



DISSIPATIVE QUASI-PARTICLES: THE GENERALIZED WAVE EQUATION APPROACH

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Generalized Wave Equations containing dispersion, dissipation and energy-production (GDWE) are considered *in lieu* of dissipative NEE as more suitable models for two-way interaction of localized waves. The quasi-particle behavior and the long-time evolution of localized solutions upon take-over and head-on collisions are investigated numerically by means of an adequate difference scheme which represents faithfully the balance/conservation laws. It is shown that in most cases the balance between energy production/dissipation and nonlinearity plays a similar role to the classical Boussinesq balance between dispersion and nonlinearity, namely it can create and support localized solutions which behave as quasi-particles upon collisions and for a reasonably long time after that.

Keywords: Solitary waves; dissipation; generalized wave equation.

1. Introduction

A well-known property of nonlinear waves in the absence of dispersion or dissipation is the tendency of their fronts to become steeper until a discontinuity is formed. A major advance in the understanding of weak nonlinear wave phenomena was made when Russell [1838] discovered the permanent (“The Great”) wave. Boussinesq put forward [Boussinesq, 1871, 1872] a new paradigm according to which nonlinearity can be balanced by dispersion, making the wave assume self-preserving localized shape propagating with constant phase speed. Because of the single-hump appearance of the wave, it was called solitary wave.

The significance of the self-preserving nonlinear dispersive waves was demonstrated by the numerical study of Fermi and coworkers (see [Fermi *et al.*, 1965] where a chain of 64 weakly nonlinearly coupled harmonic oscillators did not reach the stage of equipartition of energy between different states. Later on, [Zabusky & Kruskal, 1965] demonstrated that the initial localized wave profile actually

recurred in numerical simulations for long enough times. They coined the name “soliton” to delineate the individualized behavior of the solitary nonlinear wave. This key work spurred a flurry of mathematical research that unearthed the integrability properties of the different nonlinear wave models, such as Korteweg–de Vries equation, Schroedinger equation, and Klein–Gordon equation, among others.

The significance of the above discovery lies in the introduction of the new physical model called “quasi-particles” (or “pseudoparticles”) signifying localized waves with *corpuscular* properties. For integrable, conservative, and/or Hamiltonian system the existence of quasi-particles do not come as a surprise. However, as demonstrated by [Mauguin & Christov, 2001], the wave systems with just three conservation laws for mass, energy and wave momentum (pseudomomentum) can also exhibit full-fledged corpuscular behavior of nonlinear wave solutions. Such systems are most often encountered in real-life physical models. The last decade is marked by the realization that the

physical significance of the corpuscular behavior of nonlinear localized waves is somewhat deeper than the mere mathematical feature called integrability.

Adding dissipative effects in the models does not preclude the existence of quasi-particles. Christov and Velarde [1995] already treated the case of NEE and showed the role of the energy input/dissipation in the overall balance defining the corpuscular properties of the localized solutions. To describe the situation the coinage “Dissipative Solitons” was introduced.

In the present paper we address the issue of adding dissipation in a generalized wave equation that already contains dispersion and nonlinearity. The rationale for this is that one cannot speak about particles without taking into account the Newtons law for inertia. We study the *corpuscular* behavior of the localized solutions of Generalized Dissipative Wave Equations (small GDWE). Some preliminary results on GDWE have already been obtained in this direction in previous works [Christov & Velarde, 1994a, 2001].

Here we endeavor to answer the following questions:

1. Does the incorporation of production/dissipation terms in a conservative system preclude the corpuscular behavior of the localized wave solutions in moderate times? How large are coefficients of the energy input/dissipation terms to be in order to have significant deviation from the respective conservative system?
2. Is it possible to observe corpuscular nonlinear waves (quasi-particles) in systems without dispersion, i.e. when the only effect to balance the steepening effects of nonlinearity is the energy production/dissipation? If yes, what are the properties of the trajectories of the centers of the quasi-particles, e.g. phase shift after a collision, etc.

2. Models Involving Generalized Wave Equations

The flow in thin-liquid layer with free surface furnish is one of the best studied examples of nonlinear localized waves.

Boussinesq derived the first generalized wave equation (BE) for the flow in shallow inviscid layer. He found analytically the localized solution [Boussinesq, 1872] of his equation and demonstrated that the solitary wave propagates without

change of shape similarly to the propagation of the linear waves. Korteweg and de Vries [1895] derived in 1895 a Nonlinear Evolution Equation (KdV) in the moving frame.

