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Fourier-Galerkin Method for Interacting Localized Waves

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Abstract

We develop a Fourier-Galerkin spectral technique for computing the solutions of Fourth-order Generalized Wave Equations of type of interacting localized waves. To this end a special complete orthonormal system of functions in $L^2(-\infty,\infty)$ is used and a time-stepping algorithm implementing the spectral method is developed. The rate of convergence is shown to be exponential.

As a featuring example the head-on collision of *sech* solitary waves is investigated in the case of Proper Boussinesq Equation (PBE). It is shown that the solitons recover their exact shapes after the collision but experience phase shift. The numerically obtained signs and magnitudes of the phase shifts are in very good quantitative agreement with analytical results for the two soliton solution of PBE.

1. INTRODUCTION

In recent years a number of physical problems have led to boundary value problems in infinite domains. These are the cases when no boundary conditions are specified at certain points, but rather the solution is required to possess a integrable square in the infinite domain. Then the solution is said to belong to the $L^2(-\infty,\infty)$ space. A typical example is furnished by the problem for soliton solutions of different nonlinear evolution equations or generalized wave equations. There appear a lot of difficulties on the way of application of difference or finite-element numerical methods to the problems in $L^2(-\infty,\infty)$. The worst consists in the inevitable reducing of the infinite interval to a finite one which introduces artificial eigenvalue problems the latter being irrelevant to the original infinite domain. It can happen that each of the finite-domain approximations has only a trivial solution, while the original problem possesses a nontrivial one, or *vice versa*. Sometimes, the finite-domain problem has a solution only for some denumerable set of intervals of specific length.

This difficulty can be surmounted if a spectral method is used with basis system of localized functions which automatically acknowledge the requirement that the solution belongs to $L^2(-\infty,\infty)$ space. Here we make use of a complete orthonormal (CON) system of functions proposed in [10] which possesses an expression for the

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product of two members of the system into series with respect to the system. This property is crucial because it allows one to use a Galerkin type of expansion the latter being much simpler and faster than a pseudo-spectral algorithm.

Until now, the The Fourier-Galerkin technique based on the said CON system has been applied mainly to the problem of computing the shape of a wave which propagates stationary. The first results about dynamics/interactions of solitary waves were announced in [8] for the limiting case of two solitons of equal height. The present paper is the first detailed account of the application of the new Fourier-Galerkin technique to the general problem of interaction of two different solitons.

2. THE PHYSICAL BACKGROUND OF THE PROBLEM

John Scott Russell observed the solitary ("permanent") wave in a channel near Edinburgh. Boussinesq provided a theory where the effect of a higher-order linear dispersion was incorporated. In particular, Boussinesq found analytical expression of *sech* type for the permanent waves of long wavelength which are solutions of the equations he derived. Later on Korteweg and de Vries [16] found an equation for the shape of the slow evolution nonlinear wave in a coordinate frame moving with the characteristic velocity.

Zabusky and Kruskal [22] investigated the interaction of nonlinear waves in KdV and introduced the notion of *soliton* for solitary waves that behave as particles upon collision. The numerical investigations that followed showed that save the phase shift experienced in the course of interaction, the collision of two *solitary* particle-waves appeared elastic.

The problem with the KdV equation is that it is an equation for slow evolution of the wave system in a frame which moves with the characteristic speed. In other words, the KdV is valid only for a wave system that is slowly evolving in the moving frame. The one-way approximation completely rules out the possibility to investigate head-on collisions in the framework of KdV. An equation describing two-way waves should contain second time-derivatives as it is the case with the original Boussinesq equation. Let us mention in passing that from physical point of view the Boussinesq equation is valid under the same conditions as the KdV equations does. Yet, for the purposes of investigating the collisions of solitons, the Boussinesq equation is preferable because it allows two-ways interactions.

Consider a heavy inviscid liquid filling a layer of thickness H. Following Boussinesq and many others we introduce the dimensionless variables

$$x = \lambda x', \quad h = h_* h', \quad u = \sqrt{gH}u', \quad t = \sqrt{\frac{\lambda}{gH}}t'$$

and obtain the following dimensionless equation

$$u_{tt} = (u - \alpha u^2 + \beta u_{xx})_{xx} \,, \tag{1}$$

where σ and ρ are respectively the air-liquid surface tension and density of the fluid. Then

$$\beta = \left(\frac{1}{3} - \frac{\sigma}{\rho g H^2}\right) \frac{H^2}{\lambda^2}. \tag{2}$$

The "primes" are omitted without fear of confusion since in what follows we use only the dimensionless form of the equation. Here α stands for the dimensionless parameter of nonlinearity (amplitude parameter) and β is the dimensionless dispersion parameter. These are supposed to be small quantities of the same order. Under this assumption, the so-called Boussinesq balance between the nonlinearity and dispersion holds. The Boussinesq balance secures that the predominant behavior of the solution is as of the solution of the standard wave equation (any shape propagates without change) while additional to the wave equation terms act merely to select the possible shape of the propagating wave.

