5 SOLITARY WAVES WITH GALILEAN INVARIANCE IN DISPERSIVE SHALLOW-WATER FLOWS

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ABSTRACT

The present work deals with a recently derived nonlinear dispersive system for shallow-water flows. Unlike the classical Boussinesq models, the new one possesses Galilean invariance. It is investigated numerically by means of a conservative difference scheme. In order to understand the intrinsic physical mechanisms behind the balance between nonlinearity and dispersion larger phase speeds of the solitary waves are considered which are formally beyond the applicability of the weakly nonlinear approximation.

The pseudo-particle behavior of the solitary waves is interrogated. It is shown that the system with Galilean invariance is, in a sense, more "elastic" than the classical Boussinesq model. Snap-shots of the interactions of the localized waves are presented graphically. The phase shifts experienced by the pseudo-particles are shown to be of the opposite sign to these for systems without Galilean invariance.

INTRODUCTION

After John Scott-Russell discovered the "great wave" there were different attempts to explain its existence and to find its appropriate model. Boussinesq [4, 2, 3] introduced the fundamental idea of balance between the nonlinearity and dispersion and derived the first approximate expression for the dispersion in the case of weakly

K. Vajravelu (ed.), Differential Equations and Nonlinear Mechanics, 49–68.

nonlinear long waves. We call this balance "Boussinesq Paradigm". During the years different Boussinesq equations have been derived under the assumption of balance between weak nonlinearity and weak dispersion (the latter taking place for long waves). They are *generalized wave equations* which offer the opportunity to investigate the generic features of dispersive wave models, such as head-on collisions of localized structures (solitary waves/quasi-particles) even beyond the framework of the long-wave weakly-nonlinear assumptions. Boussinesq equations are not always fully integrable. As a rule they possess at least three conservation/balance laws – for mass, energy, and momentum.

Recently a more general form (preserving the Galilean invariance) of the dispersive shallow water equations has been derived [8, 9]. It has been shown to possess a solitary-wave solution of *sech* type which makes it very useful in paradigmatic sense for investigation of solitonic (pseudo-particle) behavior of localized solutions.

In the present work we investigate the properties of the new model as a dynamical system. To this end a special conservative difference scheme is constructed which generalizes to the case of Galilean invariant systems, the schemes previously developed by the author. A number of cases of soliton interactions are treated ranging from weakly nonlinear case to a strongly nonlinear case with nonlinear blow-up of the solution in finite time.

The model is expected to provide additional basis for soliton research especially as far as systems with Galilean invariance are concerned.

1. DISPERSIVE SHALLOW WATER SYSTEM (DSWS)

In the recent author's works the Boussinesq's derivation is revisited with the purpose of making the model conserve the total energy of the wave system. In order to make the present note self-contained we repeat part of the derivations from [8, 9].

Consider the inviscid flow in a thin layer with free surface represented by a single-valued shape function $h(x,y,t)$. The motion in the bulk is governed by the Laplace equation for the potential Φ.

Let H be the scale for the vertical spatial coordinate (the thickness of the shallow layer) and L is the characteristic wave length in longitudinal direction. We introduce dimensionless variables according to the scheme

$$\Phi = U L\phi, \quad h = H\eta, \quad z = Hz', \quad x = Lx', \quad y = Ly', \quad t = LU^{-1}t',$$

where $U = \sqrt{gH}$ is the characteristic scale for the velocity. Henceforth, the primes will be omitted without fear of confusion. In terms of dimensionless variables the Laplace equation takes the form

$$\beta \Delta \phi + \frac{\partial^2 \phi}{\partial z^2} = 0. \tag{1.1}$$

Here $\beta \equiv H^2 L^{-2}$ is called dispersion parameter. It is a small quantity for horizontal length scales L which are long compared to the depth of the layer H. The free surface in dimensionless form is given by $z = 1 + \eta$. The kinematic and dynamic conditions read

$$\frac{\partial \eta}{\partial t} + \nabla \phi \cdot \nabla \eta = \frac{1}{\beta} \frac{\partial \phi}{\partial z}, \tag{1.2}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2\beta} \left(\frac{\partial \phi}{\partial z} \right)^2 + \eta = 0. \tag{1.3}$$

Boussinesq expanded the solution of Laplace equation (1.1) into a power series with respect to β. Acknowledging the non-flux condition at the bottom of the layer he showed that the series contain only the even powers of the coordinate z, namely

$$\phi(x, y, z, t) = \sum_0^\infty (-\beta \Delta)^m f(x, y, t) \frac{z^{2m}}{(2m)!}, \tag{1.4}$$

where $f(x, y, t) \stackrel{def}{=} \phi(x, y, z = 0, t)$ is the unknown function representing the value of potential at the bottom of the layer.

Now the derivatives entering the surface conditions (1.2), (1.3) can be identified. Upon introducing the relevant expressions into the governing system for the surface motion and neglecting the terms proportional to β^m $(m \geq 2)$ one arrives at the following approximate system containing only the surface variables η, f:

$$\frac{\partial \eta}{\partial t} + \left[\nabla f - \frac{\beta}{2} \nabla \left[(1+\eta)^2 \Delta f \right] \right] \cdot \nabla \eta = -(1+\eta) \Delta f + \frac{\beta}{6} (1+\eta)^3 \Delta^2 f, \tag{1.5}$$

$$\frac{\partial f}{\partial t} - \frac{\beta}{2} \frac{\partial}{\partial t} \left[(1+\eta)^2 \Delta f \right] + \frac{1}{2} (\nabla f)^2 + \eta - \frac{\beta}{2} \nabla f \cdot \nabla \left[(1+\eta)^2 \Delta f \right] + \frac{\beta}{2} \left[(1+\eta) \Delta f \right]^2 = 0. \tag{1.6}$$

For small values of the dispersion parameter β the main idea of Boussinesq was to look for weakly nonlinear waves of amplitudes of the order of the small parameter, namely $|\eta|, |f| \simeq O(\beta)$. Then within the leading asymptotic order $O(\beta)$ the system reduces to a linear hyperbolic equation for the wave propagation. In the next order $O(\beta^2)$, two small effects – nonlinearity and dispersion – take place. The famous *sech* solution discovered by Boussinesq demonstrates that these two effects can be balanced pointwise making a wave propagate as a linear disturbance according to the linear wave equation from the leading asymptotic order. Thus a wave retains its shape unchanged if left alone (without collisions with other waves). This is what we call "Boussinesq Paradigm". Formally speaking one can seek for a solution $\eta = \beta\bar{\eta} + O(\beta^2)$, $f = \beta\bar{f} + O(\beta^2)$. Within the adopted asymptotic order $O(\beta^2)$ one gets

$$\frac{\partial\bar{\eta}}{\partial t} + \beta\nabla\cdot\bar{\eta}\nabla\bar{f} = -\Delta f + \frac{\beta}{6}\Delta^2\bar{f} + O(\beta^2), \tag{1.7}$$

$$\frac{\partial\bar{f}}{\partial t} - \frac{\beta}{2}\frac{\partial\Delta\bar{f}}{\partial t} + \frac{\beta}{2}(\nabla\bar{f})^2 + \bar{\eta} = 0 + O(\beta^2), \tag{1.8}$$

and the overbars will be omitted without fear of confusion.

