Dynamics of Patterns on Elastic Hypersurfaces. Part I. Shear Waves in the Middle Surface

C. I. Christov

Dept. of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010

Abstract. The shear motions in an incompressible elastic continuum are considered and it is shown that, when linearized, the governing equations can be rendered into Maxwell's form. The trace of the deviator stress tensor is analogous to the electric field, while the vorticity (the *curl* of the velocity field) is interpreted as the magnetic field. We show that the analogy can be extended further to incorporate the so-called Lorentz force as the counterpart of the advective nonlinearity of the elastic model. Localized shear dislocations are considered and shown to undergo Lorentz contraction in the direction of motion. Thus an interesting and far reaching analogy between the elastic continuum and the electrodynamics is established.

Keywords: Electrodynamics, Elastic Shear Waves
PACS: 03.50.De, 62.30.+d

INTRODUCTION

Electromagnetic phenomena are an epitome of action at a distance. On its turn the action at a distance is the intrinsic characteristic for any material continuum, and is a consequence of the action of internal stresses. For this reason, it is still a valid avenue of research to attempt to understand the luminiferous *field* as a material continuum in which the *internal stresses* are the transmitter of the long-range interactions.

An all-time candidate for the luminiferous field is the elastic medium because — as shown by Cauchy (see, [1]) — it gives a good quantitative prediction for the shear-wave phenomena (light) and explains quantitatively very well the experiments of Young and Fresnel.

It is only natural to consider the analogy between the shear waves in an elastic continuum and the electromagnetic theory of light based on Maxwell's equations. Building upon our previous work in [2] and [3], we show, in this paper, that the linearized equations of the Hookean elastic continuum admit a Maxwell's form provided that the trace of the stress tensor is understood as the electric field and the *curl* of the velocity vector as the magnetic field. In the present paper we present briefly the derivations pertinent to this analogy.

There is one difference, however. The model of elastic continuum naturally incorporates the Galilean invariance due to the advective part of the material time derivative, while the Maxwell's equations are not Galilean invariant and there is no feasible way to make them invariant if kept in their original form. For this reason, the term connected to the advective part of the time derivative appears in the classical electrodynamics as the Lorentz force on a moving charge. In this sense, the analogy developed here is between the elastic medium and the augmented Maxwellian electrodynamics, in which the

CP755, *ISIS: International Symposium on Interdisciplinary Science*
edited by A. Ludu, N.R. Hutchings and D.R. Fry

Lorentz force is part of the basic assumptions of the model and not an empirically added term.

We consider localized vortex-like solutions for the displacement field and call them twistons. They can propagate as patterns through the medium without changing shape, and possess topological charge. The kinematics of torsional localized waves is the object of the present short note.

In the second part of the work (see, the present proceedings) we will also consider effects connected with the curvature