Equation (1) is correct in the sense of Hadamard (well posed as an initial-value problem) only when the surface tension is strong enough to overcome the coefficient $\frac{1}{3}$ in (2). Then we have $\beta < 0$. When this is the case we call (1) Proper Boussinesq Equation (PBE).

3. DYNAMICS OF INTERACTING SOLITARY WAVES

3.1. The Equation and Boundary Conditions

In what follows we consider a Proper Boussinesq Equation with $\alpha = 1$ and $\beta = -1$, namely:

$$u_{tt} = (u - u^2 - u_{xx})_{xx} \tag{3}$$

We are looking for a solitary-wave solution of equation (3) which approaches zero at each infinity and hence all its derivatives decay automatically to zero. When treating the problem analytically one may also impose decay boundary conditions on the second, third, etc. derivative. These are called asymptotic boundary conditions (*a.b.c.*), namely

$$u(t,x) \to 0 \,, \quad \text{for} \quad x \to \pm\infty \,. \tag{4}$$

Equation (3) possesses solitary-wave analytical solution of *sech* type, namely:

$$u(x) = -\frac{3}{2}(1 - c^2)\text{sech}^2\left(\frac{x - ct}{2}\sqrt{1 - c^2}\right). \tag{5}$$

The analytical solution (5) of the PBE is important not only from physical point of view but also due to the fact that it provides the necessary check for the consistency and accuracy of numerical schemes, such as the spectral technique developed here.

Before proceeding further we mention that re-scaling the spatial variable x does not change the nature of the asymptotic boundary value problem in $L^2(-\infty,+\infty)$. Upon introducing $z=\zeta x$ we recast (3) to the following

$$u_{tt}=\zeta^2(u-u^2-\zeta^2u_{zz})_{zz}\quad \text{with a.b.c}\quad u(t,z)\to 0,\quad \text{for}\quad z\to\pm\infty\,. \tag{6}$$

The scaling parameter ζ can be used to optimize the method in the sense that its introduction allows one to bring in concert the typical length scales of the employed system of functions and that of the support of the sought localized solution. Naturally, such a coordination between the scales will results in a faster convergence of the Fourier-Galerkin series.

Before turning to numerical implementation, let us show the Hamiltonian form of the problem under consideration. For the Boussinesq Equations it was shown in [17, 18]. See also, the special discussion in [12].

We introduce an auxiliary function q and show that (6) is a corollary of the following system

$$u_t = \zeta^2 q_{zz}\,, \tag{7}$$
$$q_t = u-u^2-\zeta^2 u_{zz}\,. \tag{8}$$

Here the auxiliary function q is also a localized function but it can assume nonzero values at infinities. It has the shape of a hydraulic jump (or *kink*). Then the asymptotic boundary conditions for the system (7), (8) have the form

$$u,\ q_x\to 0\quad \text{for}\quad x\to -\infty,\infty.$$

If we define the wave mass M, the wave momentum (*pseudomomentum*) P, and the energy E (see [14]) of the wave system as

$$M=\int_{-\infty}^{\infty} u dx,\qquad P=\int_{-\infty}^{\infty} uq_x dx,\qquad E=\int_{-\infty}^{\infty}\frac{1}{2}[u^2+q_x^2+u^2+u_x^2]dx, \tag{9}$$

then we show that the following conservation/balance laws hold:

$$\frac{dM}{dt}=0,\quad \frac{dE}{dt}=0,\quad \frac{dP}{dt}=\frac{\beta}{2}u_x^2\bigg|_{-\infty}^{\infty}\equiv F,$$

where F is called sometimes "pseudo-force".

The conservation laws ensure that the mass and the energy remain constant during the evolution of the solution. For the case of asymptotic boundary conditions, the balance law for P secures that the pseudomomentum remains constant as well.

3.2. The Initial Condition

The initial condition is the superposition of two *sech*-solutions which are situated far enough from each other in order to neglect their intersection in the initial moment.

$$u(z,t) = -\frac{3}{2}(1-c_1^2)\operatorname{sech}^2\left[\frac{(z_1 - c_1 t\zeta)}{2}\frac{\sqrt{1-c_1^2}}{\zeta}\right] - \frac{3}{2}(1-c_2^2)\operatorname{sech}^2\left[\frac{(z_2 - c_2 t\zeta)}{2}\frac{\sqrt{1-c_2^2}}{\zeta}\right]. \quad (10)$$

In the last formula c_1 and c_2 are the phase speed of the respective solitons.

