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METHOD OF VARIATIONAL IMBEDDING FOR THE INVERSE PROBLEM OF BOUNDARY-LAYER THICKNESS IDENTIFICATION

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The inverse problem of identification of boundary-layer thickness is replaced by the higher-order boundary value problem for the Euler–Lagrange equations for minimization of the quadratic functional of the original system (Method of Variational Imbedding – MVI). The imbedding problem is correct in the sense of Hadamard and consists of an explicit differential equation for the boundary-layer thickness. The existence and uniqueness of solution of the linearized imbedding problem is demonstrated and a difference scheme of splitting type is proposed for its numerical solution. The performance of the technique is demonstrated for three different boundary-layer problems: the Blasius problem, flow in the vicinity of plane stagnation point and the flow in the leading stagnation point on a circular cylinder. Comparisons with the self-similar solutions where available are quantitatively very good.

1. Introduction

The attention paid to the ill-posed (inverse, incorrect, etc.) problems constantly increases during the last decade because of their practical importance. The optimization of technological processes and identification of material properties yield as a rule mathematical problems in which initial or boundary conditions are missing (or overdetermined) while additional information is available for the supposed solution (or additional unknown functions are present).

At the same time the incorrect problems have great potential for inciting development of the applied mathematics itself. According to Ref. 1: "The analysis of inverse problems is of relevant importance for mathematical modelling and, in general, for applied mathematics. With this in mind, the applied mathematician should attempt the solution of problems without artificial simplification, which may obscure the information he hopes to obtain from the real system."

Naturally, the whole variety of the mentioned "nonstandard" problems goes well beyond the framework of the Hadamard's[10] definition of incorrect problem. His definition does not cover all of them and is pertinent only to stability of a solution. For this reason when we speak of "inverse problems" we mean the whole set of problems which are unusually or inconveniently posed. To distinguish from the problems for which Hadamard's definition applies we shall call the latter "incorrect in the sense of Hadamard". In this instance we shall follow the classification from Ref. 1 shown in Table 1.

Table 1. Classification of mathematical problems (from Ref. 1).

Mathematical model	$\longrightarrow$	Mathematical problem	Well-formulated	Well-posed
				Ill-posed
			Ill-formulated	Ill-specified
				Inverse

The work of Hadamard spurred significant activity for creating regularizing procedures (see, e.g., Ref. 15) for the problems that are incorrect in the sense of Hadamard, e.g., for smoothing the data in order to evade the instability provoked by the pollution of the data. Such an approach has an important implication for the practical problems. At the same time the very notion of replacing the ill-formulated (e.g., ill-specified and inverse) or ill-posed by a well-formulated mathematical problem is of not lesser importance. Indeed, if one succeeds in doing so one arrives at a problem that is also correct in the sense of Hadamard and then it automatically regularizes the data if some pollution is present. To this end the Method of Variational Imbedding (MVI – for brevity) was proposed by the first author. The idea of MVI is to replace an incorrect problem with the well-posed problem for minimization of quadratic functional of the original equations, i.e. we "embed" the original incorrect problem in a higher-order boundary value problem which is well-posed. For the latter a difference scheme and numerical algorithm for its implementation can easily be constructed.

The first application of MVI was for identification of localized solutions (homoclinics of Lorenz system).[2] Later on it was also applied to the more complicated case of homoclinics[7] and heteroclinics[8] of Kuramoto–Sivashinsky equation.

The application to the elliptic problem of analytical continuation was sketched in Ref. 3 and to coefficient identification in parabolic equations – in Ref. 4. Note that the MVI does not require the introduction of higher-order derivatives multiplied by artificial small parameter as it is the case with "quasi-reversibility method".[11,12]

We focus our attention on the problem of coefficient identification in parabolic equation from overdetermined data. This problem falls in the subclass of inverse problems. According to Ref. 1 "an initial–boundary value problem is *inverse* (see

Ref. 1) if some information on the initial and/or boundary conditions needed for solution or/and on the parameters that characterize the model are missing and are replaced by suitable information on the solution of the mathematical problem".

Following Refs. 4 and 5 in our recent work[9] a difference scheme and algorithm are created to apply MVI to the classical problem of identification of heat-conduction coefficient as function of the spatial coordinate from overdetermined boundary data which are functions of time. The significance of this problem is that the sought coefficient is a function of the "elliptic" coordinate (the spatial coordinate) while the data is function of the "parabolic" coordinate (the temporal one). It is interesting to consider also the situation when the sought coefficient is a function of the same "parabolic" coordinate. Such a situation is offered by the boundary-layer model for flow of viscous liquid in the immediate vicinity of a wall. Matching the velocity of the boundary layer to the outer flow is asymptotic and hence the boundary-layer thickness is an artificial quantity introduced for the purposes of numerical treatment. Its identification can be made from the overdetermined boundary conditions the latter stemming from the asymptotic condition. One is to introduce "actual infinity" (or "numerical boundary-layer thickness") where the overdetermined conditions on the normal derivative of velocity are satisfied. As a result, the identification of the boundary-layer thickness appears to be the most sensitive part of the numerical procedures for treating boundary-layer flows. The natural way to derive somewhat more standard boundary value problem is to scale the normal independent coordinate by the unknown shape function of the boundary-layer thickness arriving thus at the problem of coefficient identification for a parabolic equation when overdetermined data is prescribed at the spatial boundaries.

