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IDEAS

Fourier-Galerkin Method for Time Dependent Problems of Interacting Localized Waves

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Abstract. We develop a Fourier-Galerkin spectral technique for computing solutions of type of interacting localized waves. We use a special complete orthonormal system of functions in $L^2(-\infty,\infty)$. The rate of convergence of the coefficients is shown to be exponential. As a featured example we consider the Boussinesq Paradigm Equation (BPE). We obtain results for the head-on and overtaking collisions of two or three solitons. We evaluate also the phase shifts of solitons.

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INTRODUCTION

In recent years a number of physical problems have led to boundary value problems in infinite domains. These are the cases when no boundary conditions are specified at given points, but rather the square of the solution (or some other energy norm) is required to be integrable over an infinite domain. Such solutions are said to belong to the $L^2(-\infty,\infty)$ space. A typical example is furnished by the problem for solitary wave (soliton) solutions. There are many difficulties on the way of application of difference or/and finite-element numerical methods to the problems in $L^2(-\infty,\infty)$, especially when some more subtle characteristics are sought. Very often, the finite-domain problem has a solution only for some denumerable set of intervals of specific length. It can even happen that each of the finite-domain approximations has only a trivial solution, while the original problem possesses a nontrivial one, or *vice versa*.

This difficulty can be surmounted if a spectral method with basis system that consists of localized functions is used instead of a difference method. Such expansion will automatically acknowledge the requirement that the solution belongs to $L^2(-\infty,\infty)$ space. Here we make use of a complete orthonormal (CON) system of functions proposed in [1]. These functions are orthogonal without weight and for them expression are available relating the product of two members of the system into series with respect to the system. This property is crucial because it allows one to use a Galerkin type expansion, the latter being much simpler and faster in implementation than the pseudo-spectral algorithms.

BOUSSINESQ PARADIGM EQUATION

Consider the so-called Boussinesq Paradigm equation (BPE),

$$u_{tt} = (u - 3u^2 + \beta_1 u_{tt} - \beta_2 u_{xx})_{xx}, \tag{1}$$

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which has an analytical *sech* soliton solution in the moving frame [2]

$$u = \frac{1}{2}(c^2-1)\mathrm{sech}^2\left(\frac{\mathrm{x}-\mathrm{ct}}{2}\sqrt{\frac{\mathrm{c}^2-1}{\beta_1\mathrm{c}^2-\beta_2}}\right). \tag{2}$$

As we can see from (2), the soliton exists for $c > 1$ (supercritical phase speeds).

Before proceeding further we mention that re-scaling the spatial variable x does not change the nature of the asymptotic boundary value problem in $L^2(-\infty,\infty)$. Upon introducing $z = \zeta x$ we recast (1) to the following

$$u_{tt} = \zeta^2(u - 3u^2 + \beta_1 u_{tt} - \zeta^2\beta_2 u_{zz})_{zz} \quad \text{with a.b.c} \quad u(t,z) \to 0, \quad z \to \pm\infty\,. \tag{3}$$

The scaling parameter ζ can be used to optimize the method in the sense that its introduction allows one to bring the typical length scales of the employed system of functions closer to the length of support of the sought localized solution.

We introduce an auxiliary function q and show that when localized solution is sought, eq.(3) is a corollary of the following system

$$u_t = \zeta^2 q_{zz}\,, \tag{4}$$
$$q_t = u - 3u^2 + \beta_1 u_{tt} - \beta_2\zeta^2 u_{zz}\,. \tag{5}$$

Function q is also a localized function but it can assume nonzero values at infinities. It has the shape of a hydraulic jump (or *kink*). Then the asymptotic boundary conditions for the system (4), (5) have the form

$$u,\ q_z \to 0 \quad \text{for} \quad z \to -\infty,\infty.$$

The initial condition is the superposition of two *sech*-solutions which are situated far enough from each other in order to neglect their intersection in the initial moment. We can find initial condition for the function $q(z,t)$ of each soliton after we integrate the first equation of (5) twice with respect to z in the moving frame. Then

$$q(z,t) = \frac{-3c_1}{\zeta^2}\sqrt{c_1^2-1}\,\tanh\left[\left(\frac{z_1}{2\zeta} - c_1 t\right)\sqrt{\frac{c_1^2-1}{\beta_1 c_1^2-\beta_2}}\right] + \frac{-3c_2}{\zeta^2}\sqrt{c_2^2-1}\,\tanh\left[\left(\frac{z_2}{2\zeta} - c_2 t\right)\sqrt{\frac{c_2^2-1}{\beta_1 c_2^2-\beta_2}}\right].$$

