

An energy-consistent dispersive shallow-water model

C.I. Christov*

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA

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Abstract

The flow of inviscid liquid in a shallow layer with free surface is revisited in the framework of the Boussinesq approximation. The unnecessary approximations connected with the moving frame are removed and a Boussinesq model is derived which is *Galilean invariant* to the leading asymptotic order. The Hamiltonian structure of the new model is demonstrated. The conservation and/or balance laws for wave mass, energy and wave momentum (*pseudo-momentum*) are derived. A new localized solution is obtained analytically and compared to the classical Boussinesq sech. Numerical simulation of the collision of two solitary waves is conducted and the impact of Galilean invariance on phase shift is discussed. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

After Russell discovered the “great wave” there were different attempts to explain its existence and to find its appropriate model. Boussinesq [3–5] introduced the fundamental idea of balance between nonlinearity and dispersion and derived the first approximate expression for dispersion for the case of weakly nonlinear long waves. We call this balance “Boussinesq paradigm”. The Boussinesq treatment of the moving frame led to two major deficiencies of the model. First, a linear instability was introduced in the sense that the initial value problem becomes incorrect in the sense of Hadamard. Second, the original BBE lacked Galilean invariance. In turn it is fully integrable. For long waves different equations have been derived in the literature under the assumption of balance between weak nonlinearity and small dispersion. They are *generalized wave equations* (usually called by the generic name “Boussinesq equations”) which offer the opportunity to investigate the intrinsic mechanisms of dispersive wave models, such as head-on collisions of localized structures (solitary waves/pseudo-particles). Boussinesq equations are not always fully integrable. As a rule they possess at least three conservation/balance laws: for wave “mass”; wave energy; and wave momentum. Because of the conservation/balance laws the localized waves behave as pseudo-particles. Since not all of them are fully integrable, the problem arises of how to treat them numerically. In previous papers of the author [6,7,9] the way to construct conservative schemes for the Boussinesq paradigm was outlined and their efficiency was demonstrated.

It appears important to re-derive the Boussinesq model removing the above-mentioned deficiencies without compromising the Hamiltonian structure and Galilean invariance.

To this task is devoted the present paper. We derive the dispersive shallow-water equations which are asymptotically correct to the first order of the small dispersion parameter. We call the model “dispersive shallow-water

* Tel.: +1-337-482-5273; fax: +1-337-482-5346.
E-mail address: christov@louisiana.edu (C.I. Christov).

system” (DSWS) and show that it is Galilean invariant and possesses Hamiltonian structure. For the new system we find also an expression for pseudo-momentum (see [10,11] for definition). We find an analytical solution for the solitary-wave solution and compare it to Boussinesq’s sech. A limited number of calculations of interactions of solitary waves of DSWS are performed using a conservative difference scheme.

The model is expected to provide additional basis for the investigation of the pseudo-particle behavior of the localized surface waves.

2. Dispersive shallow-water system

In this section, we revisit Boussinesq’s derivation with the purpose of making it consistent with the original model of inviscid shallow layer flows which possesses Hamiltonian structure.

2.1. Posing the problem

Consider the inviscid flow in a shallow layer with free surface. We restrict the derivations to the case when the shape function $h(x, y, t)$ of the free surface is single-valued, i.e., there is no wave breaking. The motion in the bulk is governed by the Laplace equation for the potential Φ .

Let H be the scale for the vertical spatial coordinate (the thickness of the shallow layer) and L the (yet undefined) wavelength in the horizontal plane. We introduce dimensionless variables according to the scheme

$$\Phi = UL\phi, \quad h = H\eta, \quad z = Hz', \quad x = Lx', \quad y = Ly', \quad t = LU^{-1}t',$$

where $U = \sqrt{gH}$ is the characteristic scale for the velocity. Henceforth, the primes will be omitted without fear of confusion. In terms of dimensionless variables the Laplace equation takes the form

$$\beta \Delta \phi + \frac{\partial^2 \phi}{\partial z^2} = 0. \quad (2.1)$$

Here $\beta \equiv H^2 L^{-2}$ is called dispersion parameter and it is a small quantity for waves of horizontal length scale L which is large compared to the depth H of the layer. The free surface is given by $z = 1 + \eta$ in dimensionless form. The kinematic and dynamic conditions read

$$\frac{\partial \eta}{\partial t} + \nabla \phi \cdot \nabla \eta = \frac{1}{\beta} \frac{\partial \phi}{\partial z}, \quad (2.2)$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2\beta} \left(\frac{\partial \phi}{\partial z} \right)^2 + \eta = 0. \quad (2.3)$$

The problem now is to identify the term containing the transverse derivative (z -derivative in the above adopted notations) of the potential at the free surface.

2.2. Boussinesq’s variables

Boussinesq expanded the solution of Laplace equation (2.1) into a power series with respect to the small parameter β . Acknowledging the nonflux condition at the bottom of the layer showed that the series contain only the even powers of the coordinate z , namely

$$\phi(x, y, z, t) = \sum_0^{\infty} (-\beta \Delta)^m f(x, y, t) \frac{z^{2m}}{(2m)!}, \quad (2.4)$$

where $f(x, y, t) \stackrel{\text{def}}{=} \phi(x, y, z = 0, t)$ is the unknown magnitude of the potential at the bottom of the layer. Then for the derivatives involved in the surface conditions (2.2) and (2.3) one gets

$$\left. \frac{\partial \phi}{\partial z} \right|_{z=1+\eta} = \sum_1^{\infty} (-\beta \Delta)^m f(x, y, t) \frac{(1+\eta)^{2m-1}}{(2m-1)!}, \quad (2.5)$$

$$\left. \frac{\partial \phi}{\partial t} \right|_{z=1+\eta} = \sum_0^{\infty} (-\beta \Delta)^m \frac{\partial f(x, y, t)}{\partial t} \frac{(1+\eta)^{2m}}{(2m)!}, \quad (2.6)$$

$$\nabla \phi|_{z=1+\eta} = \sum_0^{\infty} (-\beta \Delta)^m \nabla f(x, y, t) \frac{(1+\eta)^{2m}}{(2m)!}. \quad (2.7)$$

