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IDEAS III

2D Solitary Waves of Boussinesq Equation

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Abstract. In this paper, the 2D stationary-propagating localized solutions of Boussinesq's equation are investigated numerically. An algorithm for treating the bifurcation and finding a nontrivial solution is created. The scheme is validated employing different grid sizes and different size of the box that contains the solution. The results obtained show that there is pseudo-Lorentzian elongation of the scale of the solitons but it is only in the direction transverse to the propagation velocity. In longitudinal direction the scales are slightly contracted, so kind of "relative" contraction takes place. Results are shown graphically and discussed.

Keywords: Boussinesq, Soliton,2D, Numerical
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INTRODUCTION

Localized waves that propagate without change of shape at long distance and/or for long times are called solitary waves. They were first observed by Russell [1, 2] who named the phenomenon "The Great" or "The Permanent" wave. Since then, permanent waves have been found in many other fields of modern physics, such as metal lattices (phonon propagation), optical fibers, Bose–Einstein Condensate etc.

Boussinesq [3, 4] proposed an equation to model the permanent wave, considering shallow water layer with much smaller thickness compared to the horizontal length scale of the motion. He showed that the balance between the nonlinearity and dispersion maintains the shape of the permanent wave. Various Boussinesq equations were proposed during the years (see, e.g. [5, 6] for the literature).

Zabuski and Kruskal [7] showed numerically that two solitaty waves can interact without losing their identities and coined the name "soliton" to delineate the particle-like behavior of these waves (see [8] for the story). Several analytical techniques have been developed for solving soliton problems of different nonlinear dispersive equations, such as Bäcklund transformation, Inverse Scattering Method, and Hirota Bilinear Method (see [9] for extensive review on the subject). The limitations of analytical techniques necessitate the development of different numerical techniques and an extensive literature is available on this subject. However, the predominant part of the known results are concerned with the 1D cases. In this short note we investigate numerically the shapes of the stationary solitons in two spatial dimensions.

We focus here on the so-called "good" (or "proper") Boussinesq equation (PBE):

$$u_{tt} = \Delta(\gamma^2 u + \alpha u^2 - \beta \Delta u) \tag{1}$$

where $\beta > 0$ is the dispersion parameter, α is the amplitude parameter, and γ is the characteristic speed of the small disturbances (linear waves).

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In 1D, PBE possesses *sech* solution

$$u(x,t) = -\frac{3}{2}\frac{\gamma^2 - c^2}{\alpha}\operatorname{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{\gamma^2-c^2}{\beta}}\right) \tag{2}$$

where c is the phase speed of the localized wave. This solution demonstrates that the balance between the nonlinearity and dispersion can maintain the shape of the localized wave, making the latter permanent. For $\alpha > 0$, the solitons are depressions. When one considers PBE equation as a generic class, however, one can assume for convenience that $\alpha < 0$ and have bell-like shapes for the solitons.

The most salient features of the *sech* soliton (2) are that

1. it exists for subcritical phase speeds $|c| < |\gamma|$;
2. if c increases, the amplitude decreases;
3. This is due to the scale factor $\sqrt{(\gamma^2 - c^2)/\beta}$, the shape of solution spreads for increased c.

In 2D, there is no analytical solution for the above Generalized Dispersive Wave Equation (GDWE). Hence devising robust difference scheme is important.

BIFURCATION PROBLEM FOR THE 2D SHAPE

When investigating numerically the dynamics of solitary waves (see, [5]), the analytical *sech* solution (2) is taken as an initial condition. In 2D there is no analytical solution and the first task to be surmounted is to find the shape of the stationary propagating soliton. To this effect we consider the moving frame $\xi = x - c_1 t$ and $\eta = y - c_2 t$, where c_1, c_2 are the components of the velocity of the center of soliton. For the shape function in the moving frame $v(\xi,\eta)$, one gets the following stationary equation

$$\begin{aligned}0 = -(v_{\xi\xi}c_1^2 + 2c_1c_2v_{\xi\eta} + c_2^2v_{\eta\eta}) + \gamma^2(v_{\xi\xi} + v_{\eta\eta}) + \alpha[v_{\xi\xi}^2 + v_{\eta\eta}^2) \\ - \beta(v_{\xi\xi\xi\xi} + 2v_{\xi\xi\eta\eta} + v_{\eta\eta\eta\eta}).\end{aligned} \tag{3}$$

