Conservative Difference Scheme for Boussinesq Model of Surface Waves

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Abstract

An equation representing the so-called Boussinesq Paradigm is considered. A conservative fully nonlinear scheme is constructed making use of internal iterations. Head-on collisions between different kinds of solitary waves are considered as featuring example.

1 Introduction

After John Scott-Russell discovered the 'great wave' there were different attempts to find its appropriate model. Boussinesq [2] introduced the fundamental idea of balance between the nonlinearity and dispersion and derived the first approximate expression for the dispersion in the case of weakly nonlinear long waves. We call this balance "Boussinesq Paradigm" together with the set of different Boussinesq equations that are derived under the said assumption. They are *generalized wave equations* which offers the opportunity to investigate the generic features of wave systems, such as head–on collisions of localized structures (solitary waves/ quasi-particles). Some of the Boussinesq equations are fully integrable, others possess just three conservation/balance laws: for mass, energy and momentum.

In order to faithfully represent the conservation properties of the differential equation, the difference scheme used for its simulation must also be conservative. In the present paper we outline the way of constructing conservative schemes for the Boussinesq Paradigm and demonstrate their efficiency.

2 Boussinesq Paradigm

Boussinesq attempted to describe the quasi-stationary wave phenomena in the moving frame when some simplifications of the original system are possible. We split Boussinesq's derivation into two steps. The first step is to simplify the convective nonlinear terms arriving at

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u - \frac{dU}{du} + \frac{\beta}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\beta}{6}\frac{\partial^2 u}{\partial x^2}\right], \quad \frac{dU}{du} \equiv \frac{\alpha}{3}u^3 \tag{2.1}$$

which we call Boussinesq Paradigm Equation (BPE).

The second step which we will deliberately omit is to interchange $\partial^2/\partial t^2$ with $\partial^2/\partial x^2$ or vice versa. If one performs the second step of Boussinesq derivation one gets for the dispersion $\frac{\beta}{3}u_{xxxx}$.

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We call the respective equation Boussinesq's Boussinesq Equation (BBE) which is notorious for being linearly instable. The other option is to render the spatial derivative in the brackets to a temporal and to represent the dispersion as $\frac{\beta}{3}u_{ttxx}$. Then one arrives at the so-called Regularized Long–Wave Equation (RLWE) which is linearly stable but not fully integrable. It should be mentioned that the eqn (2.1) appears also in the theory of longitudinal (acoustic) vibrations of rods and lattices (see, e.g., [12]).

Eqn (2.1) can be rewritten as a system

$$u_t = q_{xx} \,, \quad q_t = u - \frac{dU}{du} + \beta_1 u_{tt} - \beta_2 u_{xx} \tag{2.2}$$

where $\beta_1, \beta_2 > 0$ are introduced for the sake of generality.

Consider b.c. at finite interval $x \in [-L_1, L_2]$, namely $u = 0$, $q_x = 0$. Then following [3, 7] we show that

$$\frac{dM}{dt} = 0 \,, \quad M = \int_{-L_1}^{L_2} u dx \tag{2.3}$$

$$\frac{dE}{dt} = 0 \,, \quad E = \frac{1}{2}\int_{-L_1}^{L_2} \left[q_x^2 + u^2 - 2U(u) + \beta_1 u_t^2 + \beta_2 u_x^2\right] dx \tag{2.4}$$

$$\frac{dP}{dt} = \left[\frac{u^2}{2} - \left(u\frac{dU}{du} - U(u)\right) - \frac{\beta_1}{2}u_t^2 - \frac{\beta_2}{2}u_x^2\right]_{-L_1}^{L^2} = -\frac{\beta_2}{2}u_x^2\Big|_{-L_1}^{L^2} \,,$$

$$P = \int_{-L_1}^{L_2} u\,(q_x + \beta_1 u_{xt})\,dx = \int_{-L_1}^{L_2} (uq_x - \beta_1 u_t u_x)\,dx \tag{2.5}$$

The quantities M, E and P are called respectively (wave)*mass*, (wave)*energy* and *pseudomomentum* (wave momentum) (see also the definitions in [10] from the general continuum mechanics point of view).

Thus we see that if the second step of Boussinesq simplifications is not performed and BPE is left in its original form it is preferable because now the wave *mass* is also conserved alongside with the *energy* which is not the case with the RLWE.

3 The Conservative Difference Scheme

Let us introduce a regular mesh in the interval $[-L_1, L_2]$, $x_i = -L_1 + (i-1)h$, $h = (L_1 + L_2)/(N-1)$, where N is the total number of grid points.