Now, function $q \notin L^2(-\infty,\infty)$ and it cannot be developed into series with respect to a CON system of functions whose members vanish at infinities. We use another auxiliary function $r(z)$ which "absorbs" the undesired behavior of $q(z,t)$, namely $p(z,t) = q(z,t) + r(z)$, where

$$r(z) = -\left[\frac{3c_1}{\zeta^2}\sqrt{\frac{c_1^2-1}{\beta_1 c_1^2-\beta_2}} + \frac{3c_2}{\zeta^2}\sqrt{\frac{c_2^2-1}{\beta_1 c_2^2-\beta_2}}\right]\tanh(z).$$

Note that the introduction of the new function $p(z,t)$ does not alter the equation (3) because $r(z)$ does not depend on t. In terms of $p(z,t)$, the system (4), (5) reads

$$u_t = \zeta^2 p_{zz} + \zeta^2 r_{zz}\,, \tag{6}$$

$$p_t = u - 3u^2 + \beta_1 u_{tt} - \zeta^2 \beta_2 u_{zz} \tag{7}$$

This is an essential point for the application of the method.

THE FOURIER-GALERKIN METHOD IN $L^2(-\infty,\infty)$

From the known spectral techniques, we choose the Galerkin method because it has the advantage of simplicity in implementation in comparison with the spectral collocation method or tau-method (see the arguments in [3]). The Galerkin technique requires explicit formulas expressing the products of members of the CON system into series with respect to the system. A system with the desired property has been proposed in 1982 in [1] and since then applied to Boussinesq equation and some other nonlinear-wave problems (see, [4, 5, 6]). The applications were limited to the stationary in the moving frame waves. A completely new application is started by the present authors [7] for the classical Boussinesq equation, and in [8] for the Reguralized Long-Wave equation [9]. Here we forward the technique for the equation with mixed fourth derivative.

The system

$$\rho_n = \frac{1}{\sqrt{\pi}} \frac{(ix-1)^n}{(ix+1)^{n+1}}, \quad n = 0, 1, 2, \ldots \tag{8}$$

was introduced by Wiener [10] as Fourier transform of the Laguerre functions (functions of parabolic cylinder). Higgins [11] defined it also for negative indices n and proved its completeness and orthogonality. The significance of (8) for nonlinear problems was demonstrated in [1], where the product formula was derived and the two real-valued subsequences of odd functions S_n and even functions C_n were introduced, namely

$$\rho_n \rho_k = \frac{\rho_{n+k} - \rho_{n-k}}{2\sqrt{\pi}}, \qquad S_n = \frac{\rho_n + \rho_{-n-1}}{i\sqrt{2}}, \quad C_n = \frac{\rho_n - \rho_{-n-1}}{\sqrt{2}}\,. \tag{9}$$

One easily shows that the product (*e.g.*, $C_n C_k$) of members of the real-valued sequences is expanded in series with respect to the system as follows (see, [1]):

$$C_n C_k = \frac{1}{2\sqrt{2\pi}}[\quad C_{n+k+1} - C_{n+k} - C_{n-k} + C_{n-k-1}] = \sum_{m=1}^{\infty} \beta_{nk,m} C_m \tag{10}$$

$$\beta_{nk,m} = \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k} + \delta_{m,|n-k|} - \delta_{m,n+k+1} - \mathrm{sgn}[|n-k| - 0.5]\delta_{m,[|n-k|-0.5]}\}.$$

Before proceeding further we discuss the way to increase the computational effectiveness of the product formulas. If we consider an even function from the $L(-\infty,\infty)$ space, say $U(x) = \sum_{n=0}^{\infty} u_n C_n$, we can show that

$$U^2(x) = \sum_{n=0}^{N}\sum_{m=0}^{N} u_n u_m C_n C_m = \sum_{l=0}^{N}\left[\sum_{n=0}^{N}\sum_{m=0}^{N}\beta_{nm,l} u_n u_m\right] C_l \stackrel{\text{def}}{=} \frac{1}{2\sqrt{2\pi}}\sum_{l=0}^{N} b_l C_l, \tag{11}$$

$$b_l = \sum_{n=0}^{l-1} u_n u_{l-1-n} - \sum_{n=0}^{l} u_n u_{l-n} - 2\sum_{n=l}^{N} u_n u_{n-l} + 2\sum_{n=l+1}^{N} u_n u_{n-l-1}. \tag{12}$$