Now, for the purposes of the identification of the auxiliary function q we can use the first equation of the governing system (7), namely

$$u_t = \zeta^2 q_{zz} \qquad q_x(t,x) \to 0\,, \quad \text{for} \quad z \to \pm\infty\,.$$

Hence, we differentiate equation (10) once with respect to t, then we integrate twice with respect to z to get an analytical solution for q:

$$q(z,t) = \frac{3c_1}{\zeta^2}\sqrt{1-c_1^2}\tanh\left[\frac{(z_1 - c_1 t\zeta)}{2}\frac{\sqrt{1-c_1^2}}{\zeta}\right] + \frac{3c_2}{\zeta^2}\sqrt{1-c_2^2}\tanh\left[\frac{(z_2 - c_2 t\zeta)}{2}\frac{\sqrt{1-c_2^2}}{\zeta}\right]. \quad (11)$$

3.3. Rendering the Auxiliary Function q to a Function from $L^2(-\infty,\infty)$

The boundary conditions on the auxiliary function q do not require that it decays to zero at $\pm\infty$. Then it cannot be developed into series with respect to CON system of function whose members vanish at infinities. We use a function $r(z)$ which "absorbs" the undesired behavior of $q(z,t)$. This means that instead of q, we use $p(z,t) = q(z,t) + r(z)$, where

$$r(z) = -\left[\frac{3c_1}{\zeta^2}\sqrt{1-c_1^2} + \frac{3c_2}{\zeta^2}\sqrt{1-c_2^2}\right]\tanh(z).$$

Note that the introduction of the new function $p(z,t)$ does not alter the equation (6) because $r(z)$ does not depend on t. For the second derivative of the function $r_{zz}(z,t)$ one gets

$$r_{zz} = -2\left[\frac{3c_1}{\zeta^2}\sqrt{1-c_1^2} + \frac{3c_2}{\zeta^2}\sqrt{1-c_2^2}\right]\tanh(z)\operatorname{sech}^2(z)$$

In terms of the new function auxiliary function, the system (7), (8) recasts to the following

$$u_t = \zeta^2 p_{zz} + \zeta^2 r_{zz}\,, \tag{12}$$

$$q_t = u - u^2 - \zeta^2 u_{zz} \tag{13}$$

where r_{zz} is a given quantity which is described in (12).

The initial condition for the function $p(z,t)$ is as follows

$$p(z,t) = -\left[\frac{3c_1}{\zeta^2}\sqrt{1-c_1^2} + \frac{3c_2}{\zeta^2}\sqrt{1-c_2^2}\right]\tanh(z) + \left\{\frac{3c_1}{\zeta^2}\sqrt{1-c_1^2}\,\tanh\left[\frac{(z_1-c_1t\zeta)}{2}\frac{\sqrt{1-c_1^2}}{\zeta}\right] + \frac{3c_2}{\zeta^2}\sqrt{1-c_2^2}\,\tanh\left[\frac{(z_2-c_2t\zeta)}{2}\frac{\sqrt{1-c_2^2}}{\zeta}\right]\right\}. \tag{14}$$

4. THE FOURIER-GALERKIN METHOD IN $L^2(-\infty,\infty)$

From the known spectral techniques we choose the Galerkin method (for other techniques, see, e.g., [6, 7]). The Galerkin method has the advantage of simplicity in implementation in comparison with the spectral collocation method or tau-method. This turns out to be crucial for constructing fast and efficient numerical algorithms. The only problem is that the Galerkin techniques requires explicit formulas expressing the products of members of the CON system into series with respect to the system. For instance, the Hermite functions and Laguerre functions do not possess that kind of explicit relation. The first systems for which a product formula does exist was proposed in [10]. A Galerkin technique based on the said system was developed in [13] and applied to Korteweg-de Vries (KdV) and Kuramoto-Sivashinsky (KS) equations with quadratic nonlinearity. It has been recently applied to 2D problems in [11] and to problems with cubic nonlinearity [9]. In a sequence of papers Boyd [4, 5] showed a general way of constructing CON systems in $L^2(-\infty,\infty)$ by means of coordinate transformation to finite interval and use of Chebishev polynomials (see [6] for references).