Although the above derived system is a straightforward asymptotic reduction of the system (1.1), (1.2), (1.3) it differs qualitatively from the latter because (1.7), (1.8) does not bring about the conservation of energy. In this sense, it is not a consistent asymptotic approximation of the original system. It means that in the asymptotic reduction a quality has been lost.

In order to find the correct energy-conserving form of the system, we introduce a new variable

$$\chi = \eta - \frac{\beta}{2}\frac{\partial\Delta f}{\partial t}$$

and upon substituting it in (1.7), (1.8) we get

$$\frac{\partial\chi}{\partial t} + \beta\nabla\cdot\chi\nabla f + \frac{\beta^2}{2}\nabla\cdot\left(\frac{\partial\Delta f}{\partial t}\nabla f\right) = -\Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\frac{\partial^2\Delta f}{\partial t^2},$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2}(\nabla f)^2 - \chi.$$

The term $-\frac{\beta^2}{2}\nabla\cdot\left(\frac{\partial\Delta f}{\partial t}\nabla f\right)$, can be neglected within the asymptotic order $O(\beta^2)$ and the system adopts the form

$$\frac{\partial\chi}{\partial t} = -\beta\nabla\cdot\chi\nabla f - \Delta f + \frac{\beta}{6}\Delta^2 f - \frac{\beta}{2}\frac{\partial^2\Delta f}{\partial t^2}, \tag{1.9}$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2}(\nabla f)^2 - \chi. \tag{1.10}$$

The following energy balance law holds

$$\begin{aligned}\frac{dE}{dt} &= \oint_{\partial D}\left[(1+\beta\chi)\, f_t\frac{\partial f}{\partial n} + \frac{\beta}{2} f_t\frac{\partial f_{tt}}{\partial n} + \frac{\beta}{2} f_t\frac{\partial\Delta f}{\partial n} - \frac{\beta}{2}\Delta f\frac{\partial f_t}{\partial n}\right]ds,\\ E &= \frac{1}{2}\int_D\left[\chi^2 + (1+\beta\chi)(\nabla f)^2 + \frac{\beta}{6}(\Delta f)^2 + \frac{\beta}{2}(\nabla f_t)^2\right]dx,\end{aligned} \tag{1.11}$$

which allows us to call the system (1.9), (1.10) "Energy Consistent Boussinesq Paradigm". We believe that this is the system that fulfills the Boussinesq program without unnecessary deficiencies stemming from the oversimplifications in the moving frame.

The most suitable set of boundary conditions are those that bring about the conservation of the total energy stem from the requirement that the right-hand side of (1.11) be equal to zero. There are three sets of conditions compatible with that requirement. We select the Dirichlet set of b.c.

$$f_t = 0 \rightarrow f = f_b(x,y) \text{ and } \frac{\partial f}{\partial n} = 0 \quad \text{for} \quad (x,y) \in \partial D \tag{1.12}$$

which also secures the balance law for the wave momentum (see below).

Here is to be mentioned that both η and χ are implicit functions of the respective systems (functions for which no boundary conditions are posed) and there are no mathematical reasons to prefer one formulation over the other. Hence, we will not use the original variables.

Another balance (or conservation) law holds for the wave momentum (called also *pseudomomentum* [12, 13]) which is defined as

$$\boldsymbol{P} = -\int_D \eta \nabla f \, dy \equiv -\int_D \left[\chi \nabla f + \frac{\beta}{2} \frac{\partial \Delta f}{\partial t} \nabla f \right] dx\, dy. \tag{1.13}$$

In [8, 9] the following balance law for the total pseudomomentum is derived

$$-\frac{d\boldsymbol{P}}{dt} = \int_D \left\{ -\left[\nabla \cdot (\nabla f \nabla f) - \frac{1}{2} \nabla (\nabla f)^2 \right] - \frac{1}{2} \nabla \chi^2 - \beta \left[\nabla f \nabla \cdot (\chi \nabla f) + \frac{\chi}{2} \nabla (\nabla f)^2 \right] \right.$$
$$\left. + \frac{\beta}{6} \left[\nabla \cdot (\nabla \Delta f \nabla f) - \frac{1}{2} \nabla (\nabla \Delta f \cdot \nabla f) \right] + \frac{\beta}{2} \Delta f_t \nabla f_t \right\} dx\, dy$$
$$= \oint_{\partial D} \left[-\frac{\beta}{2} \left(\frac{\partial f_t}{\partial n} \nabla f_t - \frac{1}{2} (\nabla f_t)^2 \right) \boldsymbol{n} - \frac{1}{2} \chi^2 \boldsymbol{n} - \left(\frac{\partial f}{\partial n} \nabla f - \frac{1}{2} (\nabla f)^2 n \right) \right.$$
$$\left. + \frac{\beta}{6} \left(\frac{\partial \Delta f}{\partial n} \nabla f - \frac{1}{2} \nabla \Delta f \cdot \nabla f \right) \boldsymbol{n} + \beta \chi \nabla f \frac{\partial f}{\partial n} \right] ds.$$

Most of the terms in the balance law for the pseudomomentum cancel for boundary conditions (1.12) and hence,

$$\frac{d\boldsymbol{P}}{dt} = \oint_{\partial D} \left\{ -\frac{\beta}{4} (\nabla f_t)^2 - \frac{1}{4} (\nabla f)^2 + \frac{\beta}{12} (\nabla \Delta f \cdot \nabla f) + \frac{1}{2} \chi^2 \right\} \boldsymbol{n}\, ds. \tag{1.14}$$