For the second derivative of the basis functions one has (see [1])

$$C_n'' = \sum_{m=0}^{\infty} \chi_{m,n} C_m\,, \qquad S_n'' = \sum_{m=0}^{\infty} \chi_{m,n} S_m\,,$$

$$\chi_{m,n} = -\tfrac{1}{4} n(n-1)\delta_{m,n-2} + n^2\delta_{m,n-1} - \tfrac{1}{4}(n+1)(n+2)\delta_{m,n+2}$$

$$-\tfrac{1}{4}n^2 + (2n+1)^2 + (n+1)^2\delta_{m,n} + (n+1)^2\delta_{m,n+1},$$

Here χ is a diagonal matrix (pentadiagonal, more specifically), which can be inverted for $O(N \ln N)$ operations. This gives another edge in the computational efficiency of the developed here technique.

An important property of the proposed expansion is its exponential convergence ([7]).

THE TIME-STEPPING ALGORITHM

We develop the sought solution u, p into series with respect to C_n and S_n namely,

$$u^l(z) = \sum_{n=0}^{\infty} a_n^l C_n(z) + b_n^l S_n(z), \quad p^{l+\frac{1}{2}}(z) = \sum_{m=0}^{\infty} d_m^{l+\frac{1}{2}} C_m(z) + e_m^{l+\frac{1}{2}} S_m(z). \tag{13}$$

Now, we insert the spectral expansion (13), into equations (6), (7) and making use the orthogonality of the system we get the discrete equations

$$\frac{a_m^{l+1} - a_m^l}{\tau} = \zeta^2 \sum_{k=0}^{\infty} d_k^{l+\frac{1}{2}} \chi_{m,k}, \qquad \frac{d_m^{l+\frac{1}{2}} - d_m^{l-\frac{1}{2}}}{\tau} = -\frac{\zeta^2}{2}\sum_{k=0}^{\infty}(a_k^{l+1} + a_k^{l-1})\chi_{k,m}$$

$$+\frac{a_m^{l+1} + a_m^{l-1}}{2} + \beta_1 \frac{a^{l+1} - 2a^l + a^{l-1}}{\tau^2} - \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(a_{k_1}^l a_{k_2}^l \beta_{k_1 k_2,m} + b_{k_1}^l b_{k_2}^l \alpha_{k_1 k_2,m})$$

$$\frac{b_m^{l+1} - b_m^l}{\tau} = \zeta^2 \sum_{k=0}^{\infty}\left(b_k^{l+\frac{1}{2}} + r_k\right)\chi_{m,k}, \qquad \frac{e_m^{l+\frac{1}{2}} - e_m^{l-\frac{1}{2}}}{\tau} = -\frac{\zeta^2}{2}\sum_{k=0}^{\infty}(b_k^{l+1} + b_k^{l-1})\chi_{k,m}$$

$$+\frac{b_m^{l+1} + b_m^{l-1}}{2} + \beta_1 \frac{b^{l+1} - 2b^l + b^{l-1}}{\tau^2} - \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(a_{k_1}^l b_{k_2}^l \gamma_{k_1 k_2,m} + b_{k_1}^l a_{k_2}^l \gamma_{k_1 k_2,m}).$$

where $\alpha_{k_1 k_2 k_3}$ and $\gamma_{k_1 k_2 k_3}$ are similar to $\beta_{k_1 k_2 k_3}$ given in (10). Respectively, r_k are the coefficients of the spectral expansion of function r_{zz}. In the numerical calculations a truncated version of the above system is used in which the infinity is replaced by N.

The initial conditions $\{a_n^0\}$, $\{b_n^0\}$ and $\{a_n^1\}$, $\{b_n^1\}$ for the Fourier coefficients are calculated for $t = 0$ and $t = \tau$ by means of numerical quadrature of the analytic solution formulas after multiplying them by C_n or S_n. In its turn, the initial conditions for the coefficients $\{d_n^{\frac{1}{2}}\}$, $\{e_n^{\frac{1}{2}}\}$ of p are computed via numerical quadrature of its analytic expression after multiplying the latter by C_n or S_n. Note that the initial conditions have to be calculated anew every time when the value ζ of the scaling parameter is changed. Having specified the initial conditions we can begin the time stepping. Let us assume that the variables $\{a_n^{l-1}\}$, $\{b_n^{l-1}\}$, $\{a_n^l\}$, $\{b_n^l\}$, $\{d_n^{l-\frac{1}{2}}\}$, $\{e_n^{l-\frac{1}{2}}\}$ are known. Then our systems for the coefficients give two coupled nine-diagonal algebraic systems for $\{b_n^{l+1}\}$, $\{e_n^{l+\frac{1}{2}}\}$ and for $\{a_n^{l+1}\}$, $\{d_n^{l+\frac{1}{2}}\}$, respectively. After these systems are solved and a time step is completed, the time index l is reset, and the process is repeated.