The most straightforward application of MVI for identification of boundary-layer thickness was proposed in Refs. 12 and 13 where the equation for the longitudinal component of velocity is treated separately as a parabolic equation with unknown coefficient and the continuity equation is added in an explicit manner. Such an approach (we call it "local equation for the thickness") offers certain simplicity in implementation because here MVI gives a functional (nondifferential) equation for the thickness. The MVI proved successful, yet the lack of derivatives in the equation for thickness makes it somewhat "stiffer" and hinders the rapid convergence. It is clear that the problem will be much less "stiff" if one constructs the MVI functional taking both the equation of continuity and the parabolic equation for the longitudinal component of velocity. We call this approach "differential equation for the thickness" and to its solution is devoted the present work.

2. Posing the Problem

Consider the boundary layer in two-dimensional viscous incompressible steady flow for high Reynolds number (Re). For bodies whose local curvature is much smaller in comparison with $\sqrt{Re}$ (i.e. plates) the flow is governed by the Prandtl set of equations (see Ref. 14) for a planar boundary layer:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + U\frac{dU}{dx}\,, \tag{2.1}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0\,, \tag{2.2}$$

where x is the longitudinal coordinate (following the surface of body) and y – the normal coordinate. Respectively, $u = u(x, y)$ and $v = v(x, y)$ are longitudinal and normal components of velocity in the boundary layer. The outer flow is parallel to x-axis and $U = U(x)$ stands for the longitudinal component of velocity of outer inviscid flow.

The non-slip boundary conditions at the body surface read:

$$u(x, 0) = v(x, 0) = 0\,, \tag{2.3}$$

and at the outer edge of the boundary layer the asymptotic condition holds

$$u(x, y) \to U(x) \quad \text{for} \quad y \to \infty\,. \tag{2.4}$$

Let X_l and X_r denote the left and the right end of the interval in which the solution is sought. The boundary-layer Eqs. (2.1) and (2.2) require for $x = X_l$ "initial" with respect to the longitudinal coordinate condition on function u (more properly called "inlet" condition):

$$u(X_l, y) = U_l(y)\,. \tag{2.5}$$

Two main cases are distinguished: $U_l(y) \equiv 0$ corresponds to boundary layer in the vicinity of a (planar or cylindrical) stagnation point; $U_l(y) \not\equiv 0$, $U_l(0) = 0$ corresponds to the flow around sharp leading edge (Blasius problem).

2.1. *Why this problem is inverse?*

We define the "numerical boundary-layer thickness" as the function $\delta = \delta(x)$ which is implicitly evaluated so as to satisfy the condition

$$u(x, \delta(x)) = \Delta U(x)\,, \tag{2.6}$$

where $\Delta = \text{const} \approx 1$, $\Delta < 1$. Note that $\Delta \equiv 1$ corresponds to $\delta = \infty$. For any other $\Delta \neq 1$, the normal derivative of the longitudinal component of velocity is not strictly equal to zero at the the rim of the boundary layer $y = \delta(x)$, i.e. the asymptotic b.c. (2.4) is replaced by the following condition at the "actual infinity", namely

$$\frac{\partial u}{\partial y} = \varepsilon, \qquad \text{for} \qquad y = \delta(x)\,, \tag{2.7}$$

where $|\varepsilon| \ll 1$. With the last condition, the b.v.p. becomes overdetermined if the function $\delta(x)$ is considered as known. The gist of the mathematical problem of boundary layer model is that $\delta = \delta(x)$ is unknown and has to be implicitly evaluated from the overdetermined boundary data (2.7).

It is convenient to scale the normal coordinate y and to replace the normal component of velocity $v(x,y)$ by another unknown function $w(x,y)$ as follows:

$$\eta = \frac{y}{\delta(x)} \quad \text{and} \quad w = \frac{v}{\delta(x)} - \frac{\delta'(x)}{\delta(x)}\eta u\,, \tag{2.8}$$

arriving thus at the problem of coefficient identification

$$\delta^2 u\frac{\partial u}{\partial x} + \delta^2 w\frac{\partial u}{\partial \eta} = \frac{\partial^2 u}{\partial \eta^2} + \delta^2 U\frac{dU}{dx} \tag{2.9}$$

$$\frac{\partial \delta u}{\partial x} + \delta\frac{\partial w}{\partial \eta} = 0 \tag{2.10}$$

with boundary conditions

$$u(x,0) = w(x,0) = 0\,, \quad u(x,1) = \Delta U(x)\,, \quad \frac{\partial u(x,1)}{\partial \eta} = \varepsilon\delta(x)\,, \tag{2.11}$$

and "inlet condition"

$$u(X_l,\eta) = U_l(\eta)\,. \tag{2.12}$$

Thus the difficulties connected with the unknown boundaries are circumvented and one arrives at a problem with fixed boundaries. Yet it is a problem of unknown coefficient $\delta(x)$. Under certain natural conditions it is possible to find a function $\delta(x)$ such that the problem (2.9), (2.10) has a unique solution $(u(x,y);v(x,y))$ and this solution also satisfies (2.11) and (2.12). In such a case we say that the functions (u,v,δ) constitute a solution to the problem (2.9)–(2.12). The problem is of an inverse nature and is similar to the problem of identification of heat-conduction coefficient from overdetermined data.[4,9]

3. Variational Imbedding

As above-mentioned, MVI makes use of the Euler–Lagrange equations for minimization of the quadratic functional of the governing equations. For the case under consideration the functional reads

$$\mathcal{J}(u,w,\delta) = \int_{X_l}^{X_r}\int_0^1 \left[\left(\delta^2 u\frac{\partial u}{\partial x} + \delta^2 v\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx}\right)^2 + \left(\delta\frac{\partial \delta u}{\partial x} + \delta^2\frac{\partial w}{\partial \eta}\right)^2\right] d\eta\, dx\,. \tag{3.1}$$

