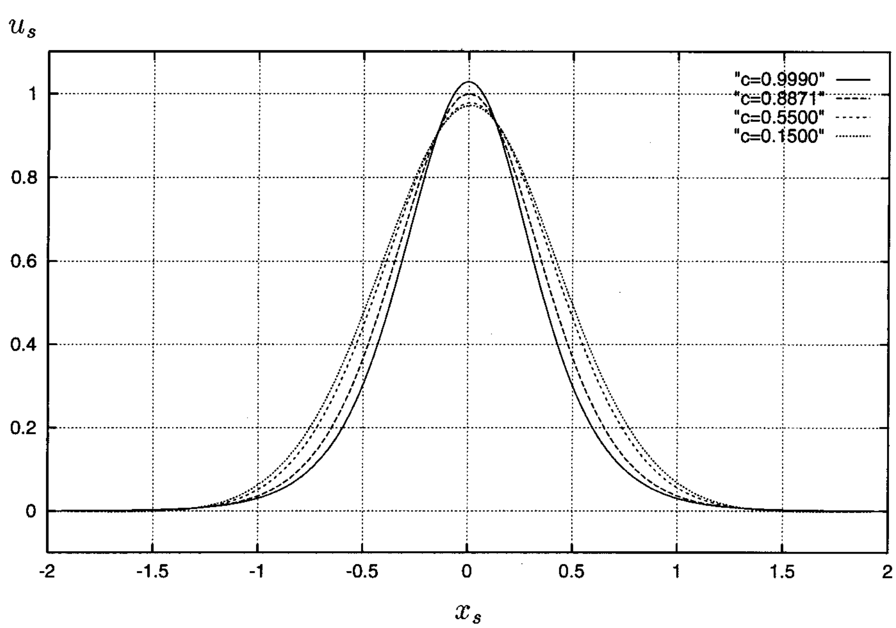

FIG. 1. Normalized $u_s=[72/105\alpha(1-c^2)]u$, $x_s=x(24/1-c^2)^{1/2}$ subsonic humplike shapes for $\beta=-1$ and different c.

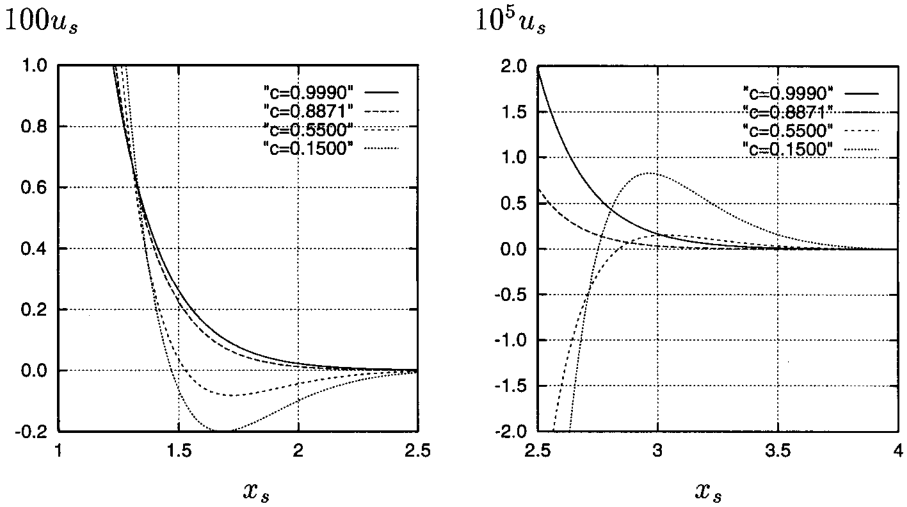

FIG. 2. Zooms of Fig. 1.

with the shortest distance between the peaks and it exists in our numerical calculations for a continuous spectrum of celerity, i.e., the bound states are valid ''quasiparticles.'' Once again the amplitude is scaled by the Lorentzian factor which in this case is $\sqrt{0.75-c^2}$. We have also found bound states with different separations of the centers of the solitons. However, to make a full taxonomy of those goes well beyond the scope of the present paper. Bound states were briefly considered also in [47] where they were called multisolitons. We believe that using such a terminology might prove misleading because multisoliton solutions (or N-soliton solutions) are called dynamical structures in which the different humps (solitons) may move with different celerities. In the bound states discussed here the soliton train moves as a rigid body, i.e., the bound state appears as a ''frozen pattern''). Unfortunately, Ref. [47] is very brief and it is not possible to make quantitative comparison with the results reported there.

The four-soliton bound state presented in Fig. 4 deserves a comment: two of the solitons form a tight bound state like a ''nucleus'' and the other two appear as two electrons in their lower orbit.

Needless to say, the results reported in the present subsection have been verified with the same scrutiny as in the preceding subsection, namely, the mesh size, the ''actual infinity,'' and initial conditions have been varied and their optimal values carefully chosen.

C. Weakly nonlocal solitons

In the supersonic case $\lambda<0$ and regardless of the sign of β the dispersion relation (5.2) has two pairs of roots (a real pair and a pair of imaginary and conjugate roots). The presence of the real pair of roots might give some expectation that a localized humplike shape could be possible, while the imaginary pair suggests that the solutions cannot be strictly localized but rather acquire oscillatory nondecaying tails. This case was thoroughly investigated by Boyd [48,33,49] who found solutions with a main hump and small wings extending to infinity. He called them *weakly nonlocal solutions* or ''nanopterons'' because of their wings. He also showed that imbrication of nanopterons can form a periodic wave which he called *nanopteroidal wave*. When the amplitude of the wings is of the same order as the amplitude of the main hump (strongly nonlocal solutions) they can only be found numerically. However, proceeding to the realm of strongly nonlocal solutions needs some more systematic investigation again beyond the scope of the present paper.

VI. PSEUDO-LORENTZIAN KINEMATICS OF ''QUASIPARTICLES''

As we are interested in 6GBE from the point of view of its field-theoretical interpretation and the ''quasiparticles'' we compile in this section the results for the three conserved quantities characterizing the localized wave-quasi-particle, namely, the *mass*, *energy*, and *pseudomomentum*.