4.1. The Complete Orthonormal (CON) system

The system

$$\rho_n = \frac{1}{\sqrt{\pi}}\frac{(ix-1)^n}{(ix+1)^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{15}$$

was introduced by Wiener, see [21] as Fourier transforms of the Laguerre functions (functions of parabolic cylinder). Higgins [15] defined it also for negative indices n and proved its completeness and orthogonality. The significance of (15) for nonlinear problems was demonstrated in [10], where the product formula was derived and the two real-valued subsequences of odd functions S_n and even functions C_n were introduced, namely

$$\rho_n\rho_k = \frac{\rho_{n+k}-\rho_{n-k}}{2\sqrt{\pi}}, \qquad S_n = \frac{\rho_n+\rho_{-n-1}}{i\sqrt{2}}, \quad C_n = \frac{\rho_n-\rho_{-n-1}}{\sqrt{2}}. \tag{16}$$

Making use of (16) one easily shows that the products of members of the real-valued sequences are expanded in series with respect to the system as follows (see, [10]):

$$C_nC_k = \frac{1}{2\sqrt{2\pi}}[\ C_{n+k+1}-C_{n+k}-C_{n-k}+C_{n-k-1}]\ , \tag{17}$$

$$S_nS_k = \frac{1}{2\sqrt{2\pi}}[\ C_{n+k+1}-C_{n+k}+C_{n-k}-C_{n-k-1}]\ , \tag{18}$$

$$S_nC_k = \frac{1}{2\sqrt{2\pi}}[-S_{n+k+1}+S_{n+k}+S_{n-k}-S_{n-k-1}]\ . \tag{19}$$

For the sake of brevity of the notation we introduce the following coefficients of the series for the products S_nS_m, C_nC_m, and S_nC_m, namely,

$$\begin{aligned}
\alpha_{nk,m} &= \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k+1}+\delta_{m,|n-k|}-\delta_{m,n+k}-\mathrm{sgn}[|n-k|-0.5]\delta_{m,[|n-k|-0.5]}\},\\
\beta_{nk,m} &= \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k}+\delta_{m,|n-k|}-\delta_{m,n+k+1}-\mathrm{sgn}[|n-k|-0.5]\delta_{m,[|n-k|-0.5]}\},\\
\gamma_{nk,m} &= \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k}+\mathrm{sgn}(n-k)\delta_{m,|n-k|}-\delta_{m,n+k+1}-\mathrm{sgn}(n-k)\delta_{m,|n-k|-1}\},
\end{aligned} \tag{20}$$

and then the nonlinear terms are formally represented as

$$S_nS_k = \sum_{m=1}^{\infty}\alpha_{nk,m}C_m, \quad C_nC_k = \sum_{m=1}^{\infty}\beta_{nk,m}C_m, \quad S_nC_k = \sum_{m=1}^{\infty}\gamma_{nk,m}S_m$$

For the second and fourth derivatives of the basis functions one has (see [10])

$$C_n'' = \sum_{m=0}^{\infty}\chi_{m,n}C_m\ , \qquad S_n'' = \sum_{m=0}^{\infty}\chi_{m,n}S_m\ , \tag{21}$$

$$\begin{aligned}
\chi_{m,n} = &-\tfrac{1}{4}n(n-1)\delta_{m,n-2}+n^2\delta_{m,n-1}-\tfrac{1}{4}(n+1)(n+2)\delta_{m,n+2}\\
&-\tfrac{1}{4}n^2+(2n+1)^2+(n+1)^2\delta_{m,n}+(n+1)^2\delta_{m,n+1}\,.
\end{aligned}$$

$$C_n^{\mathrm{IV}} = \sum_{m=0}^{\infty}\omega_{m,n}C_m\ , \qquad S_n^{\mathrm{IV}} = \sum_{m=0}^{\infty}\omega_{m,n}S_m\ , \tag{22}$$

where

$$\begin{aligned}
\omega_{m,n} \;=\;& \tfrac{1}{16}n(n-1)(n-2)(n-3)\delta_{m,n-4} - \tfrac{1}{2}n(n-1)^2(n-2)\delta_{m,n-3} \\
&+\tfrac{1}{4}n(n-1)(7n^2-7n+4)\delta_{m,n-2} - \tfrac{1}{2}n^2(7n^2+5)\delta_{m,n-1} \\
&+\tfrac{1}{8}[35n^4+70n^3+85n^2+50n+12]\delta_{m,n} \\
&-\tfrac{1}{2}(n+1)^2[7(n+1)^2+5]\delta_{m,n+1} \\
&+\tfrac{1}{4}(n+1)(n+2)[7(n+1)^2+7(n+1)+4]\delta_{m,n+2} \\
&-\tfrac{1}{2}(n+1)(n+2)^2(n+3)\delta_{m,n+3} \\
&+\tfrac{1}{16}(n+1)(n+2)(n+3)(n+4)\delta_{m,n+4}\,.
\end{aligned}$$