Clearly, the Euler–Lagrange equation for this functional possesses cubic nonlinearity with respect to the functions u and w. It means that for the numerical solution one has to linearize the said equation in order to solve it numerically. Alternatively, one can linearize the integrand in (3.1) considering the functions u and w as known (say, from the previous iteration) when they appear as coefficients, i.e. $u(x,\eta) = q(x,\eta)$ and $w(x,\eta) = r(x,\eta)$. Following the second approach for the linearization we consider the problem of minimization of the following functional

$$\mathcal{J}(u,w,\delta) = \int_{X_l}^{X_r}\int_0^1 \left[\left(\delta^2 q\frac{\partial u}{\partial x} + \delta^2 r\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx}\right)^2 + \left(\delta\frac{\partial \delta u}{\partial x} + \delta^2\frac{\partial w}{\partial \eta}\right)^2\right] d\eta\, dx\,. \tag{3.2}$$

The Euler–Lagrange equation for (u,w,δ) (the necessary condition for minimization of the functional $\mathcal{J}$ with respect to u) is linear with respect to u and v and has the form:

$$\begin{aligned}&\frac{\partial}{\partial x}\delta^2 q\left(\delta^2 q\frac{\partial u}{\partial x} + \delta^2 r\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx}\right) + \delta\frac{\partial}{\partial x}\delta\left(\delta\frac{\partial \delta u}{\partial x} + \delta^2\frac{\partial w}{\partial \eta}\right)\\ &+ \frac{\partial}{\partial \eta}\delta^2 r\left(\delta^2 q\frac{\partial u}{\partial x} + \delta^2 r\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx}\right)\\ &+ \frac{\partial^2}{\partial \eta^2}\left(\delta^2 q\frac{\partial u}{\partial x} + \delta^2 r\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx}\right) = 0\end{aligned} \tag{3.3}$$

with boundary conditions which comprise the original ones (2.11), (2.12) and the natural conditions for minimization of a functional:

$$\left[\delta^2 q\frac{\partial u}{\partial x} + \delta^2 r\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx}\right]_{\eta=0} = \left.\frac{\partial^2 u}{\partial \eta^2}\right|_{\eta=0} = 0\,, \tag{3.4}$$

$$\left[\delta^2 q\delta^2 q\frac{\partial u}{\partial x} + \delta^2 r\frac{\partial u}{\partial \eta} - \frac{\partial^2 u}{\partial^2 \eta} - \delta^2 U\frac{dU}{dx} + \delta^2\delta\frac{\partial \delta u}{\partial x} + \delta^2\frac{\partial w}{\partial \eta}\right]_{x=X_l} = 0\,. \tag{3.5}$$

The Euler–Lagrange equation for function w is also linear with respect to u and v, namely:

$$\frac{\partial}{\partial \eta}\delta^2\left(\delta\frac{\partial \delta u}{\partial x} + \delta^2\frac{\partial w}{\partial \eta}\right) = 0 \tag{3.6}$$

with boundary conditions which comprise the original ones (2.11), and the natural conditions for minimization of a functional:

$$\left[\frac{\partial \delta u}{\partial x} + \delta\frac{\partial w}{\partial \eta}\right]_{\eta=1} = 0\,, \tag{3.7}$$

which after integration with respect to η gives nothing else but the continuity equation (2.10).

Thus we have arrived at an elliptic problem for which the boundary data is not overdetermined, even when $\delta(x)$ is thought of as known – a well-posed problem. The problem which remains is to find function δ for which the minimum of the functional is exactly equal to zero. This way the solution of the original problem will be found. The essence of the MVI in the problem under consideration is the Euler–Lagrange equation for $\delta(x)$, which after fairly obvious manipulations involving the original equations adopts the form:

$$-\frac{d}{dx}\left[\frac{1}{2}A_6(x)s'(x)+A_5(x)s(x)\right] + A_5(x)s'(x)+2[A_4(x)+A_3(x)]s(x)-2A_2=0\,, \tag{3.8}$$

where $s(x)=\delta^2(x)$ and

$$A_2=\int_0^1\left(q\frac{\partial u}{\partial x}+r\frac{\partial u}{\partial \eta}-U\frac{dU}{dx}\right)\frac{\partial^2 u}{\partial x^2}d\eta\,,\qquad A_4=\int_0^1\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial \eta}\right)^2 d\eta\,,$$

$$A_3=\int_0^1\left(q\frac{\partial u}{\partial x}+r\frac{\partial u}{\partial \eta}-U\frac{dU}{dx}\right)^2 d\eta\,,\qquad A_5=\int_0^1\left(\frac{\partial u}{\partial x}+\frac{\partial w}{\partial \eta}\right)d\eta\,,$$

$$A_6=\int_0^1 u^2 d\eta\,.$$

The boundary conditions for function $s(x)$ at $x=X_l$ and $x=X_r$ are the natural conditions for minimization of a functional:

$$\frac{1}{2}A_6(x)s'(x)+A_5(x)s(x)=0\,. \tag{3.9}$$

4. Existence and Uniqueness of Solution

Following Refs. 4 and 13 we prove the existence and uniqueness of a generalized solution of the boundary-value problem (3.3), (2.11), (2.12), (3.4). For the sake of brevity we consider only the case when $U(x)=\text{const}$.