The kinematics of ''quasiparticles'' of 6GBE is dominated by what can be called *pseudo*-Lorentzian (in a sense *anti*-Lorentzian) character. In the ''real'' Lorentzian dynamics the mass and momentum of a particle increase with the increase of velocity and eventually become infinite at the characteristic speed c_0 (*speed of light* in the case of transverse vibrations or *speed of sound* for the case of longitudinal vibrations). Because of their subsonic nature, the localized waves of 6GBE have amplitudes that decrease with the increase of the phase velocity (*celerity*) c and eventually decay to zero at the characteristic speed c_0. Yet their kinematics resemble the Lorentzian in the sense that the factor

$$\gamma=\left(1-\frac{c^2}{c_0^2}\right)^{1/2},$$

enters the picture. Contrary to Lorentzian dynamics, it enters the formulas with positive powers, because for $c\to c_0$ all of the quantities must decay to zero. This anti-Lorentzian behavior appears to be characteristic of all of the different equations from the Boussinesq paradigm which contain only spatial derivatives for the dispersion and are at the same time linearly stable. In this section we perform systematic computations so as to obtain an extensive set of data for the *mass*, *pseudomomentum*, and *energy* of the stationary solitary waves.

A. The monotone sech-like shapes

In Fig. 5 the point-wise (i.e., for set of c's) numerical results for the mass, energy, and pseudomomentum are pre-

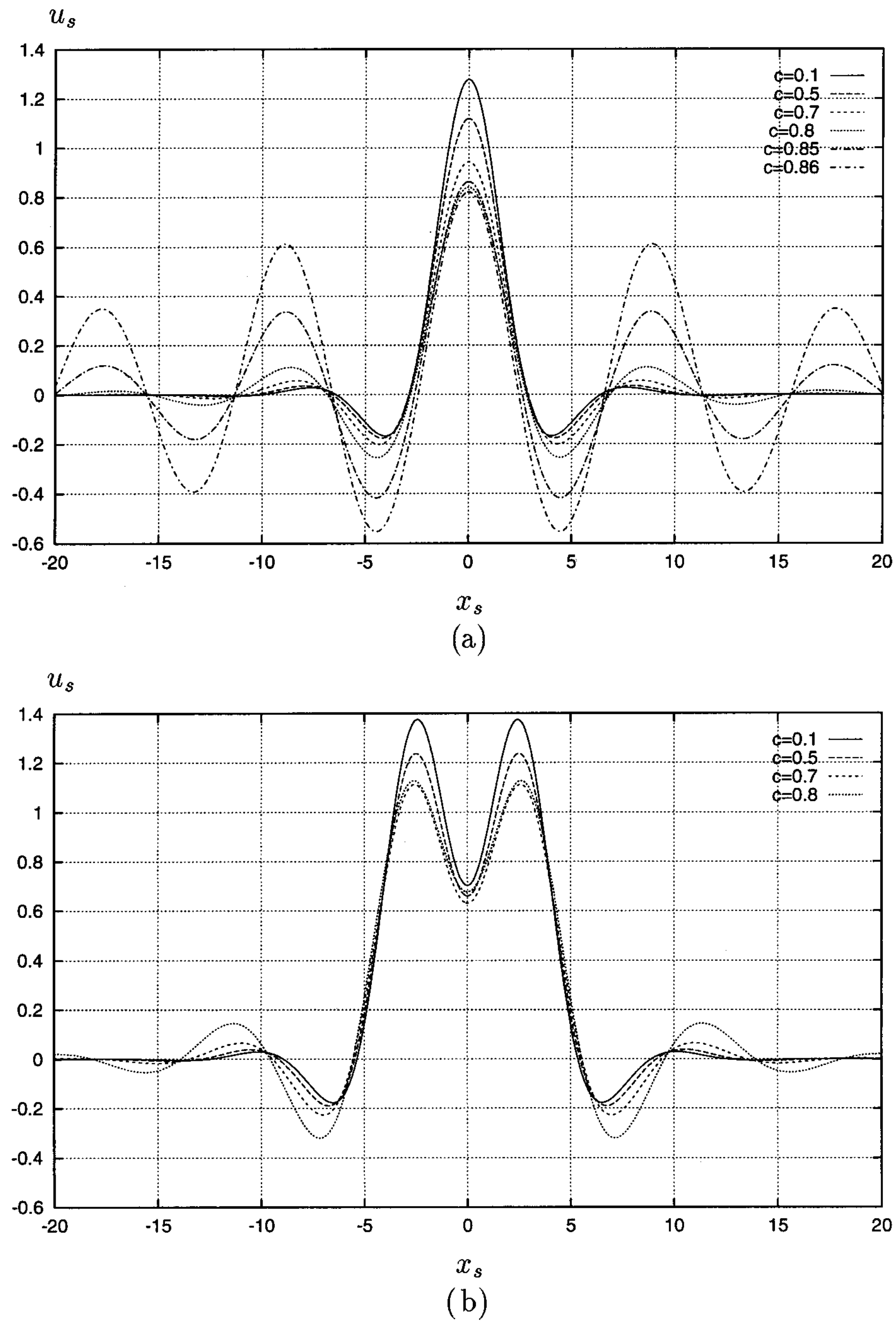

FIG. 3. Normalized subsonic oscillatory shapes for positive dispersion $\beta=1$ and different celerities. (a) Kawahara solitons; (b) two-hump bound states.

sented. One sees that the three quantities do decay to zero for $c \rightarrow c_0 = 1$.