4.2. The Spectral Expansion and Algorithm

For the solution of system (12) we consider the fully implicit time-stepping scheme:

$$\frac{u^{l+1}-u^l}{\tau} = \zeta^2 p_{zz}^{l+\frac{1}{2}} + \zeta^2 r_{zz}\,, \tag{23}$$

$$\frac{p^{l+\frac{1}{2}}-p^{l-\frac{1}{2}}}{\tau} = \frac{u^{l+1}+u^{l-1}}{2} - (u^l)^2 - \frac{\zeta^2}{2}[u_{zz}^{l+1}+u_{zz}^{l-1}]. \tag{24}$$

We develop the sought solution u and p into series with respect to the subsequences C_n and S_n namely,

$$u^l(z) = \sum_{n=0}^{\infty} a_n^l C_n(z) + b_n^l S_n(z), \quad p^{l+\frac{1}{2}}(z) = \sum_{m=0}^{\infty} d_m^{l+\frac{1}{2}} C_m(z) + e_m^{l+\frac{1}{2}} S_m(z). \tag{25}$$

Now, we insert the spectral expansion (25), into equation (23), (24) and if we make use the orthogonality of the system of our functions, the system for the even functions coefficients reads:

$$\begin{aligned}
\frac{a_m^{l+1}-a_m^l}{\tau} &= \zeta^2 \sum_{k=0}^{\infty} d_k^{l+\frac{1}{2}} \chi_{m,k}, \\
\frac{d_m^{l+\frac{1}{2}}-d_m^{l-\frac{1}{2}}}{\tau} &= \frac{a_m^{l+1}+a_m^{l-1}}{2} - \frac{\zeta^2}{2}\sum_{k=0}^{\infty}(a_k^{l+1}+a_k^{l-1})\chi_{k,m} \\
&\quad - \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(a_{k_1}^l a_{k_2}^l \beta_{k_1k_2,m} + b_{k_1}^l b_{k_2}^l \alpha_{k_1k_2,m})
\end{aligned} \tag{26}$$

Similarly, the system for the coefficients of the odd functions reads:

$$
\begin{aligned}
\frac{b_m^{l+1}-b_m^l}{\tau} &= \zeta^2\sum_{k=0}^{\infty} b_k^{l+\frac{1}{2}}\chi_{m,k} + \zeta^2\sum_{k=0}^{\infty} r_k\chi_{m,k}, \\
\frac{e_m^{l+\frac{1}{2}}-e_m^{l-\frac{1}{2}}}{\tau} &= \frac{b_m^{l+1}+b_m^{l-1}}{2} - \frac{\zeta^2}{2}\sum_{k=0}^{\infty}(b_k^{l+1}+b_k^{l-1})\chi_{k,m} \\
&\quad -\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(a_{k_1}^l b_{k_2}^l\gamma_{k_1k_2,m} + b_{k_1}^l a_{k_2}^l\gamma_{k_1k_2,m}),
\end{aligned} \tag{27}
$$

where α, β and γ are given in (20) and r_k are the coefficients of the spectral expansion of function r_{zz}.

In the numerical calculations a truncated version of the above system is used in which the infinity is replaced by N.

The initial conditions $\{a_n^0\}$, $\{b_n^0\}$ and $\{a_n^1\}$, $\{b_n^1\}$ for the Fourier coefficients are calculated for $t = 0$ and $t = \tau$ by means of numerical quadrature of formulas (10) respectively after multiplied by C_n or S_n. In its turn, the initial conditions for the coefficients $\{d_n^{\frac{1}{2}}\}$, $\{e_n^{\frac{1}{2}}\}$ of p are computed via numerical quadrature of (14) after multiplied by C_n or S_n. Note that the initial conditions have to be calculated anew every time when the value ζ of the scaling parameter is changed.

Having specified the initial conditions we can begin the time stepping. Let us assume that the variables $\{a_n^{l-1}\}$, $\{b_n^{l-1}\}$, $\{a_n^l\}$, $\{b_n^l\}$, $\{d_n^{l-\frac{1}{2}}\}$, $\{e_n^{l-\frac{1}{2}}\}$ are known. Then Des.(26) give a coupled nine-diagonal algebraic system for $\{a_n^{l+1}\}$, $\{d_n^{l+\frac{1}{2}}\}$. Respectively, Des.(27) give a coupled nine-diagonal algebraic system for $\{b_n^{l+1}\}$, $\{e_n^{l+\frac{1}{2}}\}$. After these systems are solved, one time step is completed, the time index l is reset, and the process is repeated.