2. SINGLE-EQUATION FORMULATION

Upon introducing (1.10) into (1.9) the function χ is readily excluded to obtain a single equation for the potential f, namely

$$f_{tt}+2\beta\nabla f\cdot\nabla f_t+\beta f_t\Delta f+\frac{3\beta^2}{2}(\nabla f)^2\Delta f-\Delta f+\frac{\beta}{6}\Delta^2 f-\frac{\beta}{2}\frac{\partial^2\Delta f}{\partial t^2}=0, \quad (2.1)$$

with Hamiltonian density

$$\mathcal{H}=\frac{1}{2}\left[f_t^2+(\nabla f)^2-\frac{\beta^2}{4}(\nabla f)^4+\frac{\beta}{6}(\Delta f)^2+\frac{\beta}{2}(\nabla f_t)^2\right]. \quad (2.2)$$

Note that the nonlinearity of the dynamic condition (1.10) is responsible for the cubic nonlinearity of equation (2.1). The latter is of higher order in β, but it cannot be neglected without destroying the Galilean invariance.

The system described by equation (2.1) can be re-interpreted in a field-theoretic framework (very much in the same vein as in our previous work concerning the 6th order Boussinesq equation [10]) by introducing a Lagrangian density

$$\mathscr{L}=\int_D \mathscr{L}dx\,dy,\quad P^{\omega}=-\int_D \nabla f\frac{\delta\mathscr{L}}{\delta f_t}\,dx\,dy,$$

$$\mathscr{L}=\mathcal{K}-\mathcal{W},\quad \mathcal{H}=\mathcal{K}+\mathcal{W},\quad \mathcal{K}=\frac{1}{2}f_t^2+\frac{\beta}{4}f_t(\nabla f)^2, \quad (2.3)$$

$$\mathcal{W}=\frac{1}{2}\left[(\nabla f)^2-\frac{\beta}{2}f_t(\nabla f)^2-\frac{\beta^2}{4}(\nabla f)^4-\frac{\beta}{3}(\Delta f)^2+\frac{\beta}{2}(\nabla f_t)^2\right].$$

In (2.3) one easily recognizes the pseudomomentum defined in (1.14), with an additional contribution probably due to the transport of the finite domain D when the Eulerian-Lagrangian passage is duly taken into account (see equation (4.42) from [13]). The appropriate boundary conditions yield the balance of *wave momentum*.

3. ONE-DIMENSIONAL VERSION

For two-dimensional flows the velocity potential and the surface elevation do not depend on the coordinate y. Naturally it is 1D for the surface variables. Then the system (1.9), (1.10) reduces to the following

$$\frac{\partial\chi}{\partial t}+\beta\frac{\partial}{\partial x}\left(\chi\frac{\partial f}{\partial x}\right)=-\frac{\partial^2 f}{\partial x^2}+\frac{\beta}{6}\frac{\partial^2 f}{\partial x^4}-\frac{\beta}{2}\frac{\partial^4 f}{\partial t^2\partial x^2}, \quad (3.1)$$

$$\frac{\partial f}{\partial t}+\frac{\beta}{2}\left(\frac{\partial f}{\partial x}\right)^2=-\chi. \quad (3.2)$$

Being reminded that f has the meaning of velocity potential taken at the bottom of the layer one can introduce new variables $u \stackrel{def}{=} f_x$ and $q_x \stackrel{def}{=} -\chi$. When the region under consideration is a finite interval, say $x \in [-L_1, L_2]$, then the boundary conditions (1.12) read

$$u = 0, \qquad q_x = 0, \qquad x = -L_1, L_2. \tag{3.3}$$

Upon integrating equation (3.1) once and acknowledging the boundary conditions one obtains

$$\frac{\partial q}{\partial t} + \beta u \frac{\partial q}{\partial x} = u - \frac{\beta}{6}\frac{\partial^2 u}{\partial x^2} + \frac{\beta}{2}\frac{\partial^2 u}{\partial t^2}, \tag{3.4}$$

$$\frac{\partial u}{\partial t} + \beta u \frac{\partial u}{\partial x} = \frac{\partial^2 q}{\partial x^2}. \tag{3.5}$$

4. BOUSSINESQ PARADIGM EQUATION

Boussinesq attempted to describe the nearly quasi-stationary wave phemomena in the moving frame [2, 3, 4]. He argued that for motions that evolve slowly in the moving coordinate frame the time derivatives are reasonably well approximated and can be replaced by the spatial ones.

Then two major ways of simplifications of the original system are possible. The first one is to simplify the convective nonlinear terms neglecting the nonlinearity in (1.10) or the cubic term in (2.1).

If f_t is replaced by f_x in the quadratic nonlinear term one arrives at

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[u + \frac{3\beta}{2}u^2 + \frac{\beta}{2}\frac{\partial^2 u}{\partial t^2} - \frac{\beta}{6}\frac{\partial^2 u}{\partial x^2}\right], \tag{4.1}$$

which was called in [7] "Boussinesq Paradigm Equation" (BPE). Note that it is not the equation derived by Boussinesq himself. The above simplification destroys the Galilean invariance of the system. It should be mentioned that the equation (4.1) appears also in the theory of longitudinal (acoustic) vibrations of rods (see, e.g., [15]) and in continuum limit for lattices (see, e.g., [16, 17]) where the lack of Galilean invariance can be an actual physical property.

The second simplification consists in changing the temporal derivatives to spatial ones in the linear dispersion terms. It led Boussinesq to an equation which was incorrect in the sense of Hadamard. Later on Boussinesq equation was regularized in [1] where an equation similar to (4.1) was derived (called "Regularized Long Wave Equation", or RLWE). More detailed discussion on this matter can be found in ([8, 9]). Note that RLWE is linearly stable [14, 1, 20]). It is not fully integrable (just as the original DSWS system is not), which is compliant with the physical nature of the problem.