Since there are no analytic expressions for the mass, pseudomomentum, and energy we look for the best approximation containing the powers of celerity c and the ''Lorentzian'' factor γ. Some preliminary experience with deriving analytical expressions for the pseudomomentum for other solitonic system teaches us that the expressions need not be necessarily limited only to powers of γ, but may rather contain also some transcendental functions (such as arctangent). Exhausting all the combinations with different functions is impossible and for that reason we take the usual route in the best-fit approaches resorting only to powers of the independent variables. We further restrict ourselves taking only powers of γ that are integer multiples of 1/2, 1/3, or 1/4. This reduces somewhat the flexibility of the approximation. The best fit obtained under these constraints (smooth curves in Fig. 5) is

$$M = M_0 \gamma^{5/4}, \quad M_0 = 7.4, \tag{6.1}$$

$$P = M c \gamma^2 = M_0 c \gamma^{13/4}. \tag{6.2}$$

The agreement is quite good and justifies the choice for powers of γ. We attempted some best fit approximations for the energy too, but due to the nonconvexity of the latter the

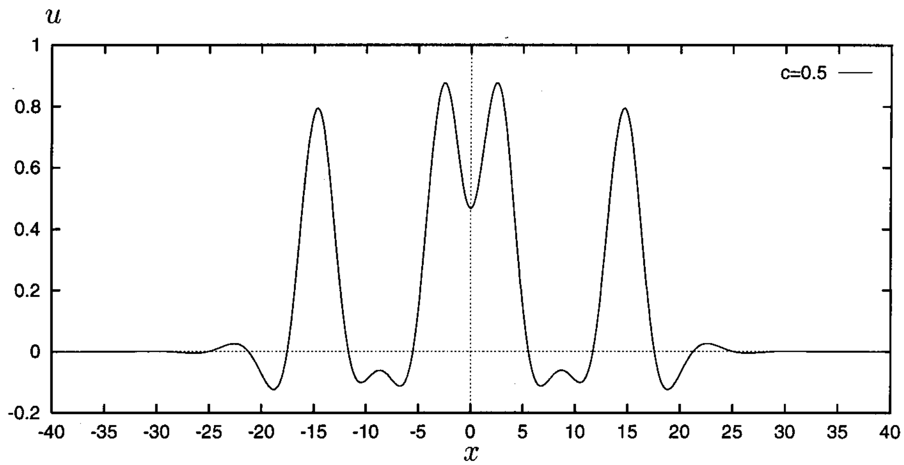

FIG. 4. A bound state of four Kawahara solitons.

number of possible different combinations of powers of c, γ, M, and P increases to such an extent that renders impossible the task to choose one expression over another because quantitatively they fit equally well the data from numerical experiments.

In the limiting case of slow celerities $c \ll c_0(\gamma \ll 1)$, as far as the mass and pseudomomentum are concerned, the dynamics of monotone shapes of 6GBE appears to be Newtonian, namely, $M \approx M_0$, $P \approx Mc$.

B. Kawahara solitons

The shapes of the subsonic (or subluminous) solitary waves of 6GBE can transform to damped oscillatory ones when changing the coefficient β of the fourth-order dispersion. Increasing β one reaches a threshold above which the localized waves acquire oscillatory tails (called Kawahara solitons). The said threshold is usually a negative value, so that if one takes $\beta > 0$ the shapes will be Kawahara solitons for the whole range of admissible subluminous celerities. So here we report the case $\beta = 1$. There is a major difference between this case and the previous one. Now the existence of the quasiparticles is not limited by the characteristic speed of the equation, but rather it is $c_0^2 = 0.75$, $c_0 \approx 0.866$.

Performing the calculations in the interval $c < c_0 \approx 0.866$ we obtain the numerical data for the quantities under consideration. These data are presented in Fig. 6 with solid lines. Once again we found a best-fit approximation guided by the above described considerations. The result (dashed lines in Fig. 6) is

$$M = M_0 \gamma^{7/4}, \quad M_0 = 2.986, \tag{6.3}$$

$$P = Mc\gamma^{5/4} = M_0 c \gamma^{12/4} \equiv M_0 c \gamma^3. \tag{6.4}$$

Here also, the selected type of approximation secures quantitatively very good results for the best fit.

There are some differences in the powers of γ between the two cases considered here. Yet, the general behavior is similar. One is to expect different behaviors from a complex system when one changes the sign of one of the dispersion coefficients. In Kawahara's case the two dispersions act against each other and this can explain the different shapes (damped and oscillatory) for the solitary waves and hence the different powers of γ in the expressions for the mass and pseudomomentum. The Newtonian limit is $M \approx M_0$, $P \approx Mc$.

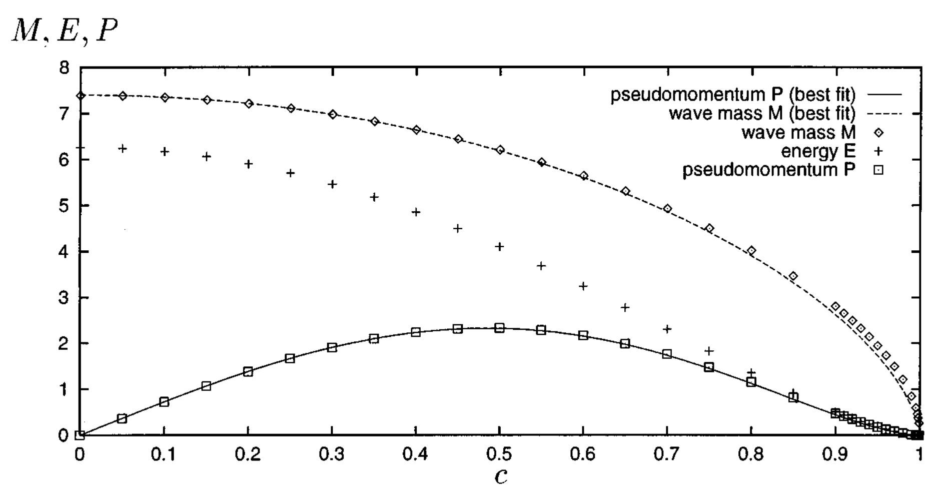

FIG. 5. The *mass*, *pseudomomentum*, and *energy* for sech shapes for negative fourth-order dispersion $\beta = -1$. Symbols: numerical result; lines: best fit approximation (6.2).