Let us consider now the space $H(D)$ comprised by the pairs of functions (α_1,α_2) which are defined in the rectangular

$$D:\{X_l\le x\le X_r;0\le\eta\le 1\}$$

and satisfy the following boundary conditions

$$\alpha_1(x,0)=\alpha_1''(x,0)=0\,, \tag{4.1}$$

$$\alpha_1(x,1)=\alpha_1'(x,1)=0\,, \tag{4.2}$$

$$\alpha_1(X_l,\eta) = 0\,, \tag{4.3}$$

$$\alpha_2(x,0) = 0\,, \tag{4.4}$$

$$\left[\delta^2 q\frac{\partial\alpha_1}{\partial x} + \delta^2 r\frac{\partial\alpha_1}{\partial\eta} - \frac{\partial^2\alpha_1}{\partial^2\eta} + \delta^2\left(\delta\frac{\partial\delta\alpha_1}{\partial x} + \delta^2\frac{\partial\alpha_2}{\partial\eta}\right)\right]_{x=X_r} = 0\,, \tag{4.5}$$

$$\left[\delta\frac{\partial\delta\alpha_1}{\partial x} + \delta^2\frac{\partial\alpha_2}{\partial\eta}\right]_{\eta=1} = 0\,. \tag{4.6}$$

We expect that the functions under consideration are as many time differentiable as necessary. The following scalar product is introduced in $H(D)$

$$[(\alpha_1,\alpha_2),(\beta_1,\beta_2)] = \int_{X_l}^{X_r}\int_0^1 [L(\alpha_1)L(\beta_1) + P(\alpha_1,\alpha_2)P(\beta_1,\beta_2)]d\eta\, dx\,, \tag{4.7}$$

where $L(\alpha) = \delta^2 q\frac{\partial\alpha}{\partial x} + \delta^2 r\frac{\partial\alpha}{\partial\eta} - \frac{\partial^2\alpha}{\partial\eta^2}$ and $P(\alpha_1,\alpha_2) = \delta\frac{\partial\delta\alpha_1}{\partial x} + \delta^2\frac{\partial\alpha_2}{\partial\eta}$. Here $\delta(x) > 0$ is a function defined for $X_l \le x \le X_r$. The functions $r(x,\eta)$, $q(x,\eta)$ are defined in D such that $r(x,\eta)$ satisfies the boundary condition (2.11), (3.7) and $q(x,\eta)$ – the boundary conditions (2.11), (2.12), (3.4), (3.5). Equation (4.7) is a scalar product since the only solution to the system of linear parabolic equations

$$\delta^2 q\frac{\partial\alpha_1}{\partial x} + \delta^2 r\frac{\partial\alpha_1}{\partial\eta} - \frac{\partial^2\alpha_1}{\partial\eta^2} = 0\,, \tag{4.8}$$

$$\frac{\partial\delta\alpha_1}{\partial x} + \delta\frac{\partial\alpha_2}{\partial\eta} = 0\,, \tag{4.9}$$

with the homogeneous boundary conditions (4.1)–(4.6) is trivial, i.e. $[(\alpha_1,\alpha_2),(\alpha_1,\alpha_2)] = 0$ is true only when $\alpha_1(x,\eta) = \alpha_2(x,\eta) \equiv 0$.

The space $H(D)$ with scalar product (4.7) is a Hilbert space.

Let us introduce the sufficiently times differentiable functions $\chi_1(x,\eta)$ and $\chi_2(x,\eta)$ defined in D and satisfying the relevant boundary conditions (2.11), (2.12). Then, a *generalized solution* of (3.3), (3.6), (2.11), (2.12), (3.4), (3.5), (3.7) is called any pair of functions (u,w) for which the following expression holds true

$$[(u,w),(\Phi_1,\Phi_2)] = \int_{X_l}^{X_r}\int_0^1 [L(u)L(\Phi_1) + P(u,w)P(\Phi_1,\Phi_2)]d\eta\, dx = 0\,, \tag{4.10}$$

where $(\Phi_1,\Phi_2) \in H(D)$ and $((u,w) - (\chi_1,\chi_2)) \in H(D)$. It is easily seen that the classical solution of (3.3), (3.6), (2.11), (2.12), (3.4), (3.5), (3.7) is also a generalized solution since

$$\int_{X_l}^{X_r}\int_0^1 \Phi_1 \frac{\partial}{\partial x}\delta^2 q\left(\delta^2 q\frac{\partial u}{\partial x}+\delta^2 r\frac{\partial u}{\partial \eta}-\frac{\partial^2 u}{\partial^2\eta}\right)d\eta\, dx$$
$$+\int_{X_l}^{X_r}\int_0^1 \Phi_1 \frac{\partial}{\partial \eta}\delta^2 r\left(\delta^2 q\frac{\partial u}{\partial x}+\delta^2 r\frac{\partial u}{\partial \eta}-\frac{\partial^2 u}{\partial^2\eta}\right)d\eta\, dx$$
$$+\int_{X_l}^{X_r}\int_0^1 \Phi_1 \frac{\partial^2}{\partial \eta^2}\left(\delta^2 q\frac{\partial u}{\partial x}+\delta^2 w\frac{\partial u}{\partial \eta}-\frac{\partial^2 u}{\partial^2\eta}\right)d\eta\, dx$$
$$+\int_{X_l}^{X_r}\int_0^1 \Phi_1 \delta\frac{\partial}{\partial x}\delta\left(\delta\frac{\partial \delta u}{\partial x}+\delta^2\frac{\partial w}{\partial \eta}\right)d\eta\, dx$$
$$+\int_{X_l}^{X_r}\int_0^1 \Phi_2 \frac{\partial}{\partial \eta}\delta^2\left(\delta\frac{\partial \delta u}{\partial x}+\delta^2\frac{\partial w}{\partial \eta}\right)d\eta\, dx$$
$$=-\int_{X_l}^{X_r}\int_0^1 \left(\delta^2 q\frac{\partial \Phi_1}{\partial x}+\delta^2 r\frac{\partial \Phi_1}{\partial \eta}-\frac{\partial^2 \Phi_1}{\partial^2\eta}\right)\left(\delta^2 q\frac{\partial u}{\partial x}+\delta^2 r\frac{\partial u}{\partial \eta}-\frac{\partial^2 u}{\partial^2\eta}\right)d\eta\, dx$$
$$-\int_{X_l}^{X_r}\int_0^1 \left(\delta\frac{\partial \delta\Phi_1}{\partial x}+\delta^2\frac{\partial \Phi_2}{\partial \eta}\right)\left(\delta\frac{\partial \delta u}{\partial x}+\delta^2\frac{\partial w}{\partial \eta}\right)d\eta\, dx$$
$$=-[(u,w),(\Phi_1,\Phi_2)]\,, \tag{4.11}$$

where the boundary conditions for (u, w) and $(\Phi_1, \Phi_2) \in H(D)$ are acknowledged.