BPE equation (4.1) can be rewritten as a system (see, [7])

$$u_t = q_{xx}, \qquad q_t = u + \frac{3\beta}{2} u^2 + \frac{\beta}{2} u_{tt} - \frac{\beta}{6} u_{xx}. \tag{4.2}$$

The Hamiltonian structure of the above system is found in [7] and shown that for the b.c. from (3.3) one has

$$\frac{dM}{dt} = 0, \qquad M = \int_{-L_1}^{L_2} u\, dx \tag{4.3}$$

$$\frac{dE}{dt} = 0, \quad E = \frac{1}{2} \int_{-L_1}^{L_2} \left[q_x^2 + u^2 + \frac{\beta}{2} u^3 + \frac{\beta}{2} u_t^2 + \frac{\beta}{6} u_x^2 \right] dx \tag{4.4}$$

$$\begin{gathered} \frac{dP}{dt} = \left[\frac{u^2}{2} + \beta u^3 - \frac{\beta}{4} u_t^2 - \frac{\beta}{12} u_x^2 \right]_{-L_1}^{L_2} = -\frac{\beta_2}{2} u_x^2 \Bigg|_{-L_1}^{L_2}, \\ P = \int_{=L_1}^{L_2} u\left(q_x + \frac{\beta}{2} u_{xt} \right) dx = \int_{-L_1}^{L_2} \left(u q_x - \frac{\beta}{2} u_t u_x \right) dx. \end{gathered} \tag{4.5}$$

BPE is preferable over RLWE because for the former the *wave mass* is conserved alongside with the *energy*. In other words, the presence of the spatial fourth derivative requires boundary conditions whose satisfaction in turn brings about the conservation of the *wave mass* which is also a property of the original hydrodynamic problem.

It is to be mentioned that though the system (4.2) looks rather similar to the original DSWS (1.9), (1.10), there is a significant difference due to the fact that the latter is Galilean invariant, while the former is not. Respectively, the Lagrangian and Hamiltonian densities for the two systems are different. BPE is our choice for a dynamical system without Galilean invariance for which the balance between nonlinearity and dispersion holds.

Note that the derivations of the present section are not restricted to 1D surface motions. One-dimensional Boussinesq equations are considered only for the sake of comparison with the classical works.

5. THE SOLITARY WAVE OF DSWS

The *sech* solitons of BPE are given by (see [7]):

$$u = \frac{a}{\cosh^2 \left[b(x-ct) \right]}, \quad a = \frac{c^2-1}{\beta}, \quad b = \sqrt{\frac{\left(c^2-1\right)}{2\beta\left(c^2 - \frac{1}{3}\right)}}, \tag{5.1}$$

where c is the phase speed or *celerity* of wave. The *sech*-es exist either for supersonic celerities $c > 1$ or for $c < \sqrt{1/3}$. Only the supersonic *sech*-es are of physical relevance to shallow-water flows because for small β the subsonic ones are not long-length waves.

Although more complex than any of the Boussinesq equations, the DSWS system (3.4), (3.5) shares with them the existence of localized solution which is stationary in the moving frame $x - ct$. In [8, 9] the following *sech*-like solution is found

$$u = \frac{a\, sign(c)}{\frac{|c|-1}{2} + \cosh^2\left[b(x-ct)\right]}, \quad a = \frac{c^2-1}{\beta}, \quad b = \sqrt{\frac{(c^2-1)}{2\beta\left(c^2 - \frac{1}{3}\right)}}, \tag{5.2}$$

which exists in the same range as the BPE solitons (see, [7]). The fact that DSWS admits a *sech*-like solution renders unnecessary the Boussinesq simplifications in the moving frame. The *sech*-like solution (5.2) is another candidate for the John Scott Russell's "Great (Permanent) Wave".

Comparing (5.1) and (5.2) reveals that the only difference is the term $\frac{1}{2}\left(|c|-1\right)$ in the denominator. This means that in the limit of weakly-nonlinear case $|c-1| \sim O(\beta) \ll 1$ they will be quantitatively very close. For arbitrary c the difference is small in the "tails" of the waves. It is significant only near the origin of coordinate system where is the smallest value of the denominator $(\cosh \approx 1)$ and then only for significantly supercritical c.

In order to keep within the long-wave approximation, we have taken the supercritical celerities $c^2 = 1 + \beta$ and calculated the shapes of the solitary waves of DSWS and BPE. Figure 1 shows the comparison between the two solutions. The DSWS wave is always of smaller amplitude than the BPE one. For really small β the differences are quantitatively very small and it is hard to distinguish which one corresponds better to the experiment. A good case for comparison with the experiments of John Scott Russell could be $\beta = 0.2$.

The subcritical case is formally overboard the shallow-water theories since for $|c| < \sqrt{1/3}$ the solitary waves are not long waves. Yet, knowing the shapes of the subcritical solitary waves is of crucial importance for understanding the results of the numerical investigation of the evolution of colliding waves which is presented in the sections to follow. The most conspicuous feature of the subcritical waves is that they are depressions.

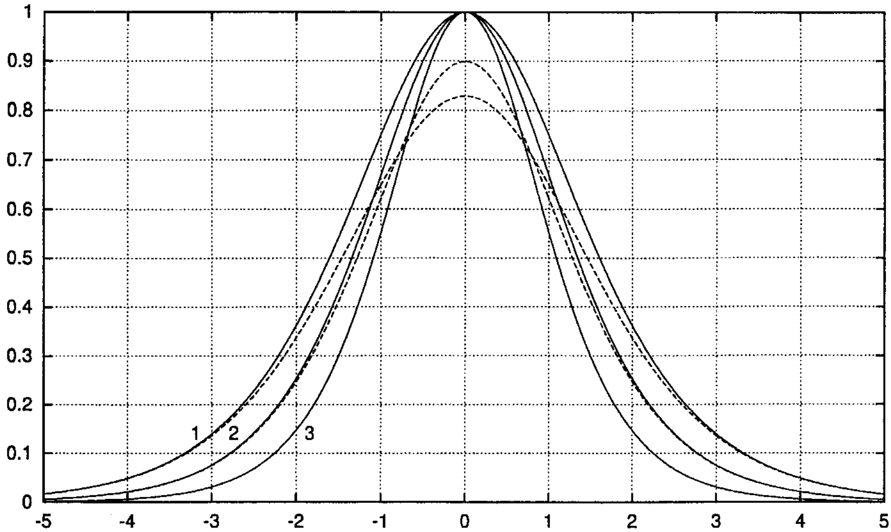

Figure 1. Comparison between BPE and DSWS solitary waves for supercritical phase speeds $c = \sqrt{1+\beta}$: 1) $\beta = 1$; 2) $\beta = 0.5$; 3) $\beta = 0.1$.