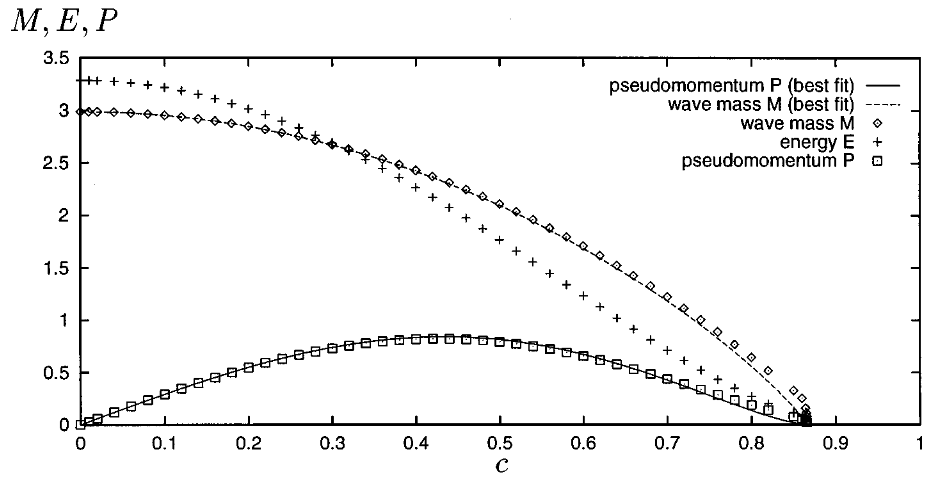

FIG. 6. The *mass*, *pseudomomentum*, and *energy* for Kawahara solitons for positive fourth-order dispersion $\beta=1$. Symbols: numerical result; lines: best fit approximation (6.4).

VII. DYNAMICS OF SOLITONS IN 6GBE

To investigate the dynamics of interactions (collisions) of the ''quasiparticles'' we employ here the conservative scheme developed in [50] as a generalization of the scheme devised for the fourth-order Boussinesq equation in [19] and used for the RLW equation in [36].

Before proceeding with the results some comments on the limitations of the scheme are due. The implicit scheme employed here has a dispersion relation of the following type:

$$4\tau^{-2}\sin^2\frac{\omega\tau}{2}\cos^{-2}\frac{\omega\tau}{2}=2^6h^{-6}\sin^6\frac{kh}{2}, \tag{7.1}$$

where ω is the frequency, k the spatial wave number, τ the time increment, and h the spacing. Here we consider for the sake of simplicity only the sixth-order spatial finite difference. At the same time the dispersion relation of the linear part of 6GBE with only sixth-order derivatives present is $\omega^2=k^6$. It is clear that the dispersion relation (7.1) of the scheme approximates reasonably well the dispersion relation of the differential equation only when $\tau\simeq h^3$ and when, in addition, $\tau\omega$ is small enough in order to replace $\sin 0.5\tau\omega$ by $0.5\tau\omega$. One should note here that the dispersion relation for an explicit scheme is quite similar save the absence of the term $\cos^{-2}(0.5\tau\omega)$. The latter means that when the above requirement for the time increment τ is not fulfilled then the solution for ω is imaginary, i.e., the scheme is linearly unstable. Naturally, the implicit scheme is always stable, but then for $hk\simeq O(1)$ and when $\tau\gg h^3$ (which is the interesting case from the point of view of efficiency of the calculations), then the phase speed of the waves with large k's is grossly misrepresented and exaggerated. This means that the implicit scheme is efficient only for $hk\ll 1$, i.e., the limitation is to have a sufficient number of grid points per wavelength. We have discovered that 20 points per wavelength ($hk\simeq\pi/10$ is the roughest resolution which can be employed without irreparable distortion of the high frequencies of the solution). Rougher meshes would require very small time increments and an intolerable amount of computational time although there are no other limitations of a theoretical nature. We have in fact selected an implicit scheme not because of the computational efficiency, but rather because of the consistent way to implement on the ''difference'' level the conservation and balance laws holding true for the differential equation.

As seen from the numerical results in Sec. V on stationary shapes, the Fourier components with very large wave numbers are of negligible amplitude for the solitary waves under consideration. However, after a collision of two solitons any wave number could be excited at least in a small region. Then due to the unfaithful dispersion relation, if a component of very high wave number appears, it is propagated at higher speed and escapes the region where the predominant part of the energy of the wave system is located (e.g., the collision site in case of multisolitons). This is the only limitation of the implicit scheme employed. In the numerical experiments we did monitor the total energy of the system before and after the collisions and found that the escape energy is negligible.

A. Humps (sech solitons)

First we begin with the case when the fourth-order dispersion term has a negative sign, i.e., the case when the fourth-order Boussinesq equation would have been correct in the sense of Hadamard. This is the case when humps of sech shape can be found. For definiteness we choose $\beta=-1$, $\alpha=1$ and $\gamma=1$. Our first objective is the head-on collision of two sechlike humps (see Fig. 1). The collisions of the analytical seches ($c=0.887\,12$) were already investigated in [50] and their solitonic behavior was confirmed. For $1>c\geq 0.9$ we discovered a practically elastic collision with extremely small transients excited in the site of collision. Respectively, M and E are conserved with an accuracy of 10^{-13}, i.e., within the round-off error of the computer. For the balance law scaled by the maximum of the solution we obtain a quantity of order of 10^{-12}. There are only slight hints of two radiative signals escaping ahead of the main two humps after the collision. The situation sharply deteriorates with decreasing celerity (increasing the amplitude of the subsonic solitons). In Fig. 7 the head-on collision is shown for $c=0.86$ where there appear considerable ''pulses.'' In fact $c=0.86$ was the smallest value for which we obtained a solution. For

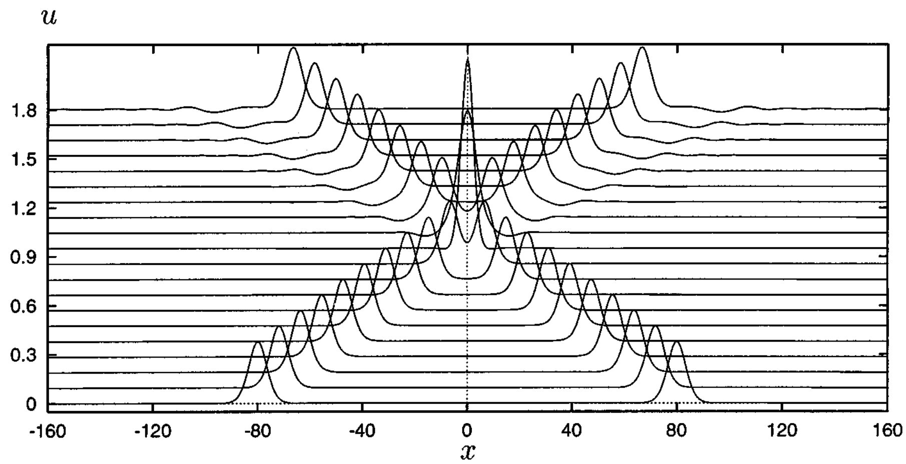

FIG. 7. Negative dispersion $\beta=-1$. Head-on collision of two monotone solitons (humps) with $c_l=-c_r=0.86$. Time from 0 to 186.