It is clear that the strictly conservative scheme is inevitably nonlinear. There exist many different ways to linearize a difference scheme. In our calculations the simplest linearization combined with an inner iteration (referred to by the index k) appeared to be the most robust one. We consider a scheme which is a descendant of the scheme proposed in [3] and used on several different occasions (e.g., [7]), namely,

$$\frac{u_i^{n+1,k} - u_i^n}{\tau} = \frac{q_{i+1}^{n+\frac{1}{2},k} - 2q_i^{n+\frac{1}{2},k} + q_{i-1}^{n+\frac{1}{2},k}}{h^2} \tag{3.6}$$

$$\begin{aligned} \frac{q_i^{n+\frac{1}{2},k} - q_i^{n-\frac{1}{2}}}{\tau} &= \frac{u_i^{n+1,k} + u_i^{n-1}}{2} - \alpha\frac{(u_i^{n+1,k})^2 + u_i^{n+1,k}u_i^{n-1} + (u_i^{n-1})^2}{3} \\ &- \frac{\beta_2}{2}\left[\frac{u_{i+1}^{n+1,k} - 2u_i^{n+1,k} + u_{i-1}^{n+1,k}}{2h^2} + \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{2h^2}\right] \\ &+ \beta_1\frac{u_i^{n+1,k} - 2u_i^{n+1,k} + u_i^{n+1,k}}{\tau^2} \,, \end{aligned} \tag{3.7}$$

with b.c.

$$u_N^{n+1,k} = u_1^{n+1,k} = 0 \ , \quad q_N^{n+\frac{1}{2},k} - q_{N-1}^{n+\frac{1}{2},k} = q_2^{n+\frac{1}{2},k} - q_1^{n+\frac{1}{2},k} = 0 \ . \tag{3.8}$$

The general idea when treating the nonlinear term is to represent it as

$$\frac{U(u^{n+1}) - U(u^{n-1})}{u^{n+1} - u^{n-1}} \tag{3.9}$$

and then to linearize it and to conduct internal iterations. The inner iterations start from the functions obtained on the previous time stage $u_i^{n+1,0} = u_i^n$ and $q_i^{n+\frac{1}{2},0} = q_i^n$, and are terminated at certain $k = K$ when

$$\max |u_i^{n+1,K} - u_i^{n,K-1}| \leq 10^{-13} \max |u_i^{n+1,K}|$$

The value 10^{-13} is selected to be great enough in comparison with the round-off error 10^{-14}. In general, the number of iterations K depends on the size of time increment (in our calculations between six and eight). After the inner iterations converge one obtains, in fact, the solution for the new time stage $n+1$ of the non-linear conservative difference scheme, namely $u_i^{n+1} \stackrel{\text{def}}{=} u_i^{n+1,K}$, $\quad q_i^{n+\frac{1}{2}} \stackrel{\text{def}}{=} q_i^{n+\frac{1}{2},K}$.

From now on we shall not refer more to the internal iterations (hence omitting the composite index k), but rather consider the general properties of the scheme (3.6), (3.7) where the iterations are considered as accomplished.

Generalizing the derivation from [3] we prove that the approximation (3.9) secures the conservation of *energy* on difference level for arbitrary potential $U(u)$, namely the difference approximations of the *mass* and *energy*

$$\begin{aligned} E^{n+\frac{1}{2}} &= \frac{h}{2} \sum_{i=2}^{N-1} \frac{(u_i^{n+1})^2 + (u_i^n)^2}{2} - U(u_i^{n+1}) - U(u_i^n) + \beta_1 \left(\frac{u_i^{n+1} - u_i^n}{\tau} \right)^2 \\ &+ \frac{1}{2h} \sum_{i=1}^{N-1} \frac{\beta_2}{2} \left[\left(u_{i+1}^{n+1} - u_i^{n+1}\right)^2 + \left(u_{i+1}^n - u_i^n\right)^2 \right] + \left(q_{i+1}^{n+\frac{1}{2}} - q_i^{n+\frac{1}{2}} \right)^2 , \\ M^{n+1} &= \sum_{i=2}^{N-1} u_i^{n+1} h \, , \end{aligned}$$

are conserved by the difference scheme (3.7), (3.6) in the sense that $M^{n+1} = M^n$ and $E^{n+\frac{1}{2}} = E^{n-\frac{1}{2}}$. As far as the satisfaction of the conservation and balance laws does not depend on the truncation error, we call it "strict in numerical sense". For comparison we mention the work [9] where the respective laws are satisfied within 10^{-5}, which though very good is not "strict in numerical sense".