Introducing these expressions into the governing system for the surface motion and neglecting the terms proportional to β^2 ($m \geq 2$) one arrives at the following approximate system containing only the surface variables η, f :

$$\frac{\partial \eta}{\partial t} + \left[\nabla f - \frac{\beta}{2} \nabla[(1+\eta)^2 \Delta f] \right] \cdot \nabla \eta = -(1+\eta) \Delta f + \frac{\beta}{6} (1+\eta)^3 \Delta^2 f, \quad (2.8)$$

$$\frac{\partial f}{\partial t} - \frac{\beta}{2} \frac{\partial}{\partial t} [(1+\eta)^2 \Delta f] + \frac{1}{2} (\nabla f)^2 + \eta - \frac{\beta}{2} \nabla f \cdot \nabla [(1+\eta)^2 \Delta f] + \frac{\beta}{2} [(1+\eta) \Delta f]^2 = 0. \quad (2.9)$$

Note that in the two-dimensional case (no dependence on y) one has $\Delta \equiv \partial^2 / \partial x^2$ and $\nabla \equiv \partial / \partial x$.

2.3. Asymptotic simplifications

For small dispersion parameter β , the dominating terms of order $O(1)$ represent the wave propagation in a nonlinear hyperbolic system, i.e. if we admit the existence of waves of amplitude of unit order $|\eta|, |f| \simeq O(1)$, then the well-known phenomenon of steepening of the surface waves takes place. That was the reason why Airy [1] ruled out the existence of permanent waves, which were actually observed by Russell [14,15]. The main idea of Boussinesq was to look for weakly nonlinear waves whose amplitude is of the order of the small parameter, namely $|\eta|, |f| \simeq O(\beta)$. Then within the leading asymptotic order $O(\beta)$ the system reduces to a linear hyperbolic equation for the wave propagation. In the next order $O(\beta^2)$, two small effects — nonlinearity and dispersion — appear. The famous sech solution discovered by Boussinesq demonstrates that these two effects can balance each other pointwise, so that a wave can propagate as a linear disturbance according to the linear wave equation from the leading asymptotic order. As a result a wave retains its shape unchanged if left alone (without collisions with other waves). Formally speaking one can rescale the sought functions assuming that $\eta = \beta \bar{\eta}$, $f = \beta \bar{f}$. Within the adopted order $O(\beta^2)$ one gets for the different terms the following equations:

$$\begin{aligned} \beta(1+\eta)^3 \Delta f &= \beta^2 \Delta \bar{f} + O(\beta^3), & \frac{1}{2} \beta \nabla[(1+\eta)^2 \Delta f] \cdot \nabla \eta &\simeq \frac{1}{2} \beta \nabla[(1+\eta)^2 \Delta f] \cdot \nabla f = O(\beta^3), \\ \frac{\beta}{2} \frac{\partial}{\partial t} [(1+\eta)^2 \Delta f] &= \frac{\beta^2}{2} \frac{\partial \Delta \bar{f}}{\partial t} + O(\beta^3), & \frac{\beta}{2} [(1+\eta)^2 \Delta f]^2 &= O(\beta^3). \end{aligned}$$

Upon introducing the last expression into (2.8) and (2.9), dividing both sides by β , and retaining only the terms within the order $O(\beta)$, the system reduces to

$$\frac{\partial \bar{\eta}}{\partial t} + \beta \nabla \cdot \bar{\eta} \nabla \bar{f} = -\Delta \bar{f} + \frac{\beta}{6} \Delta^2 \bar{f} + O(\beta^2), \quad (2.10)$$

$$\frac{\partial \bar{f}}{\partial t} - \frac{\beta}{2} \frac{\partial \Delta \bar{f}}{\partial t} + \frac{\beta}{2} (\nabla \bar{f})^2 + \bar{\eta} = 0 + O(\beta^2). \quad (2.11)$$

From now on the overbars will be omitted without fear of confusion.

2.4. Conservative form (energy-consistent approximation)

Although the above derived system is a straightforward asymptotic reduction of the system (2.1)–(2.3) it differs qualitatively from the latter because (2.10) and (2.11) does not bring about the conservation of energy. In this sense, it is an inconsistent approximation of the original system, i.e. in the asymptotic reduction a quality has been lost.

In order to find the correct energy-conserving form of the system we introduce a new variable

$$\chi = \eta - \frac{\beta}{2} \frac{\partial \Delta f}{\partial t},$$

and upon substituting it in (2.10) and (2.11), we get

$$\frac{\partial \chi}{\partial t} + \beta \nabla \cdot \chi \nabla f + \frac{\beta^2}{2} \nabla \cdot \left(\frac{\partial \Delta f}{\partial t} \nabla f \right) = -\Delta f + \frac{\beta}{6} \Delta^2 f - \frac{\beta}{2} \frac{\partial^2 \Delta f}{\partial t^2}, \quad \frac{\partial f}{\partial t} = -\frac{\beta}{2} (\nabla f)^2 - \chi.$$

The term $-\frac{1}{2} \beta^2 \nabla \cdot ((\partial \Delta f / \partial t) \nabla f)$, can be neglected within the asymptotic order $O(\beta^2)$, and the system adopts the following simple form:

$$\frac{\partial \chi}{\partial t} = -\beta \nabla \cdot \chi \nabla f - \Delta f + \frac{\beta}{6} \Delta^2 f - \frac{\beta}{2} \frac{\partial^2 \Delta f}{\partial t^2}, \quad (2.12)$$

$$\frac{\partial f}{\partial t} = -\frac{\beta}{2} (\nabla f)^2 - \chi. \quad (2.13)$$

Upon multiplying the left-hand side of Eq. (2.12) by the right-hand side of Eq. (2.13) and the right-hand side of Eq. (2.12) by the left-hand side of Eq. (2.13), adding the results and integrating over the surface region D under consideration one gets the following energy balance law:

$$E = \frac{1}{2} \int_D \left[\chi^2 + (1 + \beta \chi) (\nabla f)^2 + \frac{\beta}{6} (\Delta f)^2 + \frac{\beta}{2} (\nabla f_t)^2 \right] dx, \quad (2.14)$$

$$\frac{dE}{dt} = \oint_{\partial D} \left[(1 + \beta \chi) f_t \frac{\partial f}{\partial n} + \frac{\beta}{2} f_t \frac{\partial f_{tt}}{\partial n} + \frac{\beta}{2} f_t \frac{\partial \Delta f}{\partial n} - \frac{\beta}{2} \Delta f \frac{\partial f_t}{\partial n} \right] ds, \quad (2.14)$$

which allows us to call the system (2.12) and (2.13) “energy-consistent Boussinesq paradigm”. In our opinion, this system is a valid candidate to fulfill the Boussinesq program without unnecessary deficiencies introduced by the “simplifications” in the moving frame.