Figure 2 shows that for $c = 0$, the depressions are less steep, but of largest amplitude. With the increase of c, they become narrower and their amplitudes

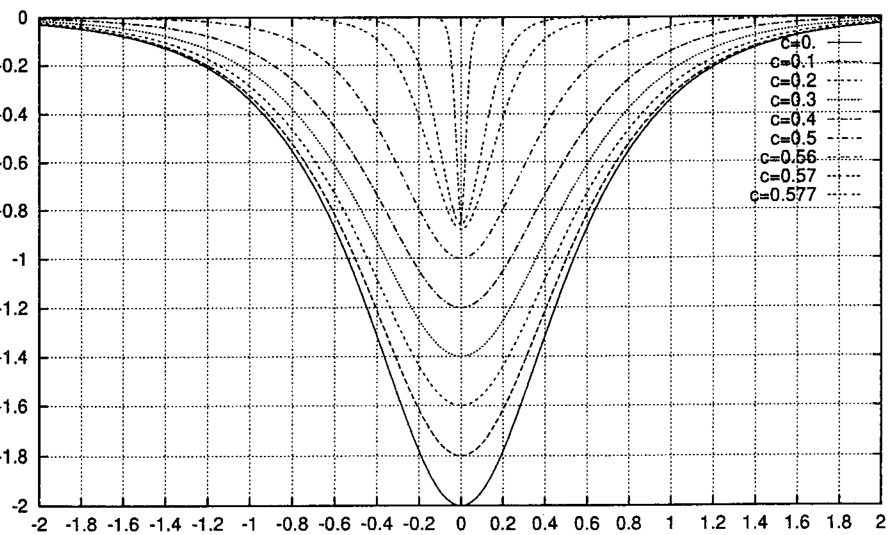

Figure 2. DSWS *seches* for subcritical phase speeds and $\beta = 1$.

somewhat decrease. For the limiting case $c \to \sqrt{1/3}$, the support of the localized solution vanishes and it becomes a solution of infinitely short length. The support for $\beta = 0.1$ is about three times shorter than for $\beta = 1$, which means that even for $c = 0$, it is significantly lesser than unity. This means that one cannot speak about long-wave solutions in this case.

In the present work, we consider DSWS in somewhat more paradigmatic way as a toy-object allowing the investigation of the interaction of pseudo-particles in a system with Galilean invariance. For this reason, we select $\beta = 0.6 \sim O(1)$ and consider the whole range of waves, e.g., strongly nonlinear short waves. The shapes of the solitary waves of DSWS and BPE are shown in Figure 3 for variety of supercritical phase speeds.

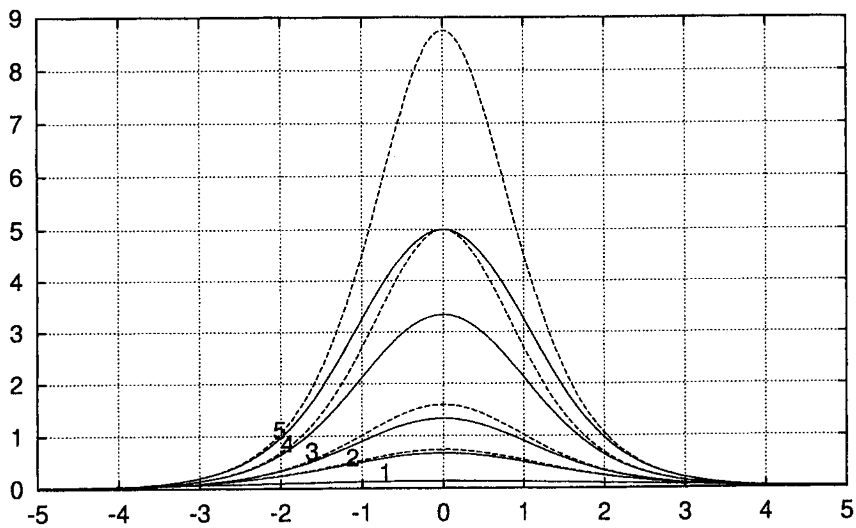

Figure 3. Solitary waves of DSWS ——— and BPE - - - for $\beta = 0.6$ and different phase speeds: 1) $c = 1.04$; 2) $c = 1.2$; 3) $c = 1.4$; 4) $c = 2.$; 5) $c = 2.5$.

Important characteristic of the solitary waves shown in Figure 3 is that the supercritical BPE *sech*-es are taller than the DSWS *sech*-es for the same magnitude of the phase speed.

6. CONSERVATIVE DIFFERENCE SCHEME

In previous author's papers [6, 11, 7], the way to construct conservative schemes for the Boussinesq Paradigm was outlined and their efficiency was demonstrated.

Following [7], we construct a conservative scheme for the Galilean invariant case treated here. We introduce a regular mesh in the interval $[-L_1, L_2]$, $x_i = -L_1 + (i-1)\,H$, $h = (L_1 + L_2)/(N-1)$, where N is the total number of grid points. We use a simplest linearization combined with an internal iteration (referred to by the composite superscipt k). It appears to be robust enough and economical.

$$\frac{u_i^{n+1,k} - u_i^n}{\tau} = \frac{q_{i+1}^{n+\frac{1}{2},k} - 2q_i^{n+\frac{1}{2},k} + q_{i-1}^{n+\frac{1}{2},k}}{h^2} - \frac{\beta}{8h}\left[\left(u_{i+1}^{n+1,k-1}\right)^2 - \left(u_{i-1}^{n+1,k-1}\right)^2 + \left(u_{i+1}^n\right)^2 - \left(u_{i-1}^n\right)^2\right] \tag{6.1}$$

$$\begin{aligned}\frac{q_i^{n+\frac{1}{2},k} - q_i^{n-\frac{1}{2}}}{\tau} = &-\frac{\beta}{8h}\left[q_{i+1}^{n+\frac{1}{2},k-1} - q_{i-1}^{n+\frac{1}{2},k-1} + q_{i+1}^{n-\frac{1}{2}} - q_{i-1}^{n-\frac{1}{2}}\right]\left(u_i^{n+1,k} + u_i^{n-1}\right)\\ &-\frac{\beta}{12}\left[\frac{u_{i+1}^{n+1,k} - 2u_i^{n+1,k} + u_{i-1}^{n+1,k}}{h^2} + \frac{u_{i+1}^{n-1} - 2u_i^{n-1} + u_{i-1}^{n-1}}{h^2}\right]\\ &+\frac{\beta}{2}\frac{u_i^{n+1,k} - 2u_i^n + u_i^{n-1}}{\tau^2} + \frac{u_i^{n+1,k} + u_i^{n-1}}{2},\end{aligned} \tag{6.2}$$

with b.c.