The scheme (3.6), (3.7) consists of two conjugated three-diagonal systems. We render them to a single five-diagonal system and to apply the specialized solver for Gaussian elimination with pivoting [6].

4 Numerical Experiments

Our first objective is the head–on collision of two *sechs*. For the Paradigm equation with $U(u) = \alpha u^3/3$ they are given by:

$$u = -\frac{3}{2} \frac{c^2 - 1}{\alpha} \text{sech}^2 \left(\frac{x - ct}{2} \sqrt{\frac{c^2 - 1}{\beta_1 c^2 - \beta_2}} \right) \ , \tag{4.10}$$

for $|c| < \min\{1, \sqrt{\beta_2/\beta_1}\}$ or $|c| > \max\{1, \sqrt{\beta_2/\beta_1}\}$. Here c is the phase velocity or *celerity* of the wave. In the numerical experiments we used $\alpha = -3$, $\beta_1 = 1.5$, $\beta_2 = 0.5$, so that the ratio β_1/β_2

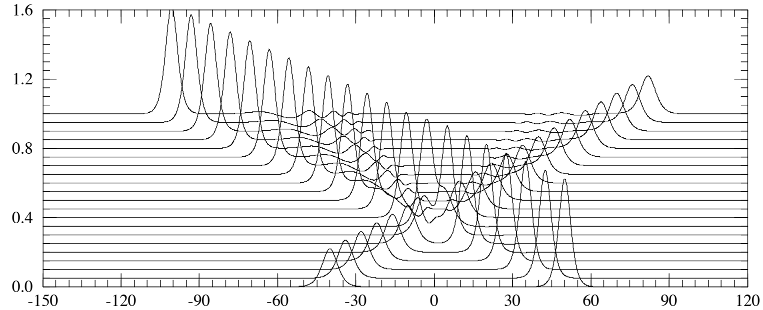

Figure 1: Interaction of two *sech*^{2}-solitons in BPE: $c_1 = 1.2,\ c_2 = -1.5$, The time ranges from 0 to 100. Ordinate refers to the lowest line.

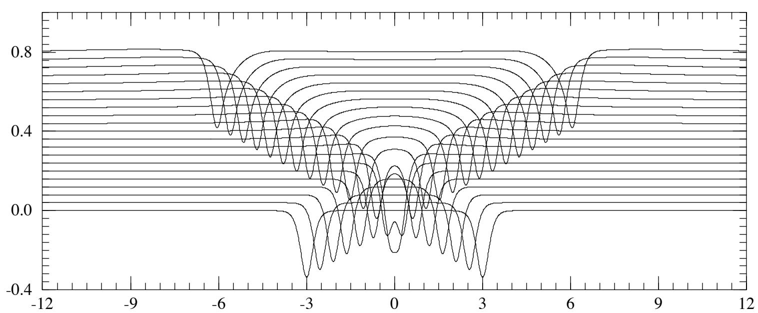

Figure 2: Head-on collision of two 'depressions': $c_1 = -0.57$, $c_2 = -0.57$. Time ranges from 0 to 20.

(the value of, say, β_1 can be re-scaled) and the sigh of the nonlinear term is the as same as for the original Boussinesq case.

We begin with the supersonic case. The interaction of *sech*es is elastic (small phase shift and no excitation of residual signals) only for very small deviations from the characteristic speed, approximately when $0.99\gamma \leq |c| < 1$ which appears to give the quantitative assessment of the validity of long-wave weakly-nonlinear approximation.

Increasing the difference from the characteristic speed increases the inelastic effects. This is an intrinsic property of the equation and not an artifact of the scheme, because the conservativeness of the difference approximation. For instance, the energy of the configuration showed in Fig. 1 kept the constant value $E = 3.9904887039946$ during the evolution. The agreement of numerically observed phase shift with the analytical two-soliton solutions is discussed in [7]. For $c > 2.1$ a nonlinear blow-up (see [13] for definition) took place in our calculations. It is explained by the fact that the energy functional of Boussinesq equations is not positive definite and in for some wave shapes the amplitude can grow while the total energy of the system is conserved.