$c=0.85$ the nonlinear blowup took place in our calculations (see, [29,30] for definition and theory and [36]—for numerical verification for BE and RLW). The coincidence between the threshold of the nonlinear blowup and the limit of existence of strictly monotone shapes is interesting and awaits its explanation.

B. Pulse formation

We proceed further and investigate the long-time evolution of the transient excited after the collision of hump solitons of 6GBE. For definiteness we consider only the solution in the right-hand side of the interval. We cut the main hump and investigate the evolution of the reminder (a pulse) in the moving coordinate frame. The *mass* of the pulse appears to be of the order of 10^{-4} of the total initial mass and the energy of order of 10^{-5}. In this sense the *mass* and *energy* of the *pulse* are virtually equal to zero. Due to the nondefiniteness of the energy functional, however, its amplitude is allowed to change while the energy remains fixed and that is what happens. It is clearly seen in Fig. 8. The *pulse* broadens with time (it experiences a ''red shift'') and decreases in amplitude, which we call ''Big-Bang'' property. It was observed in [19,36] for the quadratic Boussinesq equation and in [51,52,14]—for the case of cubic-pentic nonlinearity of fourth-order BE. This behavior can also be traced back to the relevant numerical calculations for KdV (see, [53,54,55] and the works referred in [56]). The asymptotic rate of expansion $t^{1/3}$ for KdV pulses was found in [54] (see, also [57]). The same law was confirmed for the Boussinesq equation in the numerical experiments [52].

Now one can investigate the behavior of a pulse as a solitary wave. Since it propagates with the characteristic speed and with virtually zero mass and energy we may call it *pulse photon* to distinguish it from the transient pulses generated by the dissipation in nonconservative systems. In fact, the emission of a pulse in 6GBE is exactly the same process of redistributing of the mass, energy, and pseudomomentum as the splitting of the initial signal into several seches in KdV. Then the question of solitonic nature and the ''quasiparticle'' properties of the pulse photons is raised. To answer this question we take as an initial condition a system of two pulse photons propagating towards each other. We have found that in the course of interaction they pass through each other without qualitatively changing their shapes (save a red shifting) and the mass and energy of the system of pulses are conserved. This suffices to claim that the pulse photons are also solitons.

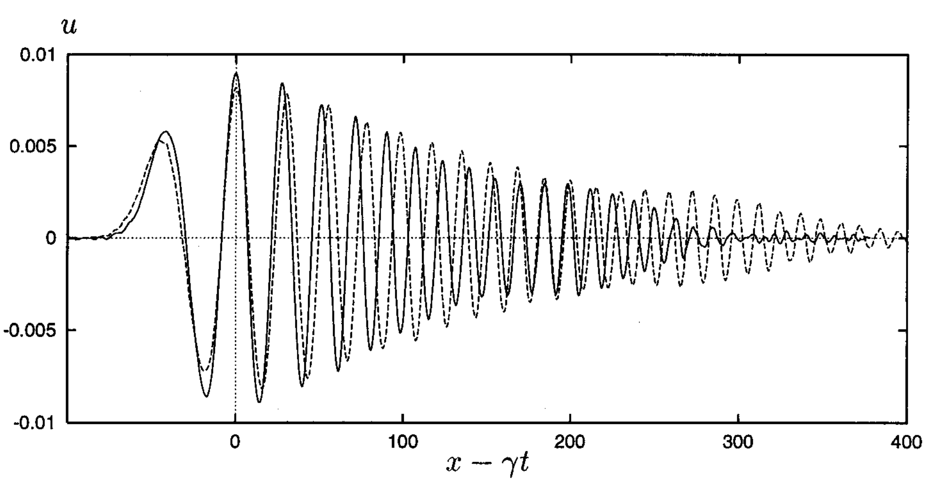

FIG. 8. Long-time evolution of the pulse created after the collision shown in Fig. 7. Solid line: time=400; dashed line: time=600.

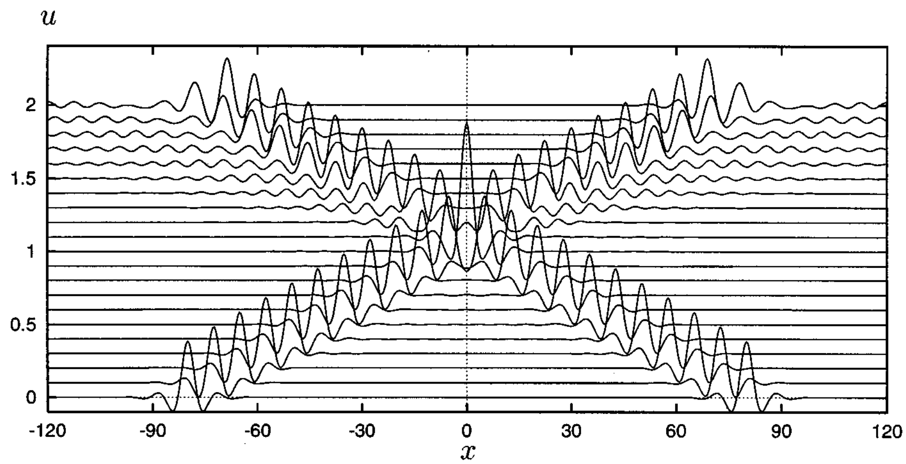

FIG. 9. Positive dispersion $\beta=1$. Head-on collision of two Kawahara solitons with $c_l=-c_r=0.75$. Time from 0 to 200.