VALIDATIONS OF THE SCHEME

The optimal ζ is different for different initial configurations of the system of solitons. When the solitons aresituated far from each other, one is faced with a wave configuration which is not tightly localized. Then the optimal value of ζ is smaller. Conversely, if in the initial configuration the solitons are close enough, the value of ζ tends to be larger. In the present work we consider initial configurations of solitons that are well separated (in order not to overlap significantly), but not excessively far from each other (not to loose the localization). After conducting extensive numerical experiments we found for these cases that the optimal value of the scaling parameter is in the vicinity of $\zeta = 0.06$. This is the value for which the convergence was faster and more accurate

As far as the time increment is concerned we found that the calculations are perfectly stable for τ as large as 0.1 even for $c = 2$, which is a very large value from the point of view of weakly-nonlinear approximation.

As already mentioned above, the rate of convergence of the Galerkin series is exponential and our calculations comply with this analytical result. In Figure 1 we present the Galerkin coefficients for the case $c_1 = 1.8$ and $c_2 = -1.4$. As we can see, the convergence remains exponential after 300 time steps even though we used a rather large time increment $\tau = 0.1$.

RESULTS AND DISCUSSION

For the Proper Boussinesq Equation, the faster solitons have smaller amplitudes, while in the case of the BPE (as in the KdV and Reguralized Long-Wave equations), the faster solitons are taller. The amplitude of the wave is getting smaller as $c \to 1, c > 1$. When $c = 1$, then $u(z,t) = 0$,

We computed the evolution of several different initial configurations using $\zeta = 0.06$ and $\tau = 0.1$. For c close to unity the interaction is completely shape preserving and after the interaction the two solitons reappear with their exact shapes and the only sign of inelasticity is the phase shift. For larger c we observe that after the separation of

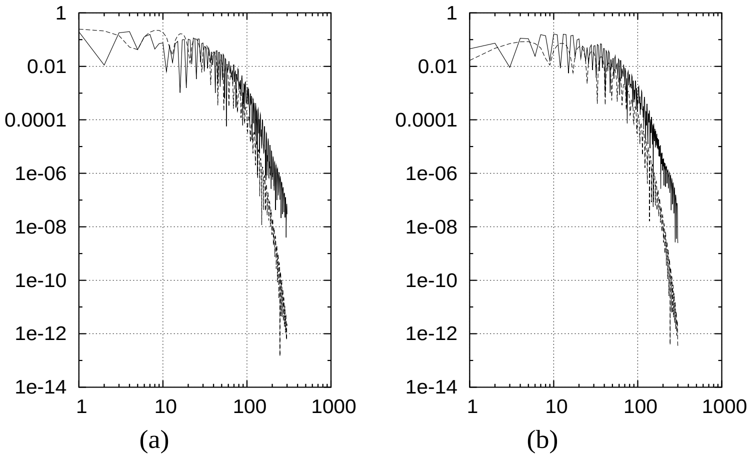


FIGURE 1. Exponential decay of the computed coefficients for $\zeta = 0.06$ and $N = 250$: (a) Even coefficients; (b) Odd coefficients.

the solitons a residual signal appears, which was also reported in the work using finite difference scheme [12].

The scheme proposed here conserves the energy of the system up to six decimal digits which made possible the investigation of the interaction for very long times. Note that even the slightest but persistent "leakage" of energy during the calculations would have led to eventual linear dispersion of the solution and disappearance of the permanent $(sech)$ shapes. The quantitative agreement with the finite-difference scheme is very good.

The cases of two equal solitons require only even functions in the expansion while the general case needs also the odd functions and reveals better the effectiveness of the method developed here. We present the interaction of two nonequal solitons in Fig. 2.