When the dissipative effects are taken into account, the models of nonlinear elastic beams yield to the same Boussinesq equation (see e.g. [Samsonov, 1990]). As shown in [Christov & Velarde, 1994b] (in somewhat heuristic fashion), the flow in thin viscous liquid layer is governed once again by a Generalized Dissipative Wave Equation (GDWE) of the following type

$$u_{tt} = \left[\gamma^2 u - \frac{\alpha}{2} u^2 - \beta \gamma u_{xx} - \alpha_4 u_{xxt} - \alpha_2 uu_t \right]_{xx}, \quad (1)$$

which encompasses also the oscillations of elastic beams. Here α is the amplitude coefficient, γ is the phase speed of the small disturbances, $(\beta\gamma)$ is the dispersion coefficient, α_2 is the coefficient of energy-production term, and α_4 is the dissipation coefficient.

We show here that in a frame which moves with the characteristic speed γ , Eq. (1) reduces to a Nonlinear Evolution Equation (NEE). To this end we consider the “left-going” moving frame and introduce new independent coordinates and sought function

$$t_1 = \frac{1}{2} t, \quad x_1 = x - \gamma t, \quad u(t, x) = u_1(t_1, x_1).$$

The different derivatives are expressed as follows

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u_1}{\partial x_1}, & \frac{\partial u}{\partial t} &= \frac{1}{2} \frac{\partial u_1}{\partial t_1} - \gamma \frac{\partial u_1}{\partial x_1}, \\ \frac{\partial^2 u}{\partial t^2} &= \frac{1}{4} \frac{\partial^2 u_1}{\partial t_1^2} - \gamma \frac{\partial^2 u_1}{\partial t_1 \partial x_1} + \gamma^2 \frac{\partial^2 u_1}{\partial x_1^2}. \end{aligned} \quad (2)$$

If we consider motions for which the evolution in the moving frame is very slow (and that is precisely the condition under which all kinds of KdV-type equations are being derived), then we can disregard the local-time derivatives with respect to the local spatial derivatives in the sense, that

$$\left| \frac{\partial^m u_1}{\partial t_1 \partial x_1^{m-1}} \right| \ll \gamma \left| \frac{\partial^m u_1}{\partial x_1^m} \right|, \quad \left| \frac{\partial^2 u_1}{\partial t_1^2} \right| \ll \gamma \left| \frac{\partial^2 u_1}{\partial t_1 \partial x_1} \right|.$$

Upon introducing (2) into Eq. (1) and neglecting the terms according to the above scheme we

arrive at the following approximate equation

$$\gamma \frac{\partial^2 u_1}{\partial x_1 \partial t_1} = \frac{\partial^2}{\partial x_1^2} \left[-\frac{\alpha}{2} \gamma u_1^2 - \alpha_2 \gamma \frac{\partial u_1}{\partial x_1} - \beta \gamma \frac{\partial^2 u_1}{\partial x_1^2} - \alpha_4 \gamma \frac{\partial^3 u_1}{\partial x_1^3} \right], \quad (3)$$

After dividing both sides of the equation by γ and disregarding the index “1”, without fear of confusion we arrive at

$$u_t + \alpha u u_x + \alpha_2 u_{xx} + \beta u_{xxx} + \alpha_4 u_{xxxx} = 0, \quad (4)$$

which presents the weakly nonlinear approximation to the equations for the capillary flow in thin viscous layer. Kapitza [1948] initiated the investigation of the role of viscosity and surface tension in thin-layer flows. Benney [1966] carried out the asymptotic simplifications for the viscous case. Homsy [1974] derived the correct weakly-nonlinear approximation in the absence of dispersion which is now known as Kuramoto–Sivashinsky KS equation. The NEE (4) was treated extensively in the literature (see, e.g. [Chang, 1987; Elphick *et al.*, 1991]). The numerical simulations of [Christov & Velarde, 1995] show that the localized solutions of KdV–KS behave in many instances like solitons which warrant the coinage “dissipative solitons”.