The boundary conditions that bring about the conservation of the total energy stem from the requirement that the right-hand side of (2.14) be equal to zero. There are three sets of conditions compatible with that requirement, namely

$$f_t = 0 \Rightarrow f = f_b(x, y), \quad \frac{\partial f}{\partial n} = 0 \quad \text{for } (x, y) \in \partial D, \quad (2.15)$$

$$f_t = 0 \Rightarrow f = f_b(x, y), \quad \Delta f = 0 \quad \text{for } (x, y) \in \partial D, \quad (2.16)$$

$$\frac{\partial f}{\partial n} = 0, \quad \frac{\partial \Delta f}{\partial n} = 0 \quad \text{for } (x, y) \in \partial D. \quad (2.17)$$

The third set of boundary conditions (2.17) are Neumann conditions and they specify the function f up to an arbitrary function of time. The most suitable set of boundary conditions will be selected after considering the balance law for the wave momentum.

Here is to be mentioned that both η and χ are implicit functions of the respective systems (functions for which no boundary conditions are posed) and there are no mathematical reasons to prefer one formulation over the other. Hence we will not use the conservative system in original variables.

2.5. Balance law for the pseudo-momentum

Another balance (or conservation) law holds for the wave momentum (called sometimes pseudo-momentum [10,11]). In order to derive it for the system under consideration we invoke the following identities:

$$\begin{aligned}\Delta f \nabla f &= \nabla \cdot (\nabla f \nabla f) - \frac{1}{2} \nabla (\nabla f)^2, & \Delta f_t \nabla f_t &= \nabla \cdot (\nabla f_t \nabla f_t) - \frac{1}{2} \nabla (\nabla f_t)^2, \\ \Delta^2 f \nabla f &= \nabla \cdot (\nabla \Delta f \nabla f) - \frac{1}{2} \nabla (\nabla \Delta f \cdot \nabla f), & \nabla \cdot (\chi \nabla f \nabla f) &= \nabla f \nabla \cdot (\chi \nabla f) + \frac{1}{2} \chi \nabla (\nabla f)^2,\end{aligned}\quad (2.18)$$

whose proof is standard. Note that when there is no sign for scalar product “ \cdot ” between two (or more) vectors then the outer product is meant, namely $\nabla f \nabla f$ is a dyad (a tensor).

For the system under consideration we define pseudo-momentum as follows (see also [8]):

$$\mathbf{P} = - \int_D \eta \nabla f \, dy \equiv - \int_D \left[\chi \nabla f + \frac{\beta}{2} \frac{\partial \Delta f}{\partial t} \nabla f \right] dx \, dy. \quad (2.19)$$

The relationship of the above quantity to the pseudo-momentum in classical continuum mechanics is corroborated in the next section.

Now making use of the equalities (2.18), we arrive at

$$\begin{aligned}- \frac{d\mathbf{P}}{dt} &= \int_D \left\{ - \left[\nabla \cdot (\nabla f \nabla f) - \frac{1}{2} \nabla (\nabla f)^2 \right] - \frac{1}{2} \nabla \chi^2 - \beta \left[\nabla f \nabla \cdot (\chi \nabla f) + \frac{\chi}{2} \nabla (\nabla f)^2 \right] \right. \\ &\quad \left. + \frac{\beta}{6} \left[\nabla \cdot (\nabla \Delta f \nabla f) - \frac{1}{2} \nabla (\nabla \Delta f \cdot \nabla f) \right] + \frac{\beta}{2} \Delta f_t \nabla f_t \right\} dx \, dy \\ &= \oint_{\partial D} \left[- \frac{\beta}{2} \left(\frac{\partial f_t}{\partial n} \nabla f_t - \frac{1}{2} (\nabla f_t)^2 \right) \mathbf{n} - \frac{1}{2} \chi^2 \mathbf{n} - \left(\frac{\partial f}{\partial n} \nabla f - \frac{1}{2} (\nabla f)^2 \mathbf{n} \right) \right. \\ &\quad \left. + \frac{\beta}{6} \left(\frac{\partial \Delta f}{\partial n} \nabla f - \frac{1}{2} \nabla \Delta f \cdot \nabla f \right) \mathbf{n} + \beta \chi \nabla f \frac{\partial f}{\partial n} \right] ds.\end{aligned}$$

Here it becomes clear that the most natural lateral boundary conditions are those from (2.15), because most of the terms in the balance law of the pseudo-momentum cancel reducing it to the following equation:

$$\frac{d\mathbf{P}}{dt} = \oint_{\partial D} \left\{ - \frac{\beta}{4} (\nabla f_t)^2 - \frac{1}{4} (\nabla f)^2 + \frac{\beta}{12} (\nabla \Delta f \cdot \nabla f) + \frac{1}{2} \chi^2 \right\} \mathbf{n} \, ds, \quad (2.20)$$

which transforms into a conservation law if one takes $f_b(x, y) = 0$ in (2.15).

2.6. Single-equation formulation

Upon introducing (2.13) into (2.12) the function χ is readily eliminated to obtain a single equation for the potential f , namely

$$f_{tt} + 2\beta \nabla f \cdot \nabla f_t + \beta f_t \Delta f + \frac{3\beta^2}{2} (\nabla f)^2 \Delta f - \Delta f + \frac{\beta}{6} \Delta^2 f - \frac{\beta}{2} \frac{\partial^2 \Delta f}{\partial t^2} = 0 \quad (2.21)$$

with Hamiltonian density

$$\mathcal{H} = \frac{1}{2} [f_t^2 + (\nabla f)^2 - \frac{1}{4} \beta^2 (\nabla f)^4 + \frac{1}{6} \beta (\Delta f)^2 + \frac{1}{2} \beta (\nabla f_t)^2]. \quad (2.22)$$

Now one sees that the nonlinearity of the dynamic condition (2.13) is responsible for the cubic nonlinearity of Eq. (2.21). The latter is of higher order in β , but it cannot be neglected without destroying the Galilean invariance. Boussinesq [3–5] did neglect it and the equation he obtained lacked the said invariance (for details, see Appendix A) even to the leading order $O(\beta)$.