The existence of a generalized solution follows from the Riesz theorem because, as has been shown above, (4.10) defines a scalar product and therefore a functional.

In order to prove the uniqueness we consider the difference $(\hat{u}, \hat{v}) = (u_1, w_1) - (u_2, w_2)$ between two supposed solutions. It is obvious that $(\hat{u}, \hat{v}) \in H(D)$. On the other hand, Eq. (4.10) holds also for $(\hat{u}, \hat{v})$. Then taking simply $(\Phi_1, \Phi_2) \equiv (\hat{u}, \hat{v})$ we have $[(\hat{u}, \hat{v}), (\hat{u}, \hat{v})] = 0$ and then $(\hat{u}, \hat{v}) \equiv (0, 0)$.

So far, it has been shown that the linearized Euler–Lagrange Eqs. (3.3), (3.6) possess a unique solution under the boundary conditions (2.11), (2.12), (3.4), (3.5), (3.7), provided that $\delta(x) \geq 0$ and $r(x,\eta)$, $q(x,\eta)$ satisfy the appropriate boundary conditions.

Equation (3.8) is a nonlocal equation for $\delta(x)$ in the sense that it contains spatial derivatives of δ. In this instance the present work is a continuation of our previous contributions[12,13] where a simplified approach has been used and the equation for thickness $\delta(x)$ did not contain spatial derivatives. Let us consider now the Hilbert space $H_1[X_l, X_r]$ composed of the functions $\alpha(x)$, which are defined for $X_l \leq x \leq X_r$ and satisfy the following boundary conditions:

$$\frac{1}{2}A_6(x)\alpha'(x) + A_5(x)\alpha(x) = 0 \tag{4.12}$$

for $x = X_l$ and $x = X_r$.

Here and henceforth the functions A_i $(i = 2–6)$ are the coefficients defined in (3.8).

The following scalar product is introduced in $H_1[X_l, X_r]$

$$[(\alpha,\beta)] = \int_{X_l}^{X_r} \left[\frac{1}{2}A_6\alpha'\beta' + A_5(\alpha'\beta + \alpha\beta') + 2(A_3 + A_4)\alpha\beta\right] dx\,. \tag{4.13}$$

Equation (4.13) is a scalar product because $[\alpha, \alpha] = 0$ is possible only when $\alpha \equiv 0$. Indeed,

$$\begin{aligned}[(\alpha,\alpha)] &= \int_{X_l}^{X_r} \left[\frac{1}{2}A_6\alpha'^2 + 2A_5\alpha'\alpha + 2(A_3 + A_4)\alpha_2\right] dx \\ &= 2\int_{X_l}^{X_r} \left[\frac{1}{4}\alpha'^2 \int_0^1 u^2 d\eta + \alpha'\alpha \int_0^1 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial \eta}\right) u\, d\eta \right.\\ &\quad \left. + \alpha^2 \int_0^1 \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial \eta}\right)^2 d\eta\right] dx \\ &= 2\int_{X_l}^{X_r}\int_0^1 \left[\frac{1}{2}\alpha' u + \alpha\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial \eta}\right) u\right]^2 d\eta\, dx \geq 0 \end{aligned} \tag{4.14}$$

and $[\alpha, \alpha] = 0$ is true only when $\alpha(x) \equiv 0$, because the boundary conditions (4.12) are homogeneous.

Then, a *generalized solution* of (3.8), (3.9) is called any functions $s(x)$ for which the following holds

$$[s, \Phi] = 2\int_{X_l}^{X_r} \Phi(x)A_2(x)\, dx\,, \tag{4.15}$$

where $\Phi(x, \eta) \in H_1[X_l, X_r]$. It is easily seen that the classical solution of (3.8), (3.9) is also a generalized solution since

$$\begin{aligned} 0 &= \int_{X_l}^{X_r} \Phi(x)\left\{ -\frac{d}{dx}\left[\frac{1}{2}A_6(x)s'(x) + A_5(x)s(x)\right]\right. \\ &\qquad \left. + [A_5(x)s'(x) + 2[A_4(x) + A_3(x)]s(x) - 2A_2]\right\} dx \\ &= [s, \Phi] - 2\int_{X_l}^{X_r} \Phi(x)A_2(x)dx\,. \end{aligned} \tag{4.16}$$

The last equation is obtained after integration by parts with the boundary conditions for $s(x)$ and $\Phi(x) \in H(D)$, being acknowledged. The existence of a generalized solution follows from the Riesz Theorem. In order to prove the uniqueness we consider the difference

$$(s_1 - s_2) \in H_1[X_l, X_r]$$

between two supposed solutions s_1 and s_2. Then taking $\Phi \equiv (s_1 - s_2)$ we have $[s_1 - s_2, s_1 - s_2] = 0$ and hence $(s_1 - s_2) \equiv 0$.