3. Mass Conservation and Energy Balance Law

In dissipative systems there is no energy conservation. Rather a balance between the energy production and dissipation holds. Upon introducing an auxiliary function q , Eq. (1) is transformed to the following b.v.p.

$$u_t = q_{xx}, \quad (5)$$

$$q_t = \gamma^2 u - \frac{\alpha}{2} u^2 - \beta u_{xx} - \alpha_4 u_{xxt} - \alpha_2 u_t,$$

$$u = q_x = 0, \quad \text{for } x = -L_1, L_2, \quad (6)$$

where $-L_1, L_2$ are the values of the spatial coordinate at which we truncate the infinite interval (so-called “actual infinities”). For the “mass” and energy of the wave system,

$$M \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} u dx,$$

$$E \stackrel{\text{def}}{=} \int_{-L_1}^{L_2} \frac{1}{2} \left[\gamma^2 u^2 + q_x^2 - \frac{1}{6} \alpha u^3 + \beta u_x^2 \right] dx,$$

the following conservation and balance laws are derived

$$\frac{dM}{dt} = 0, \quad \frac{dE}{dt} = \alpha_2 \int_{-L_1}^{L_2} u_t^2 dx - \alpha_4 \int_{-L_1}^{L_2} u_{xt}^2 dx \quad (7)$$

in a previous work [Christov & Velarde, 1994a].

Here we solve the system (5) by means of the implicit difference scheme from [Christov & Velarde, 1994a, 2001] faithfully representing the balance (conservation) laws (7).

In the interval $x \in [-L_1, L_2]$ we consider a uniform grid with spacing h and total number of points N , namely

$$x_i = -L_1 + (i-1)h, \quad h = \frac{L_1 + L_2}{N-1},$$

which is adequate for the purposes of the present work. For different cases we use different numbers of grid points ranging from 2000 to 10 000.

4. Corpuscular Solutions of GDWE

The problem under consideration has several governing parameters and it is important to reduce part of the freedom in order to concentrate on the physical aspects of the results. For instance, the nonlinear coefficient can be rescaled so that we set $\alpha = 3$. The role of the phase speed γ has been studied in [Christov & Velarde, 2001]. The larger the phase speed γ , the more stable the evolution of the quasi-particles, i.e. the GDWE is more adequate for studying the corpuscular behavior of nonlinear waves for larger γ . In the first subsection we choose $\gamma = 1$ to comply with the NEE calculations. In the second subsection we resort to the case $\gamma = 3$ which provides some advantages in the presentation in comparison with $\gamma = 1$ since the dimensionless time and all of the other parameters can be rescaled by γ .

4.1. Small energy dissipation (production) rates

The first question to be answered is about the threshold for the parameter α_4 above which the dissipation becomes significant. It is well known now that the dispersion stabilizes the structures in KdV–KS equation [Chang, 1987; Christov & Velarde, 1995; Elphick *et al.*, 1991]. Hence one can

expect a more conspicuous corpuscular behavior of the structures when dissipation (α_2) and production (α_4) terms are small in comparison with β . In these calculations we keep the dispersion coefficient of order of unity and for definiteness we set $\beta = 1$.

In the dissipationless limit, the GDWE considered in the current paper reduces to the so-called ‘‘Proper’’ Boussinesq Equation. The solitary waves (the *sech*-solitons) of the latter are subcritical, i.e. they exist for phase speeds $c < \gamma = 1$. Contrary to the intuition derived from the supercritical case, the solitons here are taller when they are slower. In a previous work on Boussinesq model [Christov & Velarde, 1994b] we showed also that a blow-up occurs when $c < c^* \approx 0.866$. For this reason we consider here the interaction of two *sech*-es with phase speeds $c_l = 0.9$ and $c_r = 0.95$, respectively. The subscript l refers to the solitary wave on the left, while r denotes the right quasi-particle.

We begin with the purely dissipative case when there is no energy input, i.e. when $\alpha_2 = 0$. It is clear that self-sustained localized structure cannot exist on long times without energy production [see the energy balance (7)].

We first investigate the interaction of two initially *sech*-shaped localized waves (‘‘humps’’) for very long times. The amplitudes of the ‘‘humps’’ do not decrease appreciably and they do not radically change their shapes (save some broadening

of the support). The phase shift is quantitatively similar to the phase shift of the conservative Boussinesq equation. It means that $\alpha_4 \leq 0.4$ defines the threshold of very small dissipation. This tendency is observed until rather large $\alpha_4 \sim O(10)$. Once again the initial *sech*-es do not decrease much in amplitude but the support broadens more significantly. For $\alpha_4 = 4$ the first notable signs of nonmonotone shapes occur at long time.

The next question to be answered is about the relative importance of the energy production, the latter being governed by the coefficient α_2 . To this end we have examined cases with different magnitudes for α_2 for a given dissipation coefficient α_4 .