Following [10,11] the system described by Eq. (2.21) can be re-interpreted in a field-theoretic framework (very much in the same vein as in our previous work concerning the sixth-order Boussinesq equation [8]). To this end we introduce the densities of the Lagrangian (\mathcal{L}), kinetic (\mathcal{K}) and potential (\mathcal{W}) energy, and Hamiltonian (\mathcal{H}), according to the following equations:

$$L = \int_D \mathcal{L} \, dx \, dy, \quad (2.23)$$

$$\begin{aligned} \mathcal{L} &= \mathcal{K} - \mathcal{W}, & \mathcal{H} &= \mathcal{K} + \mathcal{W}, & \mathcal{K} &= \frac{1}{2} f_t^2 + \frac{1}{4} \beta f_t (\nabla f)^2, \\ \mathcal{W} &= \frac{1}{2} [(\nabla f)^2 - \frac{1}{2} \beta f_t (\nabla f)^2 - \frac{1}{4} \beta^2 (\nabla f)^4 - \frac{1}{3} \beta (\Delta f)^2 + \frac{1}{2} \beta (\nabla f_t)^2]. \end{aligned}$$

The pseudo-momentum (or “wave momentum”) is defined in continuum mechanics as

$$\mathbf{P}^w = - \int_D \nabla f \frac{\delta \mathcal{L}}{\delta f_t} \, dx \, dy \quad (2.24)$$

(see [11, Eq. (4.42)]). The formulas (2.23) give that

$$\frac{\partial \mathcal{L}}{\partial f_t} = f_t + \frac{\beta}{2} (\nabla f)^2 - \frac{\beta}{2} \beta \Delta f_t \equiv -\eta.$$

In the last equality the definition of η from Eq. (2.11) is duly acknowledged. Then one easily recognizes in the first of the expressions in Eq. (2.24), the pseudo-momentum \mathbf{P} defined in Eq. (2.19). The appropriate boundary conditions yield the balance law for \mathbf{P}^w . More detailed discussions on the role of the pseudo-momentum can be found in [12]. It suffices only to mention that the potential function f plays here a role analogous to the displacement in the classical continuum theory.

A final comment is in order here. The term $\frac{1}{2} \beta f_t (\nabla f)^2$ enters the Lagrangian density but does not contribute to the Hamiltonian density. Such a term is called (see [11]) “gyroscopic momentum”.

2.7. One-dimensional version

Consider now the two-dimensional flow when the velocity potential and the surface elevation do not depend on the coordinate y . Naturally it is one-dimensional for the surface variables. Then the system (2.12) and (2.13) reduces to the following:

$$\frac{\partial \chi}{\partial t} + \beta \frac{\partial}{\partial x} \left(\chi \frac{\partial f}{\partial x} \right) = - \frac{\partial^2 f}{\partial x^2} + \frac{\beta}{6} \frac{\partial^4 f}{\partial x^4} - \frac{\beta}{2} \frac{\partial^4 f}{\partial t^2 \partial x^2}, \quad (2.25)$$

$$\frac{\partial f}{\partial t} + \frac{\beta}{2} \left(\frac{\partial f}{\partial x} \right)^2 = -\chi. \quad (2.26)$$

Being reminded that f has the meaning of velocity potential taken at the bottom of the layer one can revert to vertical velocity $u = f_x$ and introduce an auxiliary variable q according to $\chi = -q_x$. Upon integrating Eq. (2.25) once and acknowledging the boundary conditions one obtains

$$\frac{\partial q}{\partial t} + \beta u \frac{\partial q}{\partial x} = u - \frac{\beta}{6} \frac{\partial^2 u}{\partial x^2} + \frac{\beta}{2} \frac{\partial^2 u}{\partial t^2}, \quad (2.27)$$

$$\frac{\partial u}{\partial t} + \beta u \frac{\partial u}{\partial x} = \frac{\partial^2 q}{\partial x^2}. \quad (2.28)$$

When the region under consideration is a finite interval, say $x \in [-L_1, L_2]$, then the boundary conditions (2.15) are reduced to the following:

$$u = 0, \quad q_x = 0, \quad x = -L_1, L_2. \quad (2.29)$$

3. The family of Boussinesq equations

Because of the sometimes ambiguous usage of the terminology in the literature, it is useful to give a brief list of the different Boussinesq equations. This will help later on when comparing the numerical results obtained.

3.1. Boussinesq paradigm equation

Boussinesq attempted to describe the wave evolution in the moving frame when the motion is quasi-stationary. Then two major ways of simplifications of the original system are possible. One of them is to simplify the convective nonlinear terms neglecting the nonlinearity in (2.13) or the cubic term in (2.21). If one restricts oneself to an approximation valid only in a coordinate frame which moves to the right with the characteristic speed (equal to unity in our dimensionless variables), then following Boussinesq [3–5] one can argue that for motions that evolve slowly in the moving coordinate frame, the time derivatives are reasonably well approximated by the spatial ones and hence can be replaced by the latter. Replacing the time derivative f_t by f_x in the quadratic nonlinear term, Boussinesq arrived to

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left[u - \frac{3\beta}{2} u^2 + \frac{\beta}{2} \frac{\partial^2 u}{\partial t^2} - \frac{\beta}{6} \frac{\partial^2 u}{\partial x^2} \right] \quad (3.1)$$

as an intermediate stage to his final simplification. We call (3.1) Boussinesq paradigm equation (BPE). We treated it numerically by a fully implicit conservative scheme in [6]. It should be mentioned that Eq. (3.1) appears also in the theory of longitudinal (acoustic) vibrations of rods (see, e.g., [16]).

The above simplification destroyed the Galilean invariance of the system.