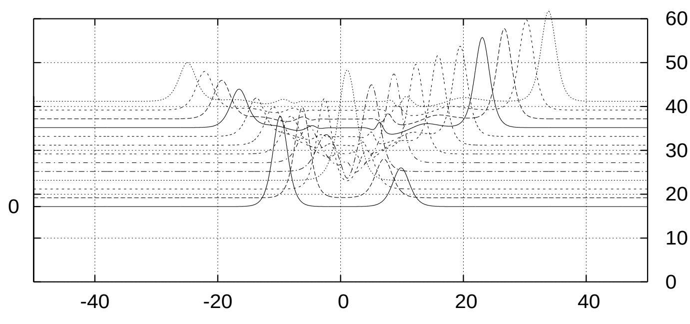


FIGURE 2. Collision of two nonequal solitons for $c_l = 1.8$ and $c_r = -1.4$

An important advance of the numerical technique here in comparison with our previous works is that we succeeded to apply it also to the overtaking collisions of solitons. The specific difficulty here is that the whole pattern moves away from the origin of the coordinate systems which decreases the role of the lower-order terms in the expansion and degrades the practical convergence. This requires larger number of terms. Yet, our calculations show that $N = 250$ is fully adequate number even for the case of tipple interaction of solitons as presented in Fig. 3.

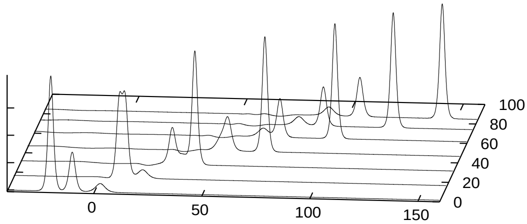
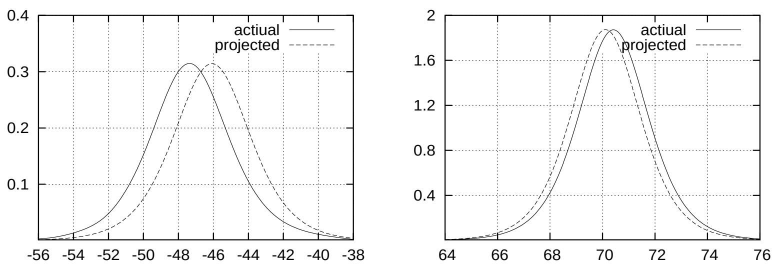


FIGURE 3. Overtaking of three nonequal solitons for $c_1 = 1.9$, $c_2 = 1.3$ and $c_3 = 1.1$.

In the end we focus our attention on the phase shift, describing the difference between the positions of a soliton with and without interaction after the same number of time steps. The idea of phase shift is illustrated in Fig. 4 for the collision of two solitons with $c_l = 1.5$ and $c_r = -1.1$ and for dimensionless time $t = 30$.



(a) $c_r = -1.1$ (b) $c_l = 1.5$

FIGURE 4. The phase shifts for a system of solitons. Solid line: actual position of a soliton; dashed line - the projected position for a noninteracting soliton at the same time $t = 30$.

Table 1. Computed phase shifts

c_1	Shift	c_2	Shift
1.4	1.2	-1.8	0.52
1.5	0.84	-1.5	0.84
1.2	1.38	-1.9	0.28
-1.2	1.24	-1.9	0.76

We have conducted extensive numerical experiments and obtained results for phase shifts for a large number of initial configurations of the solitons. A selection of phase shifts for different phase velocities of the solitons are compiled in Table 1. The last case in the table is for overtaking collision in which the taller soliton suffers relatively larger shift than in the head-on collision with the same smaller soliton.

CONCLUSIONS

A complete orthonormal (CON) basis system in $L^2(-\infty,\infty)$ is used to develop a localized solution into Fourier series with Galerkin identification of the coefficients. It is applied for finding the solitary waves for the so-called Boussinesq Paradigm Equation (BPE) which contains two kinds of dispersion. It is a development upon our previous works where only purely spatial or only purely mixed-derivative dispersions were considered.

In the two-soliton case under consideration, the localization of the solution is much less tighter and the number of terms needed for good approximation is larger. The treatment of the problem required very efficient implementation of the Fourier-Galerkin scheme. The numerical experiments confirm the exponential convergence of the method which allows to obtain highly accurate results for the time dependent problem with as few as $N = 40$ terms. This demonstrates the efficiency of the proposed technique and encourages the future use of the CON system.

It is shown that the solitons recover their exact shapes after the collision but experience phase shifts. When the phase speeds are large, the model does not comply with the weakly-nonlinear assumptions and residual signals are observed trailing the main humps.

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