The pulse formation has been observed for 6GBE in all collisions of humps while in fourth-order BE, the formation of pulses was observed only for significant enough mismatches between the initial condition and the stationary shapes. This imperfect behavior of the quasiparticles is not related to the conservativeness of the system. In all these calculations the total energy and mass were conserved to the last significant digit of the calculations. For collisionless long-wave-length systems, the sixth derivative did not change the quantitative and qualitative evolution. However, if a short-wave disturbance appears at least once, then the sixth derivative becomes the dominant feature—whence the inelasticity, radiation, and pulse formation. This is always what happens when a system is singularly perturbed (for a discussion about this, see [58]).

Paradoxically enough, the pulses can be suppressed if dissipation is introduced. But then in order to have self-sustained patterns one has to introduce energy input. The dissipation will act to smooth the short-wave-length pulses, while the energy input in the larger scales will sustain the motion preventing its decay. Some steps were already undertaken in this direction in [59] where the KdV-KSV (Kuramoto-Sivashinsky-Velarde) equation was generalized to a wave equation containing energy dissipation and energy production and the coherent structures of the proposed equation were investigated numerically. They turned out to behave as quasiparticles and can be called ''dissipative solitons'' with a proper justification [27].

C. Kawahara solitons

Let us now consider the case of positive fourth-order dispersion $\beta=1$ when the stationary shapes are not monotone.

In Fig. 9 a head-on collision is shown for a large deviation from the characteristic speed. The intuitive expectation here is that the improper sign of the fourth-order dispersion would degrade the overall stability of the process. Contrary to this expectation, the nonlinear blowup was not observed even for $c=0.75$ (compare with the case of proper sign—preceding subsection, when the blowup takes place for $c\leqslant 0.85$). Apparently, the interaction of the monotone shapes produce some unfavorable deformation of the signal, making part of it a signal of zero or negative energy. Consecutively this part of the signal blows up. The nonlinear blowup was observed in our calculations for $c_l=-c_r=0.7$. The threshold is very near that value, because as shown in Fig. 10, for $c_l=0.8$, $c_r=-0.7$ no blowup takes place. In fact, the last figure was produced in our quest for a dynamical creation of quasiparticles of type of resonances (bound states). One sees that the faster solitary wave reemerges from the collision considerably changed in shape resembling rather a bound state of two waves. We did pursue further the calculations in the moving frame of the right-going soliton but the bound state dissolved and finally the whole structure evolved into a pulse, i.e., it did not survive the collision. At the same time the bigger (left going in the figure) solitary wave did preserve its identity after the collision.

In Fig. 11 the evolution of the right-going soliton is followed after it reemerges from the collision shown in Fig. 9. Now the oscillatory soliton eventually recovers its identity and pulses photons of virtually zero energy are emitted ahead of it and propagate with the characteristic velocity. Once again a bound state was not produced by the head-on collision of two Kawahara solitons.

Finally, let us mention that we actually studied an overtaking collision for the case $c_l=0.8$, $c_r=0.7$. Once again a bound state was not found and the smaller soliton did not survive the collision.

VIII. CONCLUSIONS

The derivations of Boussinesq equations in shallow fluid layers and in nonlinear chains have been revisited. It has been shown that the correct truncation of the series representing the dispersion is after the sixth derivative. The nonlinear equation derived is called sixth-order generalized Boussinesq equation (6GBE) for which conservation laws of mass and energy and a balance law for the pseudomomentum are shown to hold.

The stationary propagating localized solutions have been investigated numerically and the two classes of solutions corresponding to the two different signs of the fourth-order dispersion term are obtained: monotone (sechlike) shapes and shapes with oscillatory tails (Kawahara solitons). These two classes are subsonic as they propagate with phase speeds slower than the characteristic speed of equation. The Kawahara solitons can form bound states.

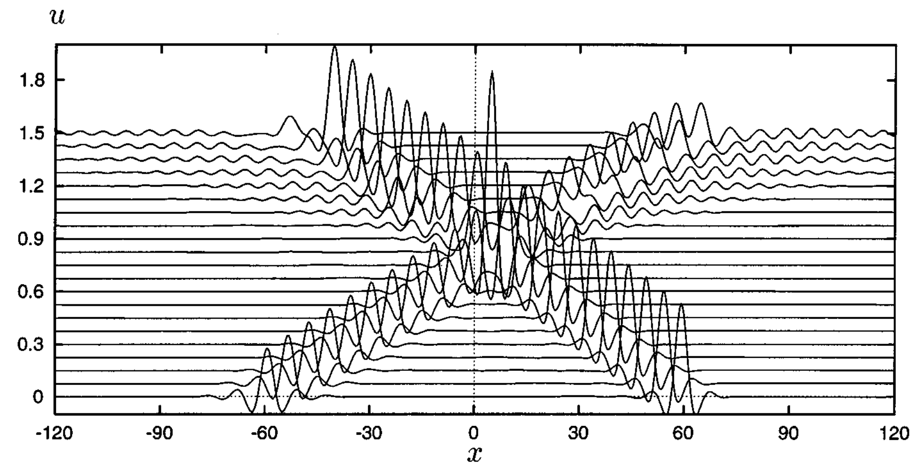

FIG. 10. Positive dispersion $\beta=1$. Head-on collision of two Kawahara solitons with $c_l=0.8$, $c_r=-0.7$. Time from 0 to 150.

The dynamics of collisions of the localized solutions has been investigated numerically by means of a difference scheme that faithfully represents the conservation and balance laws. An important feature of the collisions of solitary waves in 6GBE is inelasticity, manifesting itself in the emission of a faster pulse photon of virtually zero mass and energy which propagates with the characteristic speed. The pulse photon eventually escapes the lagging ''hump'' and the latter practically resumes its original shape, phase speed, mass, and energy. In this sense, the solitary waves of 6GBE can be called solitons, since their behavior upon collision fits well the expected behavior of the quasiparticles of the field governed by the 6GBE equation.

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APPENDIX: DIFFERENCE SCHEME

We divide the interval $x\in[-L_1,L_2]$ into $N-1$ intervals. The grid points are denoted by x_i, $i=1,\ldots,N$. They are equally spaced with spacing $h=(L_2+L_1)/(N-1)$.