If localized solutions are sought, the following boundary conditions are imposed

$$\begin{aligned}v(-L_1,\eta) = v(L_1,\eta) = 0 = v_x(-L_1,\eta) = v_x(L_1,\eta) = 0, \\ v(\xi,-L_2) = v(\xi,L_2) = v_y(\xi,-L_2) = v_y(\xi,L_2) = 0,\end{aligned} \tag{4}$$

where L_1 and L_2 are called "actual infinities" and they define the size $[2L_1 \times 2L_2]$ of the rectangular region to which the infinite domain is reduced.

The localized solution exists alongside with the trivial solution and hence we are faced with a bifurcation problem. Obtaining a numerical solution requires avoiding the trivial solution. In the present work we propose to fix the value of the function in one point, say in the middle point, namely $v(0,0) = \theta$ and to introduce a new function $v = \theta u$. The equation for new function u is the same as for function v with the only differences that the coefficient of the nonlinear term is now $\alpha\theta$ and that we have the additional constraint

$$u(0,0) = 1. \tag{5}$$

The condition (5) makes the problem overposed and in order to get a well posed problem we need some flexibility. The latter comes through the fact that θ is not known and should be adapted so that the original equation is also satisfied at the point $\xi = 0, \eta = 0$. Following this idea we use the original equation in the middle point as definitive equation for θ, namely

$$\theta = \left. \frac{\Delta(\gamma^2 u - \beta \Delta u) - (u_{\xi\xi} c_1^2 + 2c_1 c_2 u_{\xi\eta} + c_2^2 u_{\eta\eta})}{\alpha \Delta u^2} \right|_{\xi=0,\eta=0} . \tag{6}$$

Eq.(3) written for u, together with (5) and (6) form a well posed problem. Because of nonlinearity, this boundary-value problem has to be solved iteratively. One way to do this is to add artifical time derivative u_t in the governing equation and to start from arbitrary non-zero initial condition and timestep the solution until convergence is reached in a sense that two consecutive time steps do not differ significantly.

Without fear of confusion we will use once again the independent variable x, y in the moving frame *in lieu* of ξ, η.

DIFFERENCE SCHEME

We consider rectangular region $L_1 = L_2$ and specify the same number of grid intervals N in both directions. The continuous function u is replaced by the grid function

$$u_{ij}^n = u(ih, jh; nk), \qquad i = 0, 1, \ldots, N, j = 0, 1, \ldots, N,$$

where h is the spacing in both x- and y-directions and $k = \Delta t$ is time increment

We use an explicit difference scheme which is stable only when $\tau \leq \frac{1}{2} h^4$. It makes it inefficient for a bigger numerical investigation, but the purpose of this work is to investigate the practical properties of the approximation and to prepare a solution that can be used in further works as benchmark. Also, the main purpose is to find the actual shape of the wave and to discuss its physical relevance.

The iterative finite difference scheme reads

$$\begin{aligned}
\frac{u_{i,j}^{n+1} - u_{i,j}^n}{k} &= (\gamma^2 - c_1^2) \frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + (\gamma^2 - c_2^2) \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \\
&- 2c_1 c_2 \frac{u_{i+1,j+1}^n - u_{i+1,j-1}^n - u_{i-1,j+1}^n + u_{i-1,j-1}^n}{4h^2} + \alpha \left(\frac{(u_{i+1,j}^n)^2 - 2(u_{i,j}^n)^2 + (u_{i-1,j}^n)^2}{h^2} \right. \\
&+ \left. \frac{(u_{i,j+1}^n)^2 - 2(u_{i,j}^n)^2 + (u_{i,j-1}^n)^2}{h^2} \right) - \beta \left(\frac{u_{i+2,j}^n - 4u_{i+1,j}^n + 6u_{i,j}^n - 4u_{i-1,j}^n + u_{i-2,j}^n}{h^4} \right. \\
&+ 2 \frac{u_{i+1,j+1}^n - 2u_{i,j+1}^n + u_{i-1,j+1}^n - 2(u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n)}{h^2 h^2} \\
&+ \left. \frac{u_{i+1,j-1}^n - 2u_{i,j-1}^n + u_{i-1,j-1}^n}{h^2 h^2} + \frac{u_{i,j+2}^n - 4u_{i,j+1}^n + 6u_{i,j}^n - 4u_{i,j-1}^n + u_{i,j-2}^n}{h^4} \right).
\end{aligned} \tag{7}$$

A difference version of the equation for θ is coupled to (7).