3.2. Boussinesq's Boussinesq equation

The second step of the simplification proposed by Boussinesq is concerned with the linear terms. Upon substituting $\partial^2/\partial x^2$ for $\partial^2/\partial t^2$ one arrives at

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left[u - \frac{3\beta}{2} u^2 + \frac{\beta}{3} \frac{\partial^2 u}{\partial x^2} \right], \quad (3.2)$$

which we call Boussinesq's Boussinesq equation (BBE). It is linearly unstable and cannot be solved numerically. For the same reason it does not correspond to a particular physical reality. Yet it is of historical significance since it is the equations derived by Boussinesq himself who found also the analytical solution for the solitary wave. BBE (3.2) appears also in the continuum limit for lattices (see, e.g., [17,18]) and is a good toy-object for soliton research, because unlike the original DSWS system, it is fully integrable [19].

3.3. Regularized long-wave equation

Another option compatible with the original Boussinesq simplifications is to replace the dispersion term expressed by the fourth spatial derivative by a term containing the mixed fourth derivative only. Thus the total dispersion is

given by $\frac{1}{3}\beta u_{txx}$. Then

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2}{\partial x^2} \left[u - \frac{3\beta}{2} u^2 + \frac{\beta}{3} \frac{\partial^2 u}{\partial t^2} \right]. \quad (3.3)$$

Eq. (3.3) contains the mixed fourth derivative but does not involve the fourth spatial derivative. In the literature it is called regularized long-wave equation (RLWE) (see [2,20]) the very name suggesting that something had to be regularized in the shallow-layer equations. The fact is that the equation which appears naturally in Boussinesq's type of derivations (see the above derivation of BPE) is well posed and needs some effort to get it "de-regularized" and to make it incorrect. Note that RLWE is linearly stable [2,13,20], but it is not fully integrable (just as the original DSWS system is not).

3.4. Hamiltonian structure for BPE

Before turning to the numerical investigation it is important to recall here the fact (see [6]) that BPE (3.1) can be rewritten as a system

$$u_t = q_{xx}, \quad q_t = u - \frac{dU}{du} + \frac{\beta}{2} u_{tt} - \frac{\beta}{6} u_{xx}, \quad U(u) \stackrel{\text{def}}{=} \frac{\beta}{2} u^3. \quad (3.4)$$

In [6] it is shown that for boundary condition (2.29) the Hamiltonian structure for the above system reads

$$\frac{dM}{dt} = 0, \quad M = \int_{-L_1}^{L_2} u \, dx, \quad (3.5)$$

$$\frac{dE}{dt} = 0, \quad E = \frac{1}{2} \int_{-L_1}^{L_2} \left[q_x^2 + u^2 - 2U(u) + \frac{\beta}{2} u_t^2 + \frac{\beta}{6} u_x^2 \right] dx, \quad (3.6)$$

$$\begin{aligned} \frac{dP}{dt} &= \left[\frac{u^2}{2} - \left(u \frac{dU}{du} - U(u) \right) - \frac{\beta}{4} u_t^2 - \frac{\beta}{12} u_x^2 \right]_{-L_1}^{L_2} = - \frac{\beta}{12} u_x^2 \Big|_{-L_1}^{L_2}, \\ P &= \int_{-L_1}^{L_2} u \left(q_x + \frac{\beta}{2} u_{xt} \right) dx = \int_{-L_1}^{L_2} \left(u q_x - \frac{\beta}{2} u_t u_x \right) dx. \end{aligned} \quad (3.7)$$

Thus we see that BPE is preferable over RLWE because for the former the *wave mass* is conserved alongside with the *energy*. In other words, the presence of the spatial fourth derivative requires boundary conditions whose satisfaction brings about the conservation of the wave mass which is also a property of the original hydrodynamic problem.

It is to be mentioned that the system (3.4) looks rather similar to the original DSWS (2.12) and (2.13), but there is a significant difference due to the fact that the latter is Galilean invariant, while the former is not. Respectively, the Lagrangian and Hamiltonian densities for the two systems are different.

Note that the derivations of the present section are not restricted to one-dimensional surface motions. One-dimensional Boussinesq equations are considered only for the sake of comparison with the classical works.

4. The solitary wave of DSWS

The sech solution of the BPE is given by (see [6])

$$u = \frac{a}{\cosh^2[b(x - ct)]}, \quad a = \frac{c^2 - 1}{\beta}, \quad b = \sqrt{\frac{c^2 - 1}{2\beta(c^2 - \frac{1}{3})}}, \quad (4.1)$$

where c is the phase speed or *celerity* of the wave. The sech-es exist for all supercritical phase speeds $c > 1$ and for subcritical speed in the interval $0 < c < \sqrt{1/3}$. Only the supercritical sech-es are of physical relevance here because for small β the subcritical sech-es are not long waves.

Although more complex than any of the Boussinesq equations, the DSWS system (2.27) and (2.28) does admit a localized solution which is stationary in the moving frame $x - ct$. After some algebra we find a sech-like solution, namely

$$u = \frac{a \operatorname{sign}(c)}{\frac{1}{2}(|c| - 1) + \cosh^2[b(x - ct)]}, \quad a = \frac{c^2 - 1}{\beta}, \quad b = \sqrt{\frac{c^2 - 1}{2\beta(c^2 - \frac{1}{3})}}. \tag{4.2}$$

The above solitary-wave solution exists in the same range as the BPE solitons (4.1) do. The very fact that DSWS admits a sech solution, means that the Boussinesq simplifications in the moving frame are unnecessary. The sech-like solution (4.2) is the candidate for Russell’s “Great (Permanent) Wave”.

Comparing (4.1) and (4.2) reveals that the only difference is the term $\frac{1}{2}(|c| - 1)$ which means that in the weakly nonlinear limit $|c - 1| \sim O(\sqrt{\beta}) \ll 1$ the two models will give quantitatively similar predictions as far as the shape of the solitary wave is concerned. The difference between (4.1) and (4.2) in the “tails” of the solution is small even for arbitrary c . The said difference is significant only near the origin of the coordinate system and then only for significantly supercritical c .

In order to keep within the long-wave approximation we consider slightly supercritical celerities $c^2 = 1 + \beta$ and calculate the shapes of the solitary waves of DSWS and BPE. Fig. 1 shows the comparison between the two solutions. The DSWS wave is always of smaller amplitude than the BPE one. For $\beta < 0.1$ the differences are quantitatively very small and it is hard to distinguish between the two models. Hence in order to find which model corresponds better to the experiment one has to consider somewhat larger $\beta < 1$. A good case for comparison with the experiments of Russell could be $\beta = 0.2$.