The gist of our findings is that α_2 must be of order of magnitude lesser than α_4 in order to have a steady-state solution. This can be attributed to the fact that all amplitude-equation models considered here are subject to long-wave instability. This phenomena is better illustrated by the energy balance-law (7). When the wave has very long spatial scale, then the magnitude of the term u_t^2 is much larger than the last term in (7) the latter involving u_{xt}^2 . Hence, for long waves (large spatial extent L of the wave) $|u_{tx}| \sim L^{-1}|u_t|$ which means that the first term (the energy input) on the right-hand side (r.h.s) of (7) dominates the second one (the dissipation) and the r.h.s. becomes positive. Note, that this effect has nothing to do with the properties

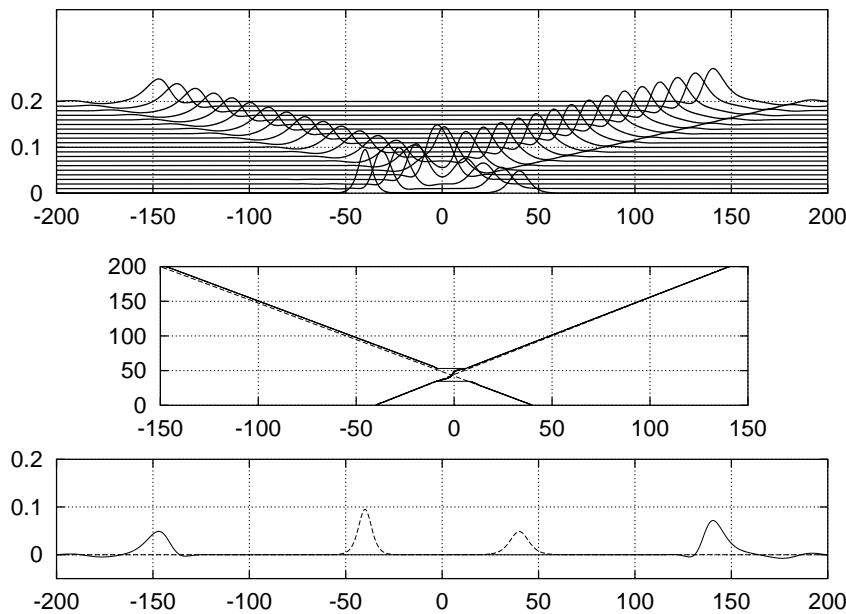


Fig. 1. $\alpha_2 = 0.5$, $\alpha_4 = 4$. Upper panel: the interaction of solitons. Middle panel: trajectories of the centers. Lower panel: the profile (dashed line) in the initial moment and (solid line) at the end of the evolution.

of the difference scheme involved. As a result, the energy increases indefinitely until the solution eventually blows-up. For the short-length waves the situation is reversed: the dissipative term dominates the energy-input term, as a result the r.h.s of (7) becomes negative and the short waves disappear in the long term. Here the eventual impurities of the difference scheme can upset this balance and lead to physically irrelevant results. Special care is taken to this end when devising the difference scheme of the present paper.

Figure 1 shows the long-wave instability scenario for a large $\alpha_4 = 4$ when even a relatively small $\alpha_2 = 0.5$ made the production term compensate and even overcome the dissipation term. As a result the long-wave instability onsets and the quasi-particles gradually accelerate after separating at collision.

4.2. Purely dissipative case ($\beta = 0$)

The dispersionless case $\alpha_3 = 0$ is the hardest to treat since the stabilizing role of the dispersion is absent. This case is of primary importance for answering the question of whether energy production/dissipation systems can give birth to localized solutions of quasi-particle type. A positive answer to this question was given by Christov and Velarde [1995] for the heteroclinic solutions (sometimes called “kinks”) of the type of hydraulic jumps. In the same work it has been shown that the NEE model does not admit stable evolution of “hump” shapes after collisions.

For the sake of definiteness we choose here $\alpha_2 = \alpha_4 = 1$.