4.3. Numerical Verification

The importance of scaling parameter ζ is understood by computing the solution for different values of ζ. Naturally, the optimal ζ is different for different initial configurations of the system of solitons. When the solitons are situated farther from each other, one is faced with a system which is not tightly localized. Then the optimal value of ζ is smaller. Conversely, if the initial configuration is tight enough, the value of ζ which brings the scales of the sought function and the CON system in a closest rapport, tends to be smaller. In the present work we consider systems of solitons that are well separated (in order not to overlap significantly), but not excessively far from each other (not to loose the localization). After extensive numerical experiments we found for these cases that the optimal value of the scaling parameter is close to $\zeta = 0.07$, for which the convergence was faster and more accurate.

Before proceeding to the presentation of the results we mention here that an extensive set of numerical experiments has been conducted in order to outline the region for the time increment τ in which the spectral solution has satisfactory approximation and is stable. We found that the calculations are perfectly stable for τ as large as 0.1 even for $c = 0.87$ the latter being on the border of existence of the soliton solution for Boussinesq equation. It turned out that in the framework of the spectral approximation of the spatial terms, the second order approximation in time produces almost insignificant phase error. Note that this is not the case with the difference approximations where the temporal approximation results in somewhat larger phase error. Most of the results presented in the next Section are obtained with $\tau = 0.1$. Some cases are treated with smaller τ for the case of completeness.

As already above mentioned, the rate of convergence of the Galerkin series is exponential. It is important to verify that we actually get this rate of convergence numerically. We consider two sets of phase velocities which correspond to two equal shallow depressions and two two non-equal depression. One of the cases is close to the threshold of existence of *sech* solitons for PBE and is supposed to be more susceptible to different errors. We begin from the initial conditions as calculated in the previous section and run the process until the solitons swap their places. In Figure 1 we present the dependence of the Fourier coefficients on their number.

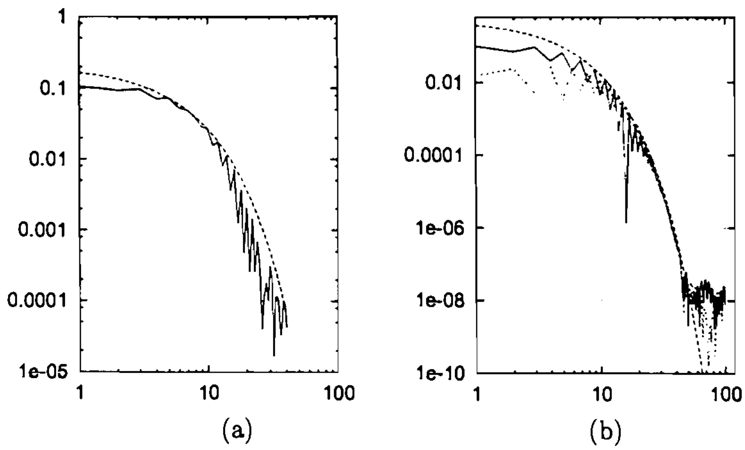


Figure 1: Exponential decay of the computed coefficients of the spectral expansion: (a) $\zeta = 0.05$ and $N = 40$. Dashed line $0.2exp(-0.2x)$; (b) $c_1 = 0.95$, $c_2 = -0.9$, $\zeta = 0.05$ and $N = 100$. Dashed line $expo(-0.34x)$.

The exponential decay of the coefficient with their number is clearly seen. Moreover, the magnitudes of the coefficients are small in the absolute sense which means that the developed algorithm exhibits no appreciable accumulation of round-off errors. To have a better quantitative description of the result we add also the exponential function which provide the best fit to the data for coefficients (dashed lines in Figure 1). Note also, that the results are obtained with values of ζ close to the optimal ones. In fact the best fit formulas offer another definition for the optimal scale parameter ζ as the parameter for which the coefficient of the exponent is most negative. A more negative coefficient means that for the same number n the relative importance of the term decreases, i.e., the convergence is faster.

5. RESULTS AND DISCUSSION

An important characteristics of the Proper Boussinesq Equation is that the localized waves can be computed numerically as initial value problem only for subcritical phase velocities. The initial value problem for the Proper Boussinesq Equation is correct in the sense of Hadamard only for $\beta < 0$, when the solitons have subcritical phase velocities. Therefore, the amplitudes are negative, which actually means that the solitons are "depressions" on the water surface. This is a kind of counterintuitive and is discussed in [14, 12]. In the KdV equations the faster solitons are taller, while in the case if PBE, the faster solitons have smaller amplitudes. When the phase velocity $c = 1$, then $u(z,t) = 0$. If we confider a value of c which is very close to unity, $c \geq 0.99$, then the amplitude of the wave is very small.