The subcritical case is formally outside the scope of the present work since it occurs for $|c| < \sqrt{1/3} \approx 0.57735$ which is not asymptotically close to unity and therefore the solitary waves are not long waves. Yet, knowing the shapes of the subcritical solitary waves is of importance for both understanding the balance of nonlinearity and dispersion and the results of the numerical investigation of the evolution of colliding solitary waves. Note that unlike the supercritical ones, the subcritical waves are depressions.

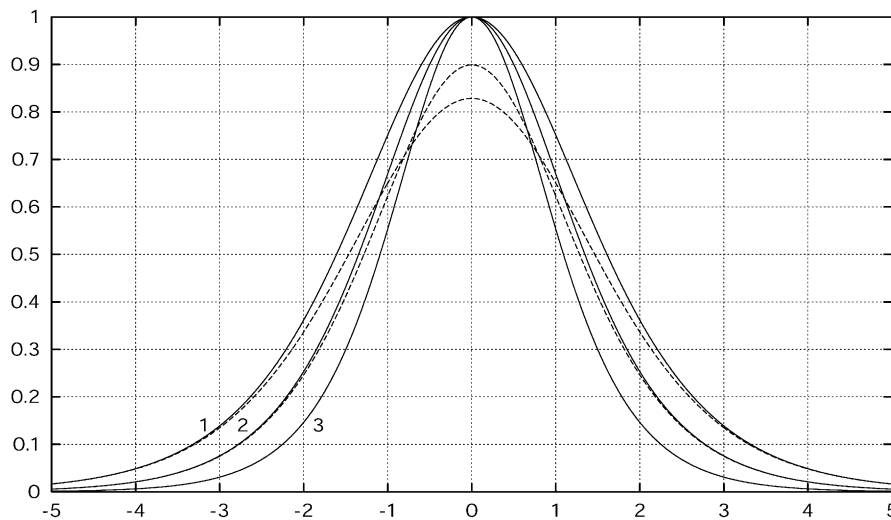


Fig. 1. Comparison between BPE (—) and DSWS (---) solitary waves for supercritical phase speeds $c = \sqrt{1 + \beta}$: (1) $\beta = 1$; (2) $\beta = 0.5$; (3) $\beta = 0.1$.

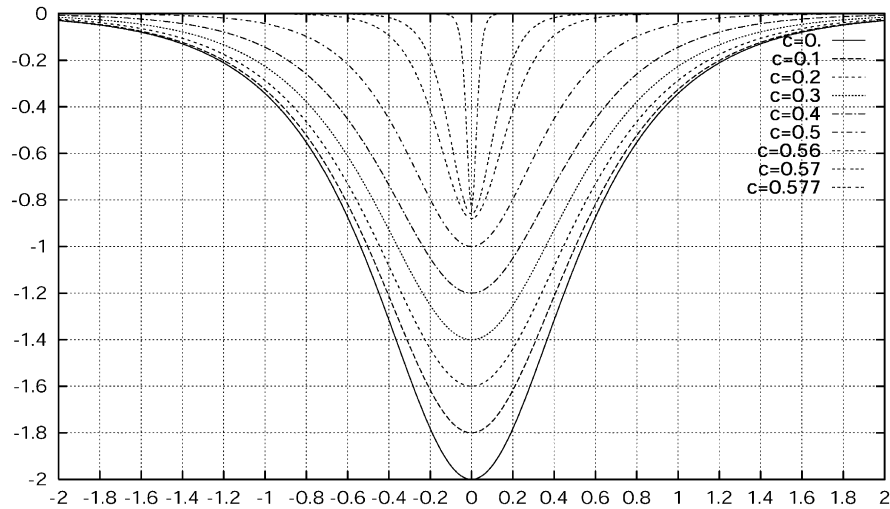


Fig. 2. DSWS sech-es for subcritical phase speeds $c < 1/\sqrt{3}$ and $\beta = 1$.

In Fig. 2, we present the shapes of subcritical solitary waves for $\beta = 1$ and different phase speeds. The figure shows that for $c = 0$ the depressions are of largest amplitude but of least steepness. With the increase of c they become narrower and their amplitudes decrease. For the limiting case $c \rightarrow 1/\sqrt{3}$ the span of the localized solution tends to zero becoming thus of infinitely short length. The support for $\beta = 0.1$ is about three times shorter than for $\beta = 1$ which means that even for $c = 0$ it is significantly lesser than unity. This means that one cannot speak about long-wave solutions in this case.

The main difference between surface elevations and surface depressions is that one can have a depression at rest ($c = 0$), while the supercritical surface elevations exist only in motion. Using the pseudo-particle analogy, the supercritical solitons are *tachyons*, while the subcritical ones are particles with rest mass. The Lorentzian kinematics of the pseudo-particles considered in the present work is rather peculiar — their mass is reduced to nil when their velocity approaches the limit $|c| < 1/\sqrt{3} \approx 0.57735$. A more general discussion on the Lorentzian kinematics and dynamics of solitons in variety of different nonlinear systems can be found in [12].

5. Numerical experiments on soliton dynamics

In this section, we present the wave profile we obtained numerically by means of a conservative scheme which is an extension of the schemes previously developed by the author. The specific numerical properties of the scheme will be reported elsewhere.

A manifestation of the advantages of the present model is that the vertical component of velocity in the left-moving wave is negative which is physically correct while all of the models without Galilean invariance predict positive vertical velocity (compare (4.1) to (4.2)). This is shown in the upper panel of Fig. 3 where the collisions of BPE- and DSWS solutions are presented together.

In order to illustrate the role of Galilean invariance we juxtapose the trajectories of the colliding solitary waves as described by BPE and DSWS (see the lower panel of Fig. 3). It is clearly seen that the oversimplification of the nonlinear term in the moving frame crippled BPE to the extent that it gives a wrong phase shift (negative). In addition the magnitude of the phase shift is twice as big as the magnitude of the positive phase shift of DSWS.

The essential properties of different Boussinesq models are summarized in Table 1. The new model (DSWS) stands apart from the others which we attribute to its Galilean invariance.