The shape of a stationary localized wave $u(x, t) = v(\xi)$, $\xi = x - ct$ is governed by the the following b.v.p.

$$(\gamma^2 - c^2)v + \alpha_1 v^2 - \alpha_4 c v''' + \alpha_2 c v' = 0, \quad v \rightarrow 0 \quad \text{for} \quad \xi \rightarrow \pm\infty. \quad (8)$$

Denoting $v = cF(\xi)$ one gets

$$-sF + \alpha_1 F^2 + \alpha_4 F''' + \alpha_2 F' = 0, \quad s = -(\gamma^2 - c^2)c^{-1}.$$

Christov and Velarde [1993] had shown that only two solutions of homoclinic types exist, say $F_+(\xi)$ and $F_-(\xi)$, that are mirror images to each other, namely $F_-(\xi) = -F_+(-\xi)$. These are shown in Fig. 2. The respective eigenvalues are $s = \pm 2\hat{s}$, $2\hat{s} \approx 1.216$.

For the solution $v(\xi)$ of GDWE there exist four different possibilities for each F_+ and F_- , e.g. $v(\xi) = c\varepsilon_1 F_+(\varepsilon_2 \xi)$, where

$$\begin{aligned} c &= -\hat{s} - \sqrt{\gamma^2 + \hat{s}^2}, & \varepsilon_1 &= -1, & \varepsilon_2 &= -1; \\ c &= \hat{s} - \sqrt{\gamma^2 + \hat{s}^2}, & \varepsilon_1 &= 1, & \varepsilon_2 &= 1; \\ c &= -\hat{s} + \sqrt{\gamma^2 + \hat{s}^2}, & \varepsilon_1 &= 1, & \varepsilon_2 &= -1; \\ c &= \hat{s} + \sqrt{\gamma^2 + \hat{s}^2}, & \varepsilon_1 &= -1, & \varepsilon_2 &= 1. \end{aligned}$$

These are supercritical “humps” going left, subcritical going left, subcritical going right and supercritical going right. From them, the initial wave configurations can be constructed.

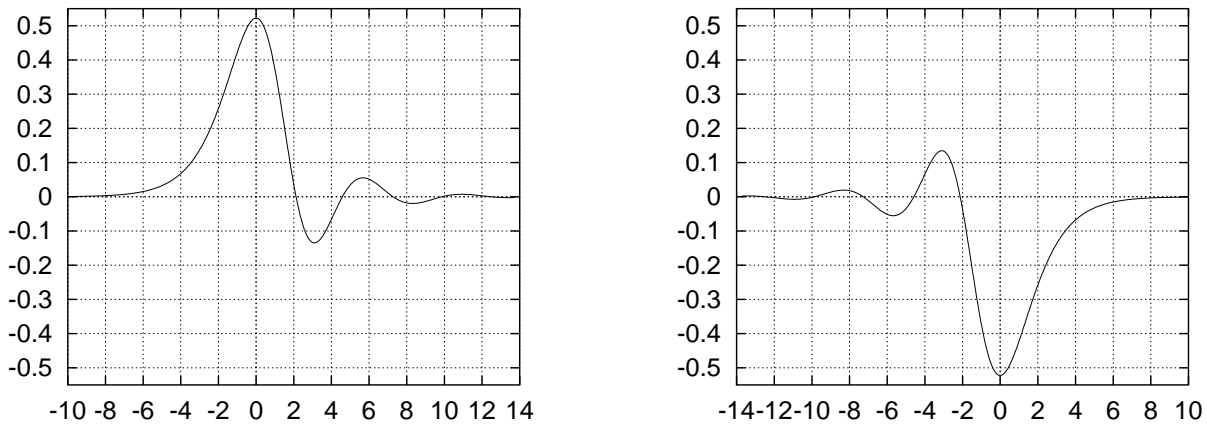


Fig. 2. The shapes of localized solutions. Left panel: right moving structure $s = 1.216$. Right panel: left moving structure with phase velocity $s = -1.216$