Upon introducing the auxiliary function $w=u''$. Eq. (5.1) is recast to a system of two second-order equations. By means of the standard central-difference approximation of the second derivatives and Newton's quasilinearization for the nonlinear term we obtain the following difference scheme for the functions on the ''new'' iterative stage (denoted by superscript $n+1$):

$$\frac{1}{h^2}\,(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})=w_i^{n+1}, \tag{A1}$$

$$\frac{1}{h^2}\,(w_{i+1}^{n+1}-2w_i^{n+1}+w_{i-1}^{n+1})+\beta w_i^{n+1}+2\alpha u_i^n u_i^{n+1}+\lambda u_i^{n+1} = \alpha u_i^{n^2} \tag{A2}$$

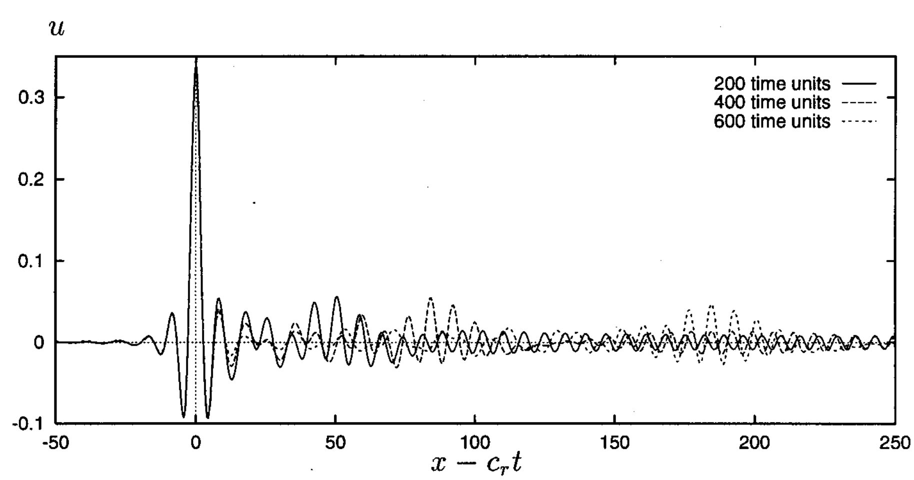

FIG. 11. Long-time evolution of the *pulse photon* created after the collision shown in Fig. 9 ($\beta=-1$, $c_l=c_r=0.75$).

for $i=2,\ldots N-1$ and with B.C.

$$u_1^{n+1}=u_N^{n+1}=w_1^{n+1}=w_N^{n+1}=0.$$

Starting from a certain initial profile u_i^0, w_i^0, the iterations are conducted until convergence is reached. The selection of the initial condition turns out to be very important due to the bifurcation nature of the problem. We consider a localized initial input of triangular shape spanning approximately one-fourth of the total grid points with various amplitudes (the height of triangle). The trivial solution to the problem always exists and when the energy of the initial condition is sufficiently small then the iterative process goes to the trivial attractor. Nontrivial solutions are obtained for sufficiently high energy levels (amplitudes) of the initial profile. Then a typical phenomenon is observed: depending on the initial energy one arrives to one-hump, two-hump, etc., localized solutions, i.e., the one-hump shapes presented here were obtained for quite narrow an interval for the amplitude of initial conditions.

In order to check the performance of the simple scheme implemented here we used also the spectral technique developed in [41–43]. The algorithm developed for the fifth-order Korteweg–de Vries equation (FKdV) [43] has been applied here without major changes save the fact that now a nontrivial term containing the second derivative is present. Limiting the number of terms in the spectral technique we reached point-wise agreement with the difference soliton within 1% from the amplitude of the soliton.

Another check was provided by the method of variational imbedding (MVI) developed in [44] for identifying homoclinic solutions (see, also [45,46]). MVI is a difference technique and if it gives a solution it must coincide with the difference solution obtained here. This has been the case and the two difference solutions agreed within the round-off error of calculations with double precision ($\approx 10^{-11}$).

Note that the inverse nature of the homoclinic problem does not show up for Eq. (5.1) and solutions have been obtained here with a simple scheme without special techniques for inverse problems, like the ones mentioned in MVI. It was not the case, however, with the homoclinic solution of the Lorenz system [44] and the Kuramoto-Sivashinsky equation [45,46] where the inverse nature of the problem of homoclinic identification showed up in a drastic form. The explanation may be that here the problem is of even order (linear part is self-adjoint) while in the mentioned cases the linear part was of odd order (third order). Thus the simple scheme with Newton's quasilinearization turns out to be instrumental in obtaining the numerical solution in the case under study.

[1] J. S. Russell, in *Report of Seventh Meeting (1837) of British Association for the Advancement of Science, Liverpool* (Murray, London, 1838), pp. 417–496.

[2] J. S. Russell, in *Report of Fourteenth Meeting (1844) of the British Association for the Advancement of Science, York* (Murray, London, 1845), pp. 311–390.

[3] J. V. Boussinesq, C.R. Acad. Sci. **72**, 755 (1871).

[4] J. V. Boussinesq, C.R. Acad. Sci. **73**, 256 (1871).

[5] J. V. Boussinesq, J. Math. Pures Appl. **17**, 55 (1872).

[6] Lord Rayleigh, Philos. Mag. **1**, 257 (1876).

[7] D. J. Korteweg and G. de Vries, Philos. Mag. **39(5)**, 422 (1895).

[8] N. J. Zabusky and M. D. Kruskal, Phys. Rev. Lett. **15**, 57 (1965).

[9] F. Calogero and A. Degasperis, *Spectral Transform and Solitons: Tools to Solve and Investigate Nonlinear Evolution Equations* (North-Holland, Amsterdam, 1982), Vol. 1.

[10] P. L. Christiansen, P. S. Lomdahl, and V. Muto, Nonlinearity **4**, 477 (1990).

[11] A. M. Samsonov, in *Frontiers of Nonlinear Acoustics: Proceedings of Twelveth ISNA*, edited by M. F. Hamilton and D. T. Blackstock (Elsevier, London, 1990), p. 583.

[12] P. A. Clarkson, R. J. LeVeque, and R. Saxton, Stud. Appl. Math. **75**, 95 (1986).