The grid b. c. are $u_{0j} = u_{1j} = u_{N-1j} = u_{Nj} = 0$, $u_{i0} = u_{i1} = u_{iN-1} = u_{iN} = 0$.

VALIDATION OF THE SCHEME

Since the iterations with respect to the artificial time are conducted until convergence, the term approximating the time derivative disappears for large number of iterations. Then the accuracy of the computations is affected by two main factors: magnitudes of spatial increment h and the cut-off value L of the region (the "actual infinity"). Our numerical experiments show that the difference of the calculated shapes for two different boxes with $L = 25$ and $L = 50$, differ only by 0.003 which means that size $2L = 100$ is fully enough to provide ample space for the solution to decay properly for large x, y.

In order to assess the consistency of the scheme we have performed calculations for the same value $2L = 100$ on grids with different N: 100, 200, and 400. Respectively, h is 1, 0.5, or 0.025. The norm of the difference between the solutions on different grids can be defined as $e(N) = \max_{i,j} |\hat{u}_{ij} - \tilde{u}_{ij}|$, where $\hat{u}$ and $\tilde{u}$ are the functions on the respective grids. The truncation error of the scheme in our case is $e(N) \simeq O(h^2)$. For a secon-order scheme, reducing the spacing twice should reduce the error four times. Indeed, we have observed that $e(100) = 0.035$ and $e(200) = 0.008$ which are approximately in the ratio 4:1.

The absolute value of error $e(200) = 0.008$ gives the estimate $e \simeq 0.03h^2$. This ensures one that the truncation error for $N = 400$ will be of order of 0.002 which is small enough to claim that a grid with 400x400 points is reliable for the problems under consideration. All results reported in what follows are obtained with grid 400×400 points.

RESULTS

In Fig. 1, the comparison of the shapes of a soliton at rest, and a soliton propagating in x-direction with phase speed $c_2 = 0.6$, is shown. The lower pannesl present the lines of sonstant elevation/depression and the negative isolines are taken much denser, because the depressions are much smaller than the elevations. In both cases there is slight depression around the main hump which is the main difference between the 2D soliton and the *sech* soliton in the 1D case.

However, when the soliton is moving, the depression is deeper in front and behind the soliton. A better quantitative description of this effect is presented in Fig. 2 where the x- and y- cross sections are presented. The upper row of graphs show the function itself, the lower row shows the absolute value of the function. The downward spikes in the lower row are in the places where the function changes it sign.

Now it is seen that the depression is present even for $c = 0$, but in the x cross section (right panels) the first change of sign is approximately in the same place, just the depth is increasing with c. In the y cross-section the passage through zero elevation takes much farther from the center which means that the soliton is elongated in y-direction. Thus the elongation of the soliton takes place in the transverse direction which is completely non-intuitive if one is guided by the 1D results where the elongation takes place in longitudinal direction.

In the end we present the cross section of the soliton shape. We chose the value 0.1 for the contour line to be compared, because the zero line spans much bigger domain and is not convenient for presentation. The left panel in Fig. 3 presents the contour for

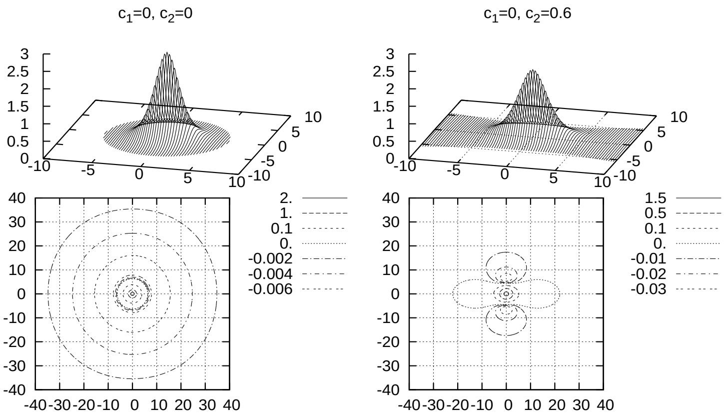

FIGURE 1. The shapes of the 2D Boussinesq soliton. Left: $c_1 = 0.c_2 = 0$. Right: $c_1 = 0.6, c_2 = 0$.