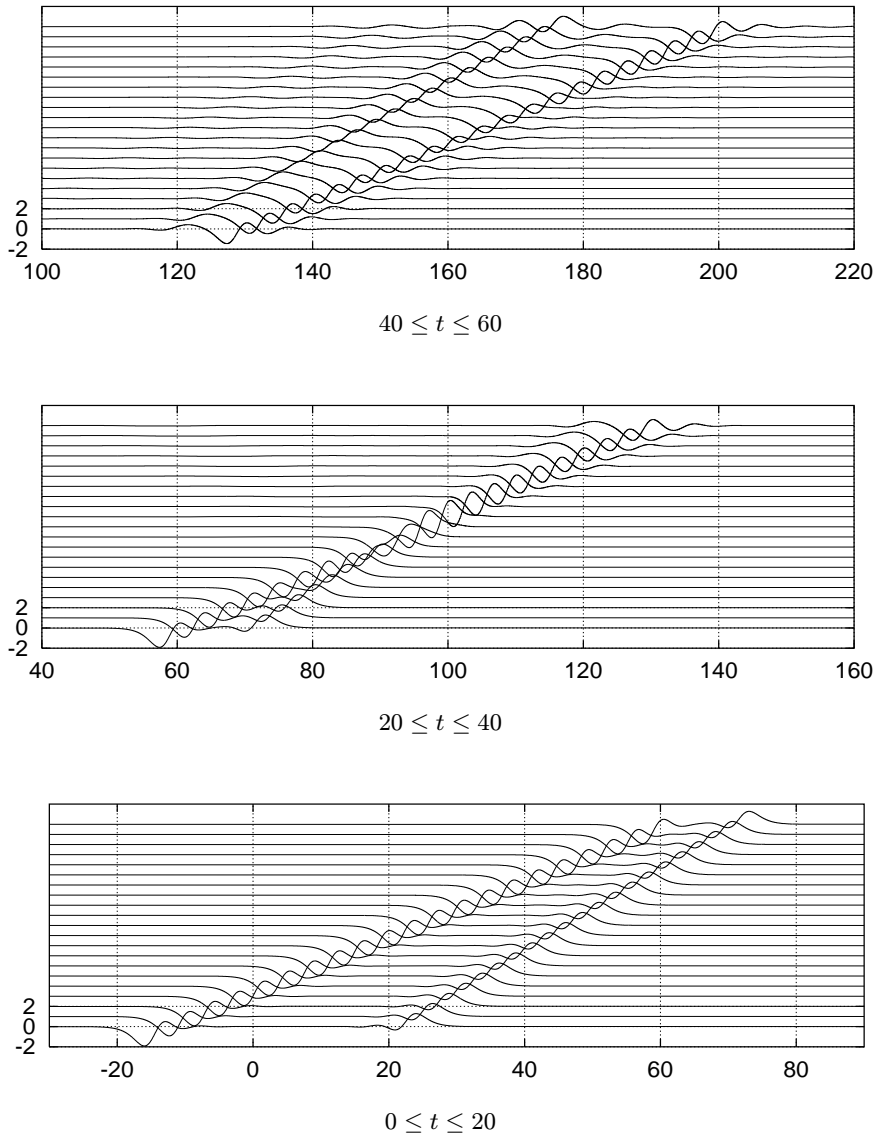


Fig. 3. Take-over interaction of a supercritical ($c = 3.669$) and a subcritical ($c = 2.453$) dissipative quasi-particles.

The first case we consider is the take-over collision of two localized waves depicted in Fig. 3. In take-over collisions the evolution is indeed slow in the moving frame and the physical situation is the one for which the NEE of type KS is derived. Yet there is a principal difference between the GDWE considered here and the NEE: in KS case, the two structures travel at the same phase speed and they form a bound state. Thus no take-over collision can be investigated. Concerning the current results one can see in the figure that the slower structure does not survive the interaction while the faster reappear from it pretty much in tact. Yet the interaction triggers the mechanism of energy production and after

long enough time the long-wave instability transforms the motion into an apparently chaotic one.

In Fig. 4 an important case of head-on collision of two localized waves of the same shape and energy (both supercritical in the sense that $c > \gamma$) is shown. This case corresponds roughly to the head-on collision investigated in [Christov & Velarde, 1995] where the localized waves were shown to stick to each other after the collision forming a profile which succumbs to the long-wave instability. The behavior observed in the present work is drastically different showing that the localized waves can successfully pass through each other and re-emerge from the collision virtually unscathed. The re-emerging

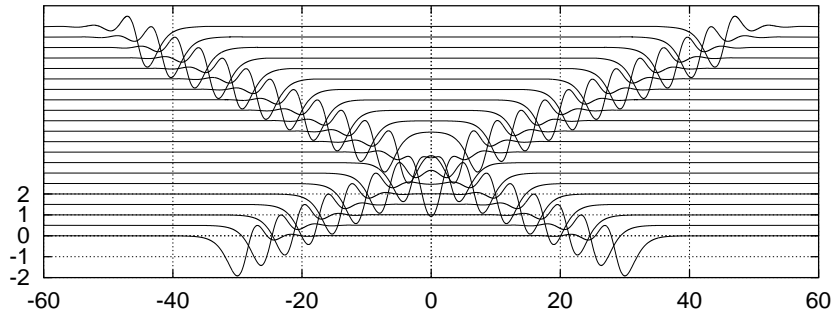


Fig. 4. Head-on collision of supercritical localized waves $c_l = c_r = \pm 3.669$.