[13] G. A. Maugin and S. Cadet, Int. J. Eng. Sci. **29**, 243 (1991).

[14] C. I. Christov and G. A. Maugin, in *Nonlinear Waves in Solids*, edited by J. L. Wegner and F. R. Norwood [American Society of Mechanical Engineers (ASME), New York, 1995)] p. 374.

[15] M. Toda, J. Phys. Soc. Jpn. **22**, 431 (1967).

[16] M. Toda, J. Phys. Soc. Jpn. **23**, 501 (1967).

[17] V. S. Manoranjan, A. R. Mitchel, and J. L. L. Moris, SIAM (Soc. Ind. Appl. Math.) J. Sci. Statist. Comput. **5**, 946 (1984).

[18] V. S. Manoranjan, T. Ortega, and J. M. Sanz-Serna, J. Math. Phys. **29**, 1964 (1988).

[19] C. I. Christov, in *Fluid Physics*, edited by M. G. Velarde and C. I. Christov (World Scientific, Singapore, 1995), p. 403.

[20] G. A. Maugin, J. Mech. Phys. Solids **29**, 1543 (1992).

[21] G. A. Maugin, *Material Inhomogeneities in Elasticity* (Chapman & Hall, London, 1993).

[22] G. A. Maugin and C. Trimarco, Acta Mecanica **94**, 1 (1992).

[23] D. H. Peregrine, J. Fluid Mech. **25**, 321 (1966).

[24] T. B. Benjamin, J. L. Bona, and J. J. Mahony, Philos. Trans. R. Soc. London **A272**, 47 (1972).

[25] G. B. Whitham, *Linear and Nonlinear Waves* (Wiley, New York, 1974).

[26] R. Grimshaw, in *Encyclopedia of Fluid Mechanics*, edited by N. P. Cheremisinoff (Gulf, Houston, 1986), p. 3.

[27] C. I. Christov and M. G. Velarde, Physica D **86**, 323 (1995).

[28] D. J. Kaup, Prog. Theor. Phys. **54**, 396 (1975).

[29] S. K. Turitzyn, Phys. Rev. E **47**, R13 (1993).

[30] S. K. Turitzyn, Phys. Lett. A **143**, 267 (1993).

[31] G. B. Airy, *Tides and Waves*, in *Encyclopaedia Metropolitana* (Taylor, London, 1845), p. 241.

[32] F. Ursell, Proc. Cambridge Philos. Soc. **49**, 685 (1953).

[33] J. P. Boyd, Adv. Appl. Mech. **27**, 1 (1990).

[34] J. W. Miles, J. Fluid Mech. **106**, 131 (1981).

[35] J. K. Hunter and J. Scheurle, Physica D **32**, 253 (1988).

[36] C. I. Christov and M. G. Velarde, Int. J. Bifurcation Chaos **4**, 1095 (1994).

[37] I. L. Bogolubsky, Comp. Phys. Commun. **13**, 149 (1977).

[38] L. Iskander and P. C. Jain, Proc. Indian Acad. Sci. Math. Sci. **54**, 171 (1980).

[39] T. Kawahara, J. Phys. Soc. Jpn. **33**, 260 (1972).

[40] Y. Yamamoto, J. Phys. Soc. Jpn. **58**, 4410 (1989).
[41] C. I. Christov, SIAM (Soc. Ind. Appl. Math.) J. Appl. Math. **42**, 1337 (1982).
[42] C. I. Christov and K. L. Bekyarov, SIAM (Soc. Ind. Appl. Math.) J. Sci. Stat. Comput. **11**, 631 (1990).
[43] K. L. Bekyarov and C. I. Christov, Solitons Fractals **1**, 423 (1992).
[44] C. I. Christov, in *Proceedings of the Fourteenth Spring Conference, Sunny Beach* (Union of Bulg. Mathematicians, Sofia, Bulgaria, 1985), p. 571.
[45] C. I. Christov and M. G. Velarde, Appl. Math. Model. **17**, 311 (1993).
[46] C. I. Christov, in *Fluid Physics*, edited by M. G. Velarde and C. I. Christov (World Scientific, Singapore, 1995), p. 353.
[47] K. A. Gorshkov, L. A. Ostrovsky, V. V. Papko, and A. S. Pikovsky, Phys. Lett. A **74**, 177 (1979).
[48] J. P. Boyd, in *Nonlinear Topics in Ocean Physics*, Proceedings of The International School of Physics ''Enrico Fermi,'' Course XXX, edited by A. R. Osborne and L. Bergamasco (North-Holland, Amsterdam, 1989) p. 527.
[49] J. P. Boyd, Physica D **48**, 129 (1991).
[50] C. I. Christov and G. A. Maugin, in *Proceedings of the Thirteenth International Symposium on Nonlinear Acoustics*, edited by H. Hobaek (World Scientific, Singapore, 1993), p. 457.
[51] C. I. Christov and G. A. Maugin, in *Nonlinear Coherent Structures in Physics and Biology*, edited by M. Peyrard and M. Remoinessenet (Springer, Berlin, 1991), p. 209.
[52] C. I. Christov and G. A. Maugin, J. Comp. Phys. **116**, 39 (1995).
[53] Yu. A. Berezin, *Numerical Investigation of Nonlinear Waves in Rarefied Plasma* (Nauka, Novosibirsk, 1977), in Russian.
[54] B. Fornberg and G. B. Whitham, Philos. Trans. R. Soc. London **289**, 373 (1978).
[55] J. L. Hammack and H. Segur, J. Fluid Mech. **84**, 337 (1978).
[56] P. G. Drazin and R. S. Johnson, *Solitons: an Introduction* (Cambridge University Press, Cambridge, 1987).
[57] A. C. Newell, *Solitons in Mathematics and Physics* (SIAM, Philadelphia, 1985).
[58] Y. Pomeau, A. Ramani, and B. Grammaticos, Physica D **31**, 127 (1988).
[59] C. I. Christov and M. G. Velarde, Phys. Scr. **T55**, 101 (1994).
[60] R. Grimshaw, B. Malomed, and E. Benilov, Physica D **77**, 473 (1994).