$$u_N^{n+1,k} = u_1^{n+1,k} = 0, \qquad q_N^{n+\frac{1}{2},k} - q_{N-1}^{n+\frac{1}{2},k} = q_2^{n+\frac{1}{2},k} - q_1^{n+\frac{1}{2},k} = 0. \tag{6.3}$$

The inner iterations start from the functions obtained at the previous time stage $u_i^{n+1,0} = u_i^n$ and $q_i^{n+\frac{1}{2},0} = q_i^n$, and are terminated at certain $k = K$ when

$$\max\left|u_i^{n+1,K} - u_i^{n,K-1}\right| \le 10^{-13}\max\left|u_i^{n+1,K}\right|.$$

The value 10^{-13} is selected to be large enough in comparison with the round-off error 10^{-14}. In general, the number of iterations K (in our calculations we keep them around six to eight by means of adjusting the time increment to the "swiftness" of the motion) depends on the size of time increment. After the inner iterations converge one obtains, in fact, the solution for the new time stage $n+1$ of the nonlinear conservative difference scheme, namely $u_i^{n+1} \overset{def}{=} u_i^{n+1,K}$, $q_i^{n+\frac{1}{2}} \overset{def}{=} q_i^{n+\frac{1}{2},K}$.

From now on we shall not refer any more to the internal iterations (hence, omitting the composite index k), but rather consider the general properties of the scheme (6.1), (6.2) where the iterations are considered as accomplished.

Generalizing the derivation from [6], we prove that the above approximation secures the conservation of *energy* on difference level for arbitrary potential $U(u)$, namely the difference approximations of the *mass* and *energy*

$$E^{n+\frac{1}{2}} = \frac{h}{2}\sum_{i=1}^{N-1}\frac{\left(u_i^{n+1}\right)^2+\left(u_i^n\right)^2}{2} - U\left(u_i^{n+1}\right) - U\left(u_i^n\right) + \frac{\beta}{2}\left(\frac{u_i^{n+1}-u_i^n}{\tau}\right)^2$$
$$+\frac{1}{2h}\sum_{i=1}^{N-1}\frac{\beta}{12}\left[\left(u_{i+1}^{n+1}-u_i^{n+1}\right)^2+\left(u_{i+1}^n-u_i^n\right)^2\right]+\left(q_{i+1}^{n+\frac{1}{2}}-q_i^{n+\frac{1}{2}}\right)^2,$$
$$M^{n+1} = \sum_{i=2}^{N-1} u_i^{n+1}\, h,$$

are conserved by the difference scheme (6.2), (6.1) in the sense that $M^{n+1} = M^n$ and $E^{n+\frac{1}{2}} = E^{n-\frac{1}{2}}$. As far as the satisfaction of the conservation and balance laws does not depend on the truncation error, we call it "strict in numerical sense".

The scheme (6.1), (6.2) consists of two conjugated tridiagonal systems. We render them to a single five-diagonal system and apply the specialized solver for Gaussian elimination with pivoting [5].

7. NUMERICAL EXPERIMENTS ON SOLITON DYNAMICS

In this section we present the calculations obtained for different wave systems as well as the comparison to the BPE results. We chose $\beta = 0.6 \sim O(1)$ which is consistent with our aim to investigate the system beyond the range of its formal relevance to shallow-water flows.

In the figures to follow the material is organized to show snap-shots of the wave system in the upper panel and the trajectories of the centers of the solitary waves – in the lower panel. The time interval is the same for the two panels, but the panel with trajectories is appropriately zoomed for convenience. The scale for vertical axis pertains to the amplitude of the waves in the initial moment of time. In the lower panel, the solid lines represent BPE solution, the dashed lines – DSWS, and the dotted lines are the trajectories, which the solitons could have followed were they not to interact with each other. Since numerically a center of a "hump" is defined as the local maximum of the absolute value of the wave profile, it is impossible to locate it precisely during the collision of the two localized waves. For this reason the trajectories of the DSWS "humps" look somewhat erratic during the collision. Yet after the two main "humps" resume their identity, the results for the trajectories are smoother.

First we begin with the weakly-nonlinear case $c = 1.04$ and $c = 1.02$. It is depicted in Figure 4. One sees that the interaction is virtually elastic with no residual signals in the place of the collision, and no significant phase shift. As already found in [9], the phase shift for the Galilean invariant system is approximately twice as small and positive while the BPE predicts negative phase shift.