It has already been shown that each of Eqs. (3.3), (3.6) and (3.8) possesses a unique solution, when the other two functions are thought of as known. Hence the system (3.3), (3.6), (3.8) has a unique solution. Thus the functional $\mathcal{J}$ has a stationary point because Eqs. (3.3), (3.6) and (3.8) are necessary conditions for the existing of a stationary point of a functional. On the other hand, the quadratic functional $\mathcal{J}$ is convex and this unique stationary point is the global minimum of the functional.

So far, we have proved the correctness of the linearized problem. The solution of the full nonlinear problem is obtained by means of iterations after replacing (q, r) with the functions (u, v) calculated at the previous iteration.

5. Method of Solution for Imbedding Problem

5.1. *The splitting*

For solving the above fourth-order boundary value problem we employ an economic splitting scheme (see Ref. 16 for general theory) as developed for the problem of coefficient identification in heat-conduction equation in Ref. 5. Let us introduce for the sake of brevity the following notations for the difference approximations of the second- and forth-order operators:

$$\Lambda_{xx}u \sim \frac{\partial}{\partial x}\delta^4 q^2 \frac{\partial u}{\partial x}, \qquad \Lambda_{yyyy}u \sim -\frac{\partial^4 u}{\partial \eta^4},$$

Λ stands for the full operator of Eq. (2.12). Introducing now a fictitious time we break the full time step into two half-time steps according to the scheme of "stabilizing correction" (see Ref. 16):

$$\frac{\hat{u} - u^n}{\Delta\tau} = \Lambda_{xx}(\hat{u} - u^n) + \Lambda u^n, \tag{5.1}$$

$$\frac{u^{n+1} - \hat{u}}{\Delta\tau} = \Lambda_{yyyy}(u^{n+1} - u^n), \tag{5.2}$$

where $\hat{u}$ is a half-time-step variable and u^{n+1} is a full-time-step one.

The natural boundary condition (3.4) at the outlet of the computational domain $x = X_r$ is in fact a nonlocal condition and if one is to use splitting-type method, the said condition can also be split (as suggested in Refs. 3 and 5 and developed in Ref. 12), namely

$$\frac{\hat{u} - u^n}{\Delta\tau} = \Lambda_x\hat{u} + \Lambda_y u^n + \Lambda_{yy}u^n, \tag{5.3}$$

$$\frac{u^{n+1} - \hat{u}}{\Delta\tau} = \Lambda_{yy}(u^{n+1} - u^n), \tag{5.4}$$

where the notations for the respective operators are self-explanatory.

Now we show that the schemes (5.1), (5.2) approximate the full-step scheme for Eq. (3.3). After introducing the fictitious time, Eq. (3.3) adopts the form of a parabolic equation for which an implicit scheme should have the following form:

$$\frac{u^{n+1}-u^n}{\Delta\tau} = \Lambda u^{n+1} . \tag{5.5}$$

To this end we rewrite the fractional-step scheme as follows:

$$(E - \Delta\tau\Lambda_{xx})\hat{u} = (E - \Delta\tau\Lambda_{xx})u^n + \Lambda u^n , \tag{5.6}$$

$$(E - \Delta\tau\Lambda_{yyyy})u^{n+1} = E\hat{u} - \Delta\tau\Lambda_{yyyy}u^n . \tag{5.7}$$

We act on Eq. (5.7) by the operator $(E - \Delta\tau\Lambda_{xx})$ and add the result to (5.6) excluding thus the half-time-step variable $\hat{u}$:

$$(E-\Delta\tau\Lambda_{xx})(E-\Delta\tau\Lambda_{yyyy})u^{n+1} = (E-\Delta\tau\Lambda_{xx})u^n + \Lambda u^n - (E-\Delta\tau\Lambda_{xx})\Delta\tau\Lambda_{yyyy}u^n$$

and hence

$$(E + \Delta\tau^2\Lambda_{xx}\Lambda_{yyyy})\frac{u^{n+1}-u^n}{\Delta\tau} = \Lambda u^{n+1} , \tag{5.8}$$

which is an $O((\Delta\tau)^2)$ approximation of (5.5), i.e. it is within the order of approximation $O(\Delta\tau)$ of the implicit scheme in full steps. It is interesting to note that the splitting scheme is more stable than the scheme in full steps because the operator on the left-hand side has a norm greater than 1.

5.2. *The mesh pattern and difference approximations*

Figure 1 shows the mesh pattern for each sought set function. For Eq. (3.3) we employ a mesh that is staggered in η-direction in order to secure second-order of approximation for the second boundary condition and $\eta = 1$.

The mesh is defined according to the following laws:

$$x_i = (i-1)h_x + X_l , \qquad \text{for } i = 1, 2, \ldots, N_x ,$$
$$\eta_j = (j-2)h_\eta , \qquad \text{for } j = 1, 2, \ldots, N_\eta ,$$

where $h_x = (X_r - X_l)/(N_x - 1)$ is the spacing in x-direction, N_x is the number of points in x-direction, $h_\eta = 1/(N_\eta - 2.5)$ is the spacing in η-direction, N_η is the number of points in η-direction.

The differential operators of second- and fourth-order are approximated with standard central differences on three- or five-point stencils with second-order of approximation. The mixed derivatives are also approximated with central differences of second-order approximation with respect to both spatial coordinates. The first half-step of the boundary condition (3.4) is approximated with first-order with respect to x.

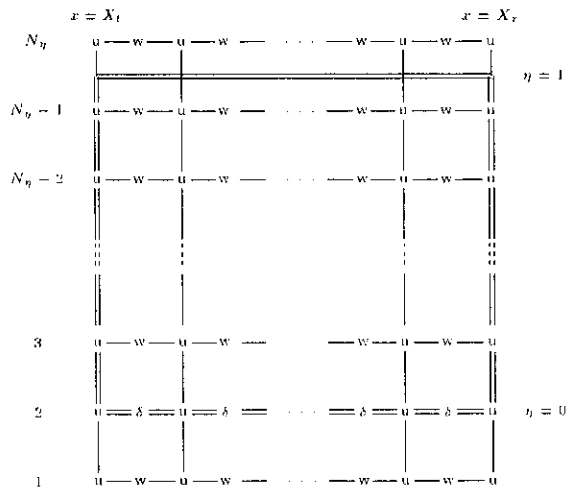

Fig. 1. The mesh definition.