In Figure 2 is presented a head-on collision of two equal *sech* Boussinesq solitons of the PBE with very small amplitudes and phase speeds $c = 0.99$. Note that when presenting the results, the sign of the function u is changed, so that the depressions appear like humps in our figures. After the collision the profile overlap completely for $t = 0$ and $t = 810$.

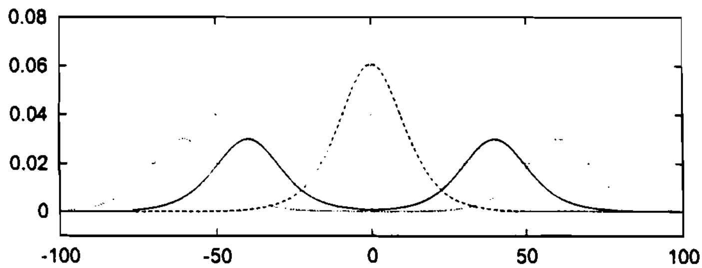


Figure 2: Profiles of two equal depressions for $c_1 = -c_2 = 0.99$ and four different times $t = 0, 410, 810, 1020$ with $\zeta = 0.05$ and $\tau = 0.1$.

Note that the amplitude of the maximum of the wave system is virtually twice as big as the amplitudes of the individual solitons. This means that this configuration of solitons is indeed a case with very small amplitude.

Figure 3 shows the interaction of a short soliton with a tall one. Once again the superposition is almost linear.

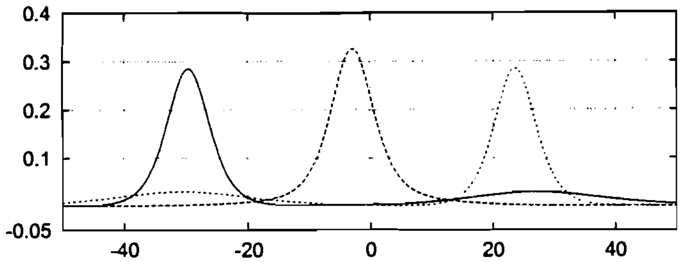


Figure 3: Profiles of two nonequal negative *seches* for $c_1 = 0.99$ and $c_2 = -0.9$ for 3 different times $t = 0$, $t = 2250$ and $t = 4200$ with $\zeta = 0.05$ and $\tau = 0.01$.

In Figure 4 is depicted a case for which one of the solitons is already on the border of existence $c = 0.87$. This case is essentially similar to the previous one but is presented here to show the effectiveness of the technique developed. Our algorithm performs robustly even for parameters very close to the threshold of non-existence of an analytical two-soliton solution.

Proceeding along these lines in Figure 5 we present the hardest case when both phase speeds are on the threshold, i.e., $c_l = -c_r = 0.87$. In order to prevent a premature break-up of the solution due to insufficient temporal approximation, we use in the last case the finer time increment $\tau = 0.01$.

In all of the above presented cases, the shapes and the energy of the solitons are preserved with high order of accuracy upon their collisions. Note that even the slightest but persistent "leakage" of energy during the calculations would have led to eventual linear dispersion of the solution and disappearance of the permanent (*sech*) shapes. It is important to note here that our calculations allow one to establish "experimentally" the region of existence of the solution. As already above mentioned, the threshold for existence of the analytical two-soliton solution of [20] is approximately $c = 0.867$ (see [14] for details).

We discovered that the smallest phase velocity for which the calculations for equal solitons were possible was $c = 0.867$. Below that threshold we encountered a numerical blow-up which is in good qualitative agreement with the analytical results from [23].

Thus the following conjecture has been made based on the results obtained here: The value $c = 0.867$ is both the threshold of the existence of the analytical two-soliton solution, on one side and the threshold of blow-up of the numerically computer two-soliton solution – on the other.

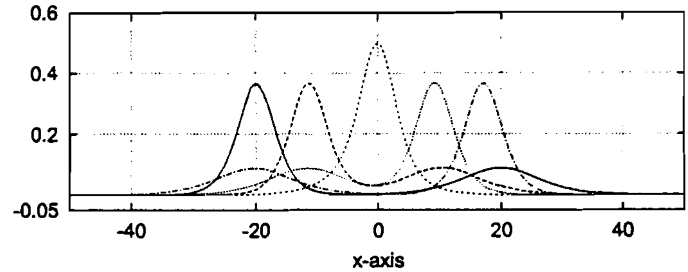


Figure 4: Profiles of two nonequal negative *seches* for $c_1 = 0.87$ and $c_2 = -0.97$ for 5 different times $t = 0$, $t = 100$, $t = 240$, $t = 350$ and $t = 440$ with $\zeta = 0.05$ and $\tau = 0.1$.