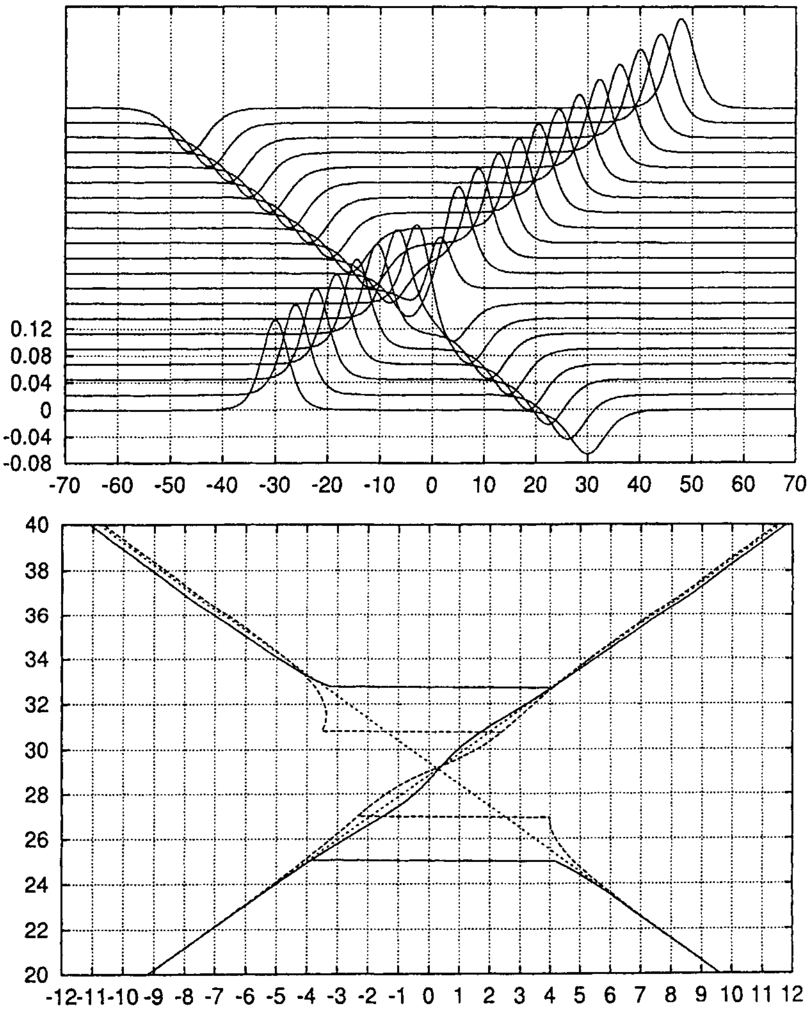

Figure 4. The weakly nonlinear case for $\beta = 0.6$, $c_l = 1.04$, and $c_r = -1.02$. Upper panel: Snapshots of the wave system for different times in the interval $0 \le t \le 75$. Lower panel: Trajectories of the centers of solitary waves. BPE: ——, DSWS: - - -, trajectories of free solitons: · · · · ·.

Next we move to moderately nonlinear case presented in Figure 5. The general tendency in the phase shift is the same, but now it is almost twice as large as in the weakly nonlinear case. In addition, after the collision, some wriggles appear propagating with the characteristic speed (they are linear waves of small amplitudes) and trail the two main humps.

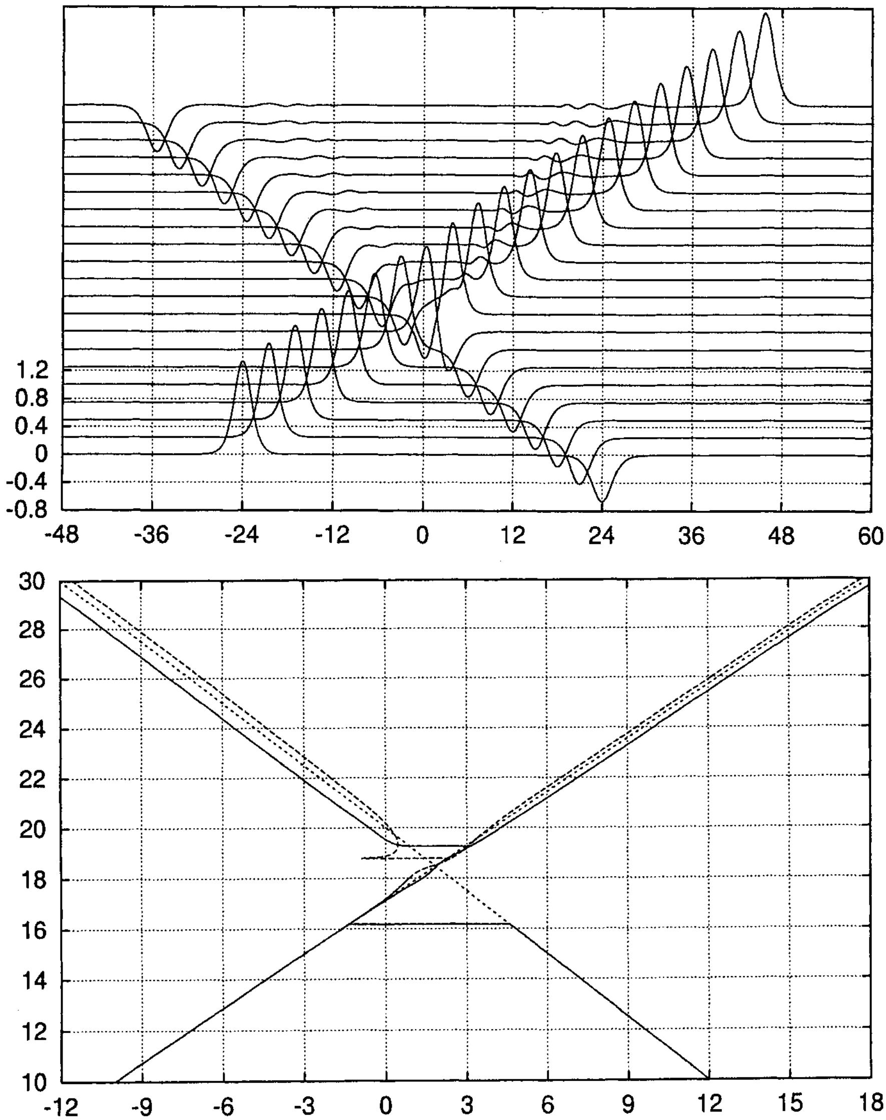

Figure 5. A moderately nonlinear case for $\beta = 0.6$, $c_l = 1.4$, and $c_r = -1.2$. Upper panel: Snapshots of the wave system for different times in the interval $0 \leq t \leq 50$.

Lower panel: Trajectories of the centers of solitary waves. BPE: ——, DSWS: - - -, trajectories of free solitons: · · · · ·.

A further increase of the phase speeds (as shown in Figure 6) doubles the magnitude of the phase shift, but now the phase shifts for DSWS and BPE have almost the same absolute values remaining of opposite signs.