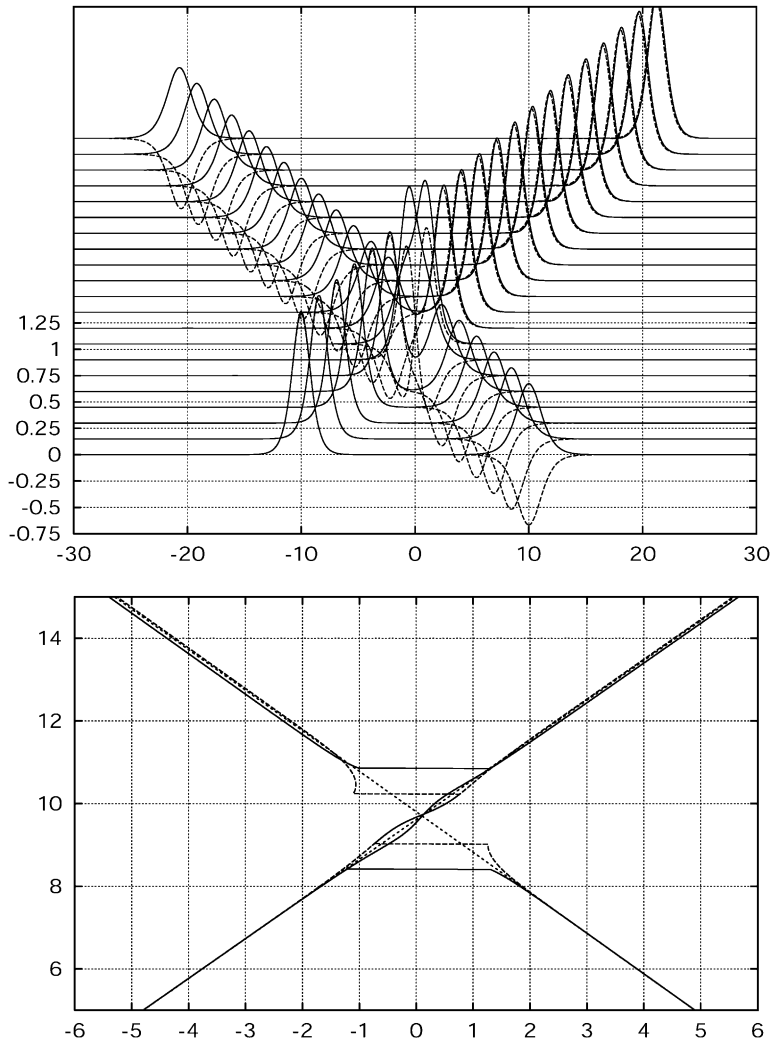


Fig. 3. The head-on collision (upper panel) and the phase shift (lower panel) for $\beta = 0.06$, $c_l = 1.04$, and $c_r = -1.02$: (—) BPE; (---) DSWS; (···) projected trajectories for noninteracting solitary waves.

Table 1
Comparison between the different models of Boussinesq paradigm

Model	BBE	RLWE	BPE	DSWS
Dispersion	Incorrect	Incorrect	Correct	Correct
Linear stability	No	Yes	Yes	Yes
Conservation of wave “mass”	Yes	No	Yes	Yes
sech solution	Yes	Yes	Yes	Yes
Phase shift	Negative	Negative	Negative	Positive
Galilean invariance	No	No	No	Yes
Vertical velocity in right wave	Positive (incorrect)	Positive (incorrect)	Positive (incorrect)	Negative (correct)

6. Conclusions

The long-wave approximation is considered for the flow of inviscid liquid in shallow layer. The Boussinesq derivation is applied for obtaining an approximate expression for dispersion when $\beta \equiv H^2/L^2 \ll 1$.

The variables are scaled by the small parameter and a special substitution is used replacing the surface elevation by another implicit function. After neglecting the terms of order $O(\beta^2)$ an asymptotically equivalent system is obtained for which we show the Hamiltonian structure and prove that a conservation law for energy holds (energy-consistent derivation). Thus a novel Boussinesq system is obtained which provides the basis for a pseudo-particle approach with Galilean invariance which is an advantage over the different Boussinesq equations available from the literature.

A new analytical solution of type of solitary wave is obtained and compared to classical Boussinesq sech. Interaction of two solitary waves as pseudo-particles is simulated numerically. It is shown to be quantitatively close to the known simulations of Boussinesq solitons as far as the shape of the wave system is concerned. The Galilean invariance of the system brings about a physically correct negative vertical velocity in the left-running wave and a positive phase shift which is a qualitative improvement over the ubiquitous Boussinesq models.

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Appendix A. Galilean invariance in terms of velocity potential

In order to facilitate the reader and to make the paper self-contained we outline here the notion of Galilean invariance in terms of the potential function, ϕ , and the surface elevation function, η .

By definition a system is said to be Galilean invariant if the governing equations remain the same when changing to any coordinate system moving with a constant velocity. Throughout the present section we use an arrow over a given symbol to designate vectorial variables. Here \vec{x} or $\vec{\xi}$ is the basis-vector associated with the coordinates in the plane of the bottom of layer, and z is the normal coordinate. Then the Galilean invariance of the physical laws means that the original velocity vector, \vec{v} , and shape function, η , are related to the respective variables in the moving frame $\vec{\xi} = \vec{x} - \vec{c}t$ according to the transformation

$$\vec{v}(\vec{x}, z; t) = \vec{V}(\vec{\xi}, z; t) + \vec{c}, \quad \eta(\vec{x}, z; t) = \hat{\eta}(\vec{\xi}, z; t).$$

Since the potential functions in the original and the moving frames are defined, respectively, as $\nabla_x \phi(\vec{x}, z; t) \stackrel{\text{def}}{=} \vec{v}(\vec{x}, z; t)$ and $\nabla_{\vec{\xi}} \hat{\phi}(\vec{\xi}, z; t) \stackrel{\text{def}}{=} \vec{V}(\vec{\xi}, z; t)$, then the Galilean invariance requires that

$$\nabla_x \phi(\vec{x}, z; t) = \nabla_{\vec{\xi}} \hat{\phi}(\vec{\xi}, z; t) + \vec{c}.$$