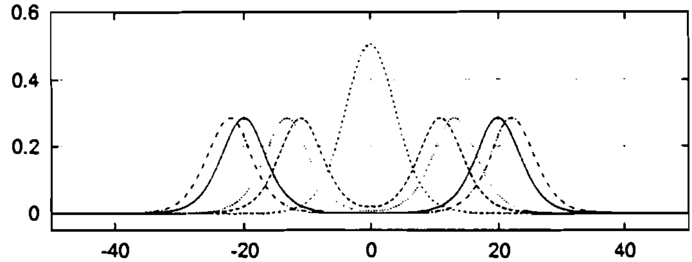
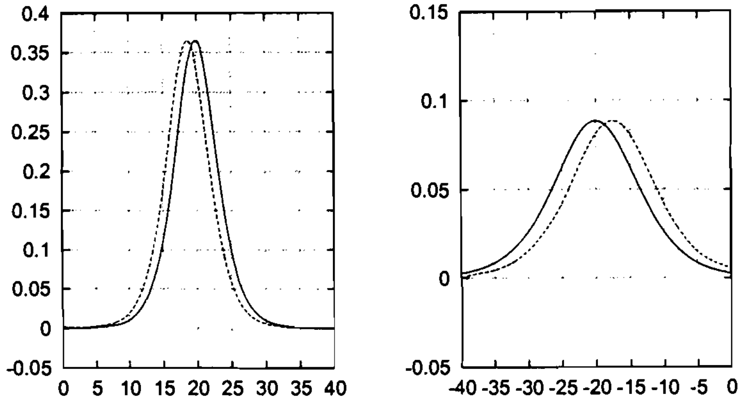


Figure 5: Profiles of two equal negative *seches* for $c_1 = -c_2 = 0.87$ for 5 different times $t = 0, 1000, 2000, 4000, 5000$, with $\zeta = 0.1$ and $\tau = 0.01$.

In the end we focus our attention on the phase shift experienced by the solitons upon collisions. The idea of phase shift is illustrated in Figure 6. In the left panel (a) is presented the slower soliton $c_1 = 0.87$ (larger amplitude), while in the right panel is the faster one with $c_2 = 0.97$ (smaller amplitude).

In each of the panels of Figure 6 the profile of the soliton in its actual position is juxtaposed to the profile of the respective soliton taken in the position in which it would have ended-up if no interaction with another soliton took place. Since both solitons were initially at distance of 20 units from the origin of the coordinate system, then the slower soliton would have taken time $t = 400/0.87 \approx 46$ (460 time steps) to reach the position $x = 20$. Respectively the faster soliton would have needed time $t = 400/0.97 \approx 41.2$ time units (or 412 time steps) to clear up the distance from $x = 20$ to $x = -20$. As it is clearly seen the larger soliton is shifted on 1.22 time units, while the phase shift for the smaller one is 2.48.

We have conducted extensive numerical experiments and obtained results for phase shifts for a large number of initial configurations of the solitons. They appear to be in very good agreement with the finite-difference results of [14] (Table 1) and analytical results [20].



(a) $c_l = 0.87$, shift 1.22, 458 steps; (b) $c_r = 0.97$, shift 2.48, 412; steps

Figure 6: The phase shifts for a system of solitons with $c_l = 0.87$ and $c_r = 0.97$.

6. CONCLUSIONS

In the present paper a Fourier-Galerkin spectral technique is developed for calculating the interaction of localized Waves. The interaction of solitary waves in the Proper Boussinesq Equation is considered as a featuring example.

A complete orthonormal (CON) basis system in $L^2(-\infty, \infty)$ is used to develop the solution into Fourier series with Galerkin identification of the coefficients. This is an important development of the technique in comparison with our previous works where the shape of a single stationary solitary wave was the object of investigation. This is the first work in which a time stepping algorithm involving the CON system is developed.

In the two-soliton case under consideration the localization of the solution is much less tighter and the number of terms needed for good approximation is larger. The treatment of the problem required very efficient implementation of the Fourier-Galerkin scheme.

A scaling parameter, ζ, is introduced which allows fine-tuning and optimization of the technique proposed. Numerical experiments with different number of terms are conducted for different values of the scaling parameter. These experiments establish the practical convergence of the method and indicate an exponential decay of the coefficients with the increase of their number (exponential convergence). Highly accurate results are obtained for the time dependent problem with as few as $N = 40$ terms. This demonstrates the efficiency of the proposed technique and encourages the future use of the CON system of functions for more applications.

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