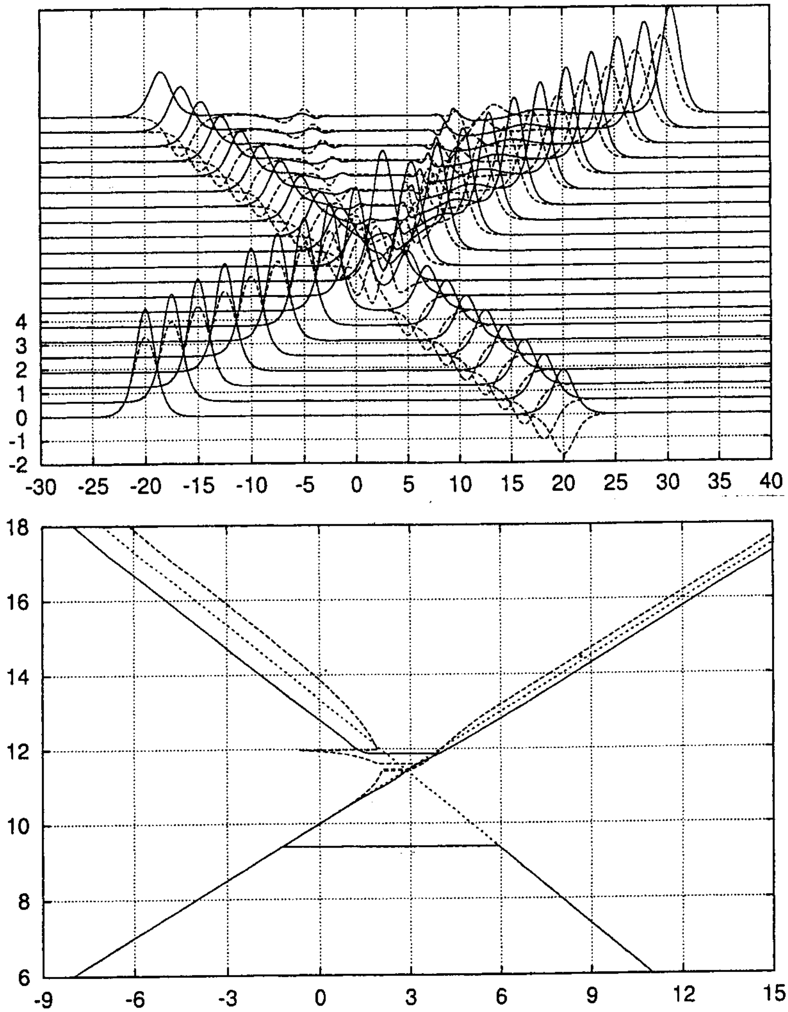

Figure 6. A nonlinear case for $\beta = 0.6$, $c_l = 2.$, and $c_r = -1.5$. Upper panel: Snapshots of the wave system for different times in the interval $0 \le t \le 25$. Lower panel: Trajectories of the centers of solitary waves. BPE: ——, DWSW: - - -, trajectories of free solitons: · · · · ·.

In order to give some more tangible information on the behavior of the two systems, we juxtapose directly the snapshots of the wave systems for the two systems under consideration. One sees that the profiles are similar, save the fact that the left-going wave in DSWS is negative (which is the physically correct case) while the same wave for BPE is positive (one of the deficiencies of the simplifications in the moving frame). The positions of the wriggles excited by the collision are roughly the same for the two systems, but their amplitudes are larger in DSWS rather than in BPE. In turn, the DSWS-wriggles are smoother which implies stronger elasticity of the system. In Figure 6 the difference in the signs of the phase shifts is also well seen.

Finally, we treat a case with very strong nonlinearity (shown in Figure 7). For

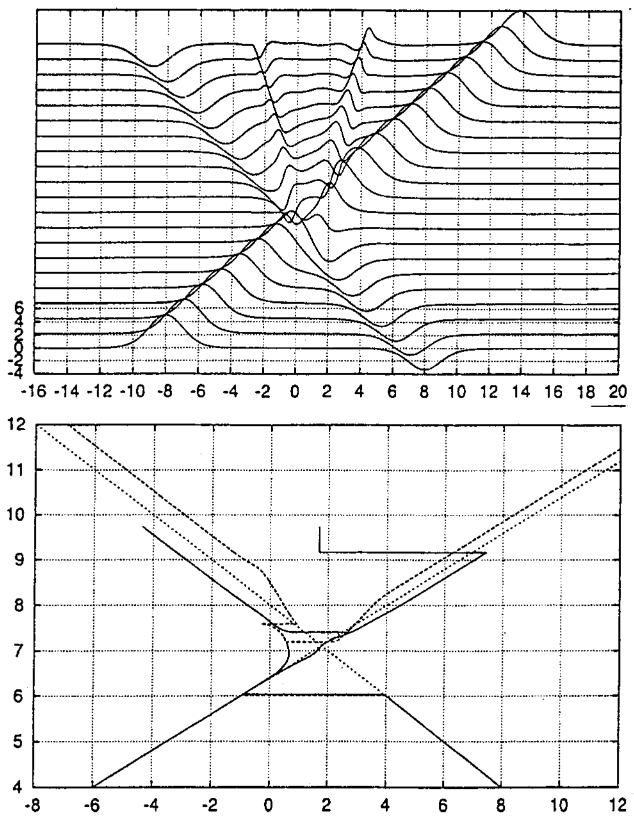

Figure 7. Strongly nonlinear case for $\beta = 0.6$, $c_l = 2.5$, and $c_r = -2.0$. Upper panel: Snapshots of the wave system for different times in the interval $0 \le t \le 18$. Lower panel: Trajectories of the centers of solitary waves. BPE: ——, DSWS: - - -, trajectories of free solitons: · · · · ·.

this set of parameters, the BPE solution ends up in finite-time nonlinear blow-up (see, e.g., [18, 19]) while the DSWS solution survives. This is another sign of the stronger elasticity of the system with Galilean invariance (DSWS).

Figure 7 shows that (curiously enough) in BPE the solitary wave of smaller amplitude (the left-going one in this case) blows-up earlier than the wave of larger amplitude and larger phase speed.

8. CONCLUSIONS

A recently derived (see [8, 9]) dispersive shallow-water system is considered which originally appears in the long-wave models of the flow of inviscid liquid with free surface. The new system preserves the Galilean invariance of the original problem which is its main advantage over the Boussinesq equations. Its Hamiltonian structure is derived and new analytical solution of type of solitary wave is obtained. A conservative difference scheme is constructed and an algorithm for its implementation is developed.

The interactions of solitary localized waves are investigated for the case of significantly supercritical phase speeds of the solitary waves. This case is formally outside the region of applicability of long-wave weakly-nonlinear Boussinesq derivations. Hence, the new model is considered in a paradigmatic fashion and the intrinsic mechanisms of interactions of the pseudo-particles (solitary waves) are interrogated for the first time in the literature for a system with Galilean invariance. It is shown that for moderate nonlinearities, the behaviour is qualitatively similar to the weakly nonlinear case. The interactions are fairly elastic and the signals excited after the collision of two solitary waves are small. The phase shift is smaller than those for a system without Galilean invariance (so-called Boussinesq Paradigm Equation) and of opposite sign.

The strongly nonlinear cases reveal the same relationship between these signs of the phase shifts, but the system with Galilean invariance DSWS tends to be more robust in the sense that its solution exists for phase speeds for which a nonlinear blow-up takes place for BPE. Together with the smaller phase shift for moderate phase speeds, the last property underscores somewhat stronger elasticity of the system with Galilean invariance.

Acknowlegments. This work is supported by Grant LEQSF (1999-2002)-RD-A-49 from the Louisiana Board of Regents and partly by Grant NZ-611/96 of the National Science Foundation of Bulgaria.

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