We employ the staggered in both x- and η-direction mesh when solving Eqs. (2.10) and (3.8). The integrals for evaluating the thickness δ are approximated by means of the *trapezoidal rule* with second-order approximation. Equation (2.10) is approximated with central differences of second-order approximation in x-direction and with first order in η-direction.

5.3. *General consequence of algorithm*

(I) With given q, r and δ, the fourth-order boundary value problem (3.3) is solved for function u. The iterations with respect to the fictitious time are terminated when two consecutive iterations for u differ by less than ε_0;

(II) With δ given and with the newly calculated u, the function w is evaluated from (2.10). If the difference between the new and old fields for w is less than ε_0, then go to (III), otherwise to (I);

(III) The deviation of the calculated u and w from q and r is calculated and if it is smaller than ε_0, then proceed to (IV), otherwise q, r are replaced by u, w and then go to (I);

(IV) The function δ is calculated according to (3.3). If the difference between the new and old fields for δ is less than ε_0, then the calculations are terminated, otherwise return to (I).

The optimal value of ε_0 was found to be 5×10^{-5}.

6. Results and Discussion

In order to assess the practical applicability of MVI we solve here three classical problems which are known to possess self-similar solutions and compare our calculations with the well-known results for those problems.

The accuracy of the developed difference scheme is checked with the mandatory tests involving different grid spacing h_x and h_y. For the three different problems, we conducted calculations with different values of mesh parameters and compared them in order to verify the practical convergence, approximation and consistency of the difference scheme. For grid spacings $h_x \leq 0.05$ and $h_y \leq 0.05$, the difference between the present results and the self-similar solutions is indistinguishable within the accuracy of calculations with ordinary precision.

The first case we consider is the problem of the plane-stagnation-point flow when $U = x$. This flow is more amenable because of the continuous "inlet condition" $u(0, y) = U(y) \equiv 0$. In addition, the thickness is expected to be $\delta(x) = c = \text{const}$.

In Fig. 2 the velocity profiles obtained by the MVI are rescaled according to the longitudinal position and compared with the self-similar solution (see Ref. 12). The very good quantitative agreement is evident. Figure 3 shows the respective result for the thickness. The important conclusion from the figure is that the profile of the boundary-layer thickness resembles fairly well a constant.

The boundary condition for normal derivative of $u(x, y)$ defines the numerical value of the boundary-layer thickness δ as implicit function of ε. For this reason we solve the problem with several different values of ε. Table 2 presents the magnitudes for c extracted from the self-similar solution.[14] As can be easily seen by the inspection of Fig. 3, the agreement is very good for the average behavior of δ.

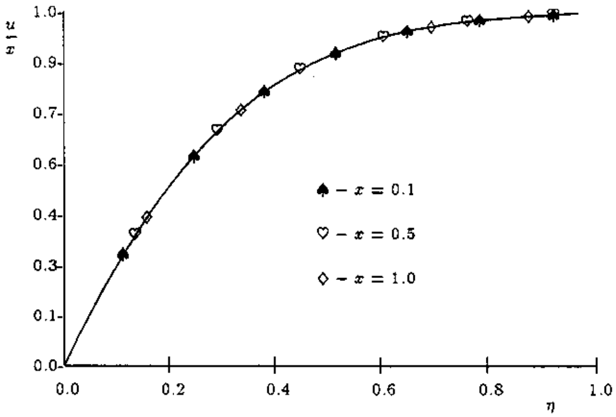

Fig. 2. Obtained profiles of $u(x, \eta)/x$ for stagnation point flow ($U = x$) at three different spatial positions and comparison with the self-similar solution.

Table 2. The values of c for three different values of ε for planar-stagnation-point flow as extracted from the self-similar solution.[14]

ε	0.0260	0.0156	0.0090
c	2.4	2.6	2.8

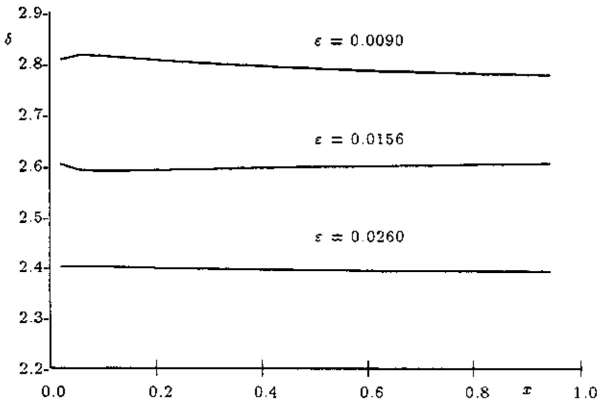

Fig. 3. Obtained boundary layer thickness for cylindrical-stagnation-point flow ($U = x$).

The second case considered here is the flow in the vicinity of the leading stagnation point on circular cylinder when $U = \sin x$. This case differs only quantitatively from the previous one. The boundary-layer thickness now is not constant with the longitudinal coordinate x. However, the behavior of $\delta(x)$ near $x = 0$ must be the same as in the previous case when $U = x$, which were presented in Table 2. Figure 4 shows the result obtained by the present algorithm. It is seen that the obtained thickness near $x = 0$ comply very well with Table 2. Note that here the results are obtained with $X_r = 1.5$.