After integration the latter gives

$$\phi(\vec{x}, z; t) = \hat{\phi}(\vec{\xi}, z; t) + \vec{c} \cdot \vec{\xi} - c^2 t,$$

where the integration “constant” (function of t only) is specified as $-c^2 t$. Then

$$\frac{\partial \phi}{\partial t} = \frac{\partial \hat{\phi}}{\partial t} - \vec{c} \cdot \nabla_{\vec{\xi}} \hat{\phi} - c^2, \quad \frac{\partial \phi}{\partial z} = \frac{\partial \hat{\phi}}{\partial z}, \quad \frac{\partial \eta}{\partial t} = \frac{\partial \hat{\eta}}{\partial t} - \vec{c} \cdot \nabla_{\vec{\xi}} \hat{\eta}, \quad \frac{\partial \eta}{\partial z} = \frac{\partial \hat{\eta}}{\partial z}.$$

For the convective terms we get

$$\begin{aligned}\frac{\partial \eta}{\partial t} + \nabla_x \phi \cdot \nabla_x \eta &= \frac{\partial \hat{\eta}}{\partial t} - \vec{c} \cdot \nabla_\xi \hat{\eta} + (\nabla_\xi \hat{\phi} + \vec{c}) \cdot \nabla_\xi \hat{\eta} \equiv \frac{\partial \hat{\eta}}{\partial t} + \nabla_\xi \hat{\phi} \cdot \nabla_\xi \hat{\eta}, \\ \frac{\partial \phi}{\partial t} + \frac{1}{2} (\nabla_x \phi)^2 &= \frac{\partial \hat{\phi}}{\partial t} - \vec{c} \cdot \nabla_\xi \hat{\phi} - c^2 + \frac{1}{2} (\nabla_\xi \hat{\phi})^2 + (\nabla_\xi \hat{\phi}) \cdot \vec{c} + c^2 \equiv \frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} (\nabla_\xi \hat{\phi})^2.\end{aligned}$$

Let us mention in passing that Boussinesq neglected the term $\frac{1}{2} (\nabla_\xi \hat{\phi})^2$ on the heuristic basis that its effect is smaller than the contribution of the other nonlinear term $\nabla_\xi \hat{\phi} \cdot \nabla_\xi \hat{\eta}$ (see Section 2.6). Yet, it is clearly seen that the said term is crucial for preserving the Galilean invariance of the system to the leading asymptotic order $O(1)$.

Upon introducing the above formulas into (2.2) and (2.3), we get the same system for the variables $\hat{\phi}$, $\hat{\eta}$ in the moving frame, namely

$$\frac{\partial \hat{\eta}}{\partial t} + \nabla_\xi \hat{\phi} \cdot \nabla_\xi \hat{\eta} = \frac{1}{\beta} \frac{\partial \hat{\phi}}{\partial z}, \quad (\text{A.1})$$

$$\frac{\partial \hat{\phi}}{\partial t} + \frac{1}{2} (\nabla_\xi \hat{\phi})^2 + \frac{1}{2\beta} \left(\frac{\partial \hat{\phi}}{\partial z} \right)^2 + \hat{\eta} = 0. \quad (\text{A.2})$$

In the same fashion the system for the surface variables $\bar{\eta}$, \bar{f} can be treated. In doing this one should be reminded that $\eta = \beta \bar{\eta}$, $f = \beta \bar{f}$. Hence the Galilean transformation reads

$$\bar{f}(\vec{x}, t) = \frac{1}{\beta} [\beta \hat{f}(\vec{\xi}, t) + \vec{c} \cdot \vec{\xi} - c^2 t] = \hat{f}(\vec{\xi}, t) + \frac{\vec{c} \cdot \vec{\xi}}{\beta} - \frac{c^2 t}{\beta}.$$

Then

$$\begin{aligned}\frac{\partial \bar{f}}{\partial t} &= \frac{1}{\beta} \left[\frac{\partial \beta \hat{f}}{\partial t} - \vec{c} \cdot \nabla_\xi (\beta \hat{f}) - c^2 \right] = \frac{\partial \hat{f}}{\partial t} - \frac{\vec{c} \cdot \nabla_\xi \hat{f}}{\beta} - \frac{c^2}{\beta}, \quad \nabla_x \bar{f} = \nabla_\xi \hat{f} + \frac{\vec{c}}{\beta}, \\ \frac{\partial (\Delta \bar{f})}{\partial t} &= \frac{\partial (\Delta_\xi \hat{f})}{\partial t} - \vec{c} \cdot \nabla_\xi (\Delta_\xi \hat{f}), \quad \Delta \bar{f} = \Delta_\xi \hat{f}, \quad \frac{\partial \bar{\eta}}{\partial t} = \frac{\partial \hat{\eta}}{\partial t} - \vec{c} \cdot \nabla_\xi \hat{\eta},\end{aligned}$$

and the system (2.10) and (2.11) has the same shape for the surface variables $\hat{\eta}$, \hat{f} , namely

$$\frac{\partial \hat{\eta}}{\partial t} + \beta \nabla_\xi \cdot \hat{\eta} \nabla_\xi \hat{f} = -\Delta_\xi \hat{f} + \frac{\beta}{6} \Delta_\xi^2 \hat{f}, \quad (\text{A.3})$$

$$\frac{\partial \hat{f}}{\partial t} - \frac{\beta}{2} \left[\frac{\partial (\Delta_\xi \hat{f})}{\partial t} - \vec{c} \cdot \nabla_\xi (\Delta_\xi \hat{f}) \right] + \frac{\beta}{2} (\nabla \hat{f})^2 = -\hat{\eta} \quad (\text{A.4})$$

save the term $-\vec{c} \cdot \nabla_\xi (\Delta_\xi \hat{f})$ which contributes to the order $O(\beta)$. In the same time, what is said in the precedence about the importance of the $O(1)$ -term $\frac{1}{2} (\nabla_\xi \hat{\phi})^2$, also holds true for the term $\frac{1}{2} (\nabla_\xi \hat{f})^2$ in the last equation. Thus we arrive at a system which is Galilean invariant to the leading order $O(1)$ for the scaled variables \bar{f} , $\bar{\eta}$ (or \bar{f} , $\bar{\chi}$).

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