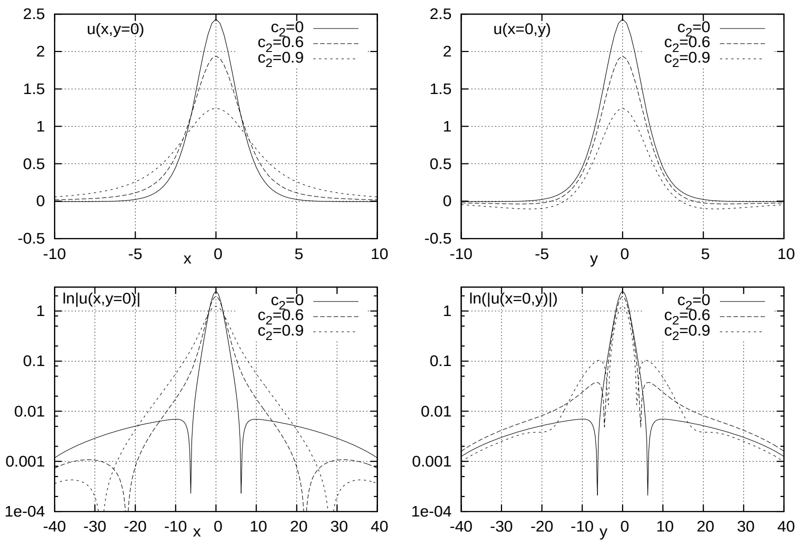

FIGURE 2. The comparison between thecross-sections of the soliton's shape. Left: $y = o$. Right: $x = 0$. Upper panels show the shape, the lower panels show the logarithm of the absolute value.

$c_1 = 0$ and different c_2. The shape becomes elliptical in a similar fashion as in Lorentz contraction. It is seen that the transverse scale is slightly contracted. The conclusion is that the 2D Boussinesq quasi-particle undergo relative pseudo-Lorentz contraction in the direction of motion.

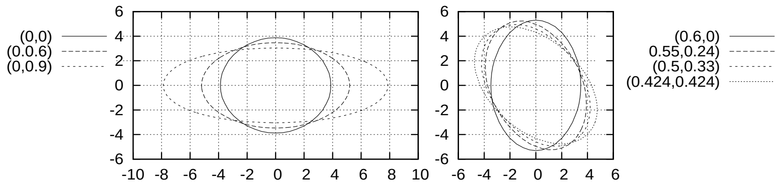

FIGURE 3. Left: Contour line 0.1 for motion in y-direction with phase speeds $c = 0, 0.6, 0.9$. Left: Contour line 0.1 for different c_1 and c_2, but for the same magnitude $c = \sqrt{c_1^2 + c_2^2} = 0.6$ of the phase speed.

The right panels of the Fig. 3 depicts the orientation of the elliptic cross section for different combination of the components of the phase velocity, but when the amplitude of the latter is kept constant $c = \sqrt{c_1^2 + c_2^2} = 0.6$. It is seen that within the error of approximation we have the same ellipse, with its major and minor axis aligned exactly with the direction of the phase velocity.

CONCLUSION

In the present paper the shape of the 2D Boussinesq soliton is calculated numerically. This is a result not known in the literature and shows two main physical differences in comparison with the 1D case: there is pseudo-Lorentzian elongation in the transverse direction, and there are depressions in front and in the rear of the soliton. This shows that one has to proceed with caution when comparing the qualitative features of quasi-particles in 2D to the well studied 1D cases.

The solution is obtained on high density grid 400×400 and can be used in future works as initial condition for numerical experiments for collision of Boussinesq solitons.

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