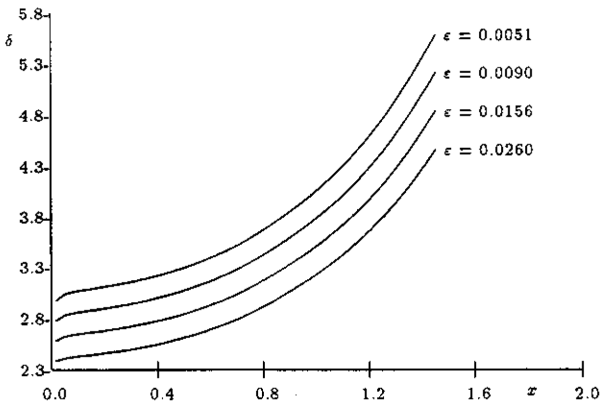

Fig. 4. Obtained boundary layer thickness for stagnation point flow ($U = \sin x$).

The third is the Blasius problem when $U(x) = \text{const.} = 1$. In this case the calculations in the leading point are complicated by the discontinuous "inlet" condition $u(0, y) = 1, y > 0$ and $u(0, 0) = 0$. Then the solution has singularities in its derivatives which immediately provokes lost of accuracy when solving the elliptic imbedding problem. We have actually solved the problem with $X_l = 0$ and the agreement was better than 5% in the immediate vicinity of $X_l = 0$ and rapidly improved inside the region. In order not to obscure the main points concerned with the imbedding itself we set the "inlet" condition at the position $X_l = 1$ and impose there the well-known self-similar profile (see e.g. Ref. 14). Respectively, the "outlet" condition is imposed at $X_r = 2$ with the appropriately rescaled self-similar profile. Such a test is not meaningless, because imposing self-similar profiles in the "inlet" and "outlet" spatial positions by no means secures the self-similar profile inside the region where we solve the fourth-order elliptic imbedding boundary value problem. The very good compliance of the calculated and the rescaled self-similar profile in an interior cross-section x speaks of the adequacy of the MVI. A number of calculations with different values of mesh parameters were conducted and similarly to the case of planar-stagnation-point flow the results for the velocity profile were found to compare very well with the self-similar results.

Now the expected behavior of the thickness is $\delta(x) = c\sqrt{x}$, where in general $c = c(\varepsilon)$. Table 3 gives the values of c for four different values of ε as calculated from the self-similar profile of the longitudinal velocity component (see Ref. 14). Figure 5 shows the obtained results for the boundary-layer thickness scaled for convenience by $\sqrt{x}$. Respectively, Fig. 6 shows the result for the velocity profile. The differences between them and the values extracted from the self-similar solution presented in Table 3 are very small.

Table 3. The values of c for four different values of ε for Blasius problem as extracted from the self-similar solution.[14]

ε	0.01591	0.00543	0.00155	0.00037
c	5.0	5.6	6.2	6.8

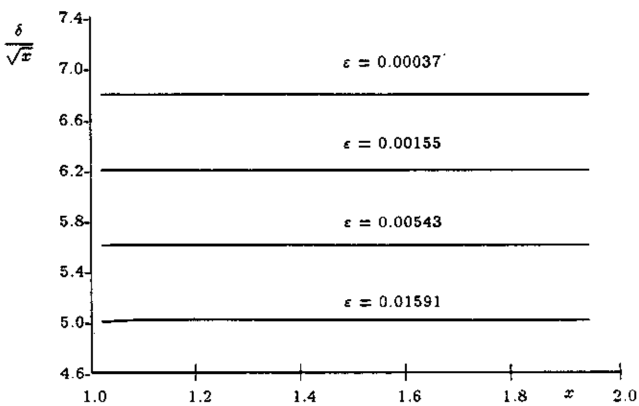

Fig. 5. Obtained boundary layer thickness scaled by $\sqrt{x}$ for Blasius problem (U = const.).

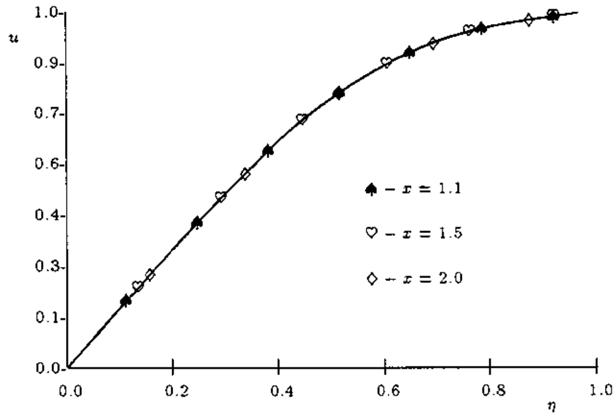

Fig. 6. Obtained profiles of $u(x, \eta)$ for Blasius problem ($U = \text{const.}$).

Thus the calculations for three different boundary-layer flows demonstrate the efficiency of MVI for translating the inverse problem into well-posed one for which more or less standard numerical methods are capable for providing the efficient tool for solution.

7. Conclusions

The method of variational imbedding is employed to render the ill-formulated (inverse) problem of identification of the boundary-layer thickness into a well-posed elliptic problem of identification of fourth-order with explicit differential equation of second-order for the thickness.

The correctness of the linearized "imbedding" problem is demonstrated. For the numerical solution the method of coordinate splitting is applied after fictitious time is added in the elliptic equations.

Three different boundary-layer problems are treated: planar and cylindrical stagnation points and Blasius flow. The good quantitative agreement with the asymptotic solutions shows that the MVI generated robust numerical procedures for identifying the boundary-layer thickness as an inverse problem.

Acknowledgment

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