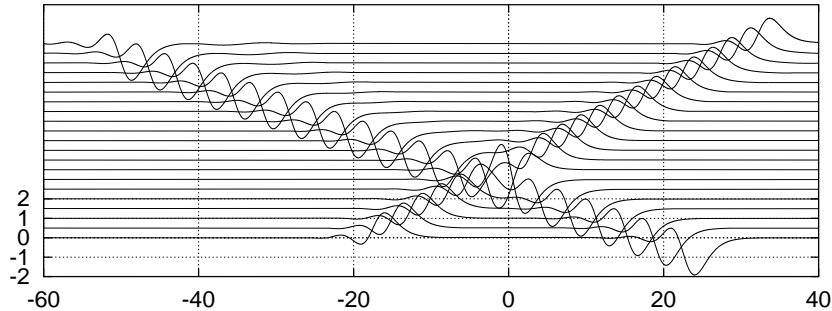


Fig. 5. Head-on collision of a supercritical ($c_r = -3.669$) and a subcritical ($c_l = 2.453$) structures.

quasi-particles retain their identity long after the collision. This means that their shapes are almost identical to the original ones since only the shapes $F_{\pm}(\xi)$ exist as solution of the ODE in the moving frame. Thus we show that retaining the Newtonian inertia (the second time derivative) is essential for the adequate modeling of the head-on collisions. Note that the phase shift is rather small in this purely inelastic model. It means that the notion of “inelastic” collision defined solely on the base of the appearance of a phase shift needs refinement.

The same conclusion can be drawn from Fig. 5 where the head-on collision of two different localized waves (supercritical and subcritical ones) is shown. The structures retain their shapes and the phase shift exhibits the same properties as in the case of a conservative system (see [Christov & Velarde, 1994b]), namely the smaller structure is shifted more.

Figures 4 and 5 suggest that the corpuscular behavior and the almost-elastic collision can be observed in purely inelastic system of production/dissipation type. Note that this is only true for supercritical structures.

Completely different is the situation with the head-on collision of subcritical structures. Upon collision they stick to each other and form a single bulge. Then the resulting wave shape grows and blows-up in finite time as shown in Fig. 6. Note that the “hump”-solution of NEE investigated by Christov and Velarde [1995] triggered the long-wave instability after a head-on collision, while the hydraulic jumps went to a perfect still which warranted calling the latter inelastic quasi-particles.

The last result offers one more analogy between the dissipative systems and the conservative ones. Turitsyn [1993] showed that for certain initial conditions a blow-up could occur for the solution of Boussinesq equation. Christov and Velarde [1994b] confirmed this finding in thorough numerical experiments with specially designed conservative difference scheme. Now one sees in Fig. 6 that a rather similar effect takes place for the strictly dispersionless dissipative system. It means that the balance between nonlinearity and energy dissipation/production has similar limitations to those of the classical Boussinesq balance of nonlinearity and dispersion.