The *sech* depressions are possible for $c < c_d = 3^{-\frac{1}{2}} \approx 0.57745$ but $\beta_1, \beta_2 \ll 1$ they are not long-wave-length solutions because $|c^2 - 1| \gg \beta_1 c^2 - \beta_2$. In this instance they are not of physical relevance to the surface-wave phenomenon. Yet, it is important to investigate the properties of BPE also in the subsonic case. The threshold of nonlinear blow-up for BE was found [7] to be near $c \approx 0.866$ which is much larger than c_d. Although in a very narrow range $0.575 > c > 0.568$ we did find head-on collisions of depressions in PBE without nonlinear blow-up taking place (see Fig. 2).

We present here also a case which is specific only for eqn (2.1), namely an interaction between a

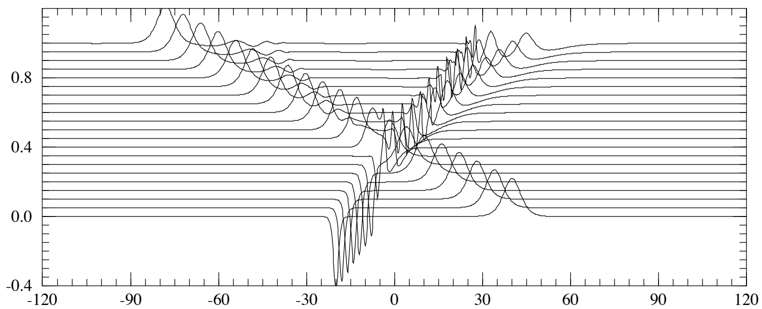

Figure 3: Head-on collision of a 'depression': $c_1 = -0.4$ and elevation $c_2 = 1.2$. Time ranges from 0 to 100.

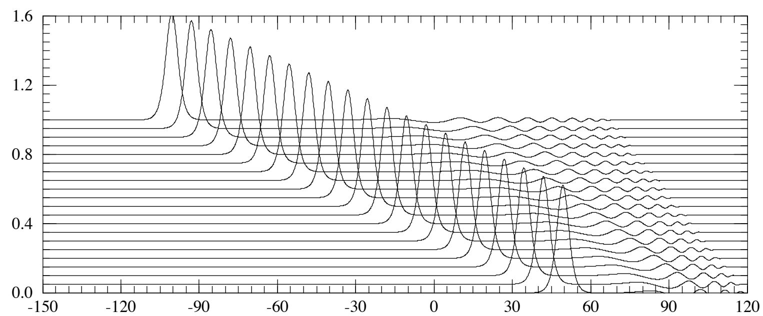

Figure 4: The long-time evolution of the left-going wave system from Fig. 1

depression and elevation (Fig. 3). It is seen that the depression does not survive the collision while the supersonic elevation is split into two humps which eventually become two separate *sech* solitons.

Now we address the problem of formation pulses. Their appearance was observed for KdV as early as in [8, 1]; for BE – in [12] and for RLW —in [9]. The pulses did not attract much attention because of the lack of analytical expression for them. Our calculations confirm that in the wake of collision of two humps oscillatory pulses occur without violation of the conservation of the *mass* and *energy*. The energy of the pulses is approximately 0.002 of the total energy, i.e., they appear to be localized waves of non-zero amplitude, but with virtually zero energy which is another demonstration of the fact that the energy functional of PBE is not positive definite.

It is shown in Fig. 4 that the pulse is not a stationary localized wave. An important feature of a *pulse* is that its amplitude decreases with time while its support increases ("red shift"). This self-similar (we call it also "Big-Bang") behaviour of the pulses was observed in our previous calculations [5, 4] with a non-conservative scheme for the case of cubic-quintic nonlinearity. The self-similar scaling for BE found in [5, 4] is the same as the scaling for self-similar solutions of KdV ([11]). We performed numerical experiments with *pulse*es and discovered that they did preserve their shapes upon collisions, save some "aging" on a time scale longer than the time scale (cross–section) of collision. This allows us to call them "aging solitons".

5 Conclusions

For Boussinesq Paradigm Equation we have constructed conservative nonlinear scheme of second order of approximation in space and time with internal iterations on each time step.

The head–on collisions between the different kinds of solitary waves (depressions and/or elevations) have been investigated and their solitonic properties verified numerically. For phase velocities (*celerities*) close to the characteristic speed, the collisions are virtually elastic save the appearance of a phase shift.

For larger deviations from the characteristic speed a residual signal is also excited in the cite of bygone collision. The conservativeness of the scheme allows us to claim that this is a generic property of the considered class of equations. Depending on the initial energy of the wave system, the residual signal either yields a nonlinear blow-up or transforms into a pulse that is "red-shifted" (spreading in space) and decreases in amplitude during the time.

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