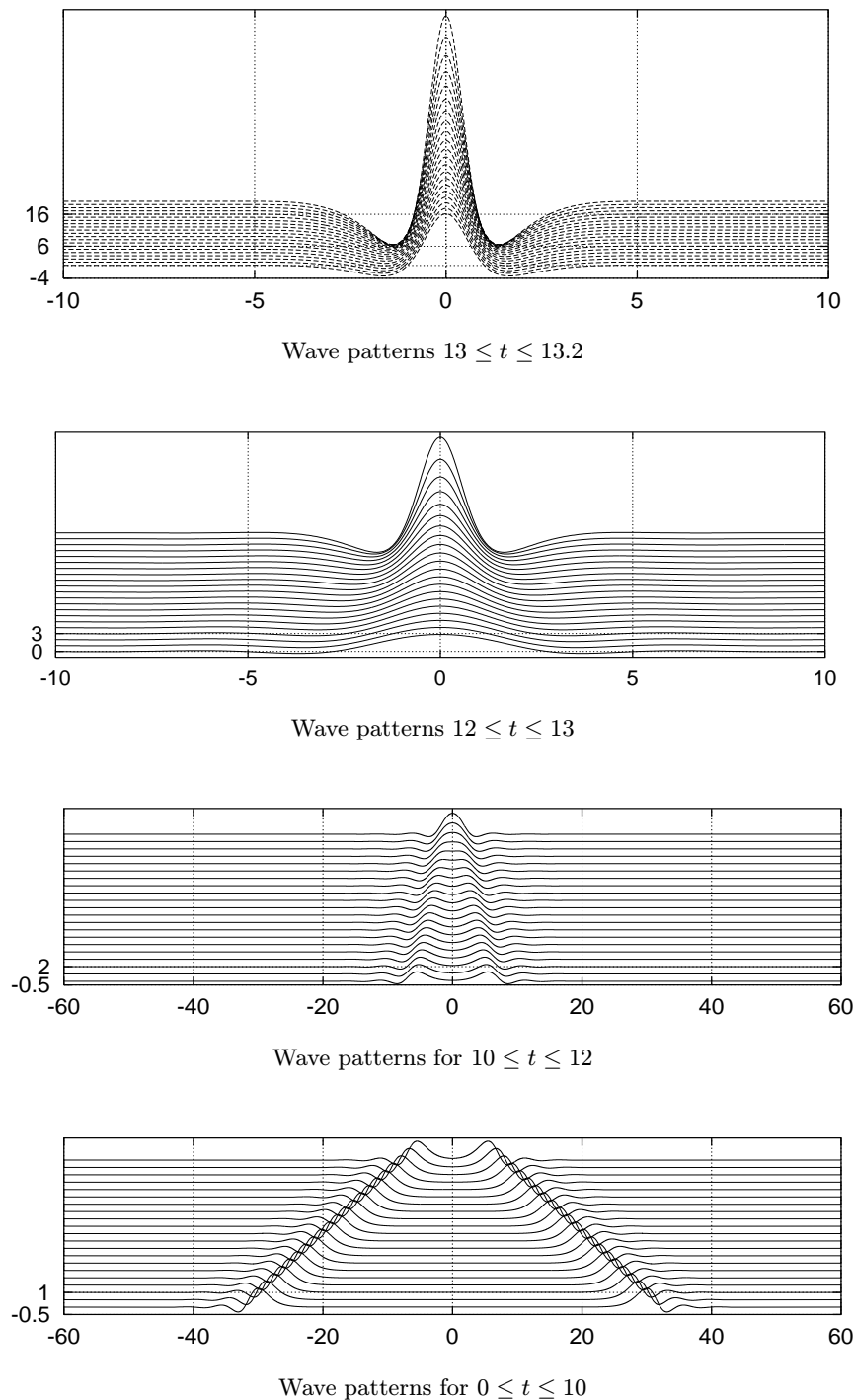


Fig. 6. Blow-up in the head-on collision of two subcritical localized waves: $c = \pm 2.453$.

5. Conclusions

We have studied in the present paper the corpuscular behavior of the localized solutions of Generalized Dissipative Wave Equations (GDWE). The second-time derivative (the linear Newton inertia)

in GDWE stabilizes the shapes of the quasi-particles in comparison with the one-way wave models described by Nonlinear Evolution Equation (NEE) such as dissipation-modified KdV equation.

We conducted a thorough numerical experiment and the most typical and informative

results are presented graphically. Our experiments show that a quasi-particle (corpuscular) behavior is clearly demonstrated at relatively long times in GDWE. Given enough time, the energy-input mechanism always brings about the long-wave instability. The quasi-particle behavior is unequivocally demonstrated for moderate times unlike NEE of KS type, where the long-wave instability onsets immediately after a collision and the shapes of localized waves are destroyed before even emerging from the collision.

The important observation is that even in the case without dispersion the localized solutions of the GDWE behave as quasi-particles for reasonably long time after the collision.

In head-on collisions of supercritical localized waves (or a super- and sub-critical) the wave structures behave like quasi-particles. They retain their identity after collision and the phase shift is small. Hence the collisions are almost elastic (see the comments after Fig. 4 despite the fact that the original system is purely dissipative. The very fact that the shapes are structurally stable in the dispersionless limit shows that the concept of quasi-particles may have a much broader application than merely conservative systems.

In a take-over collision the supercritical structures survive while the subcritical evolve after the collision into a nonlocalized wave shape. The new shape is defined by the balance between energy production and dissipation.

After a head-on collision the subcritical localized waves form a single “bulge” whose shape triggers a nonlinear blow-up in finite time. This signifies the onset of the long-wave instability.

The present paper shows that a dissipative system can possess corpuscular solutions which interact almost elastically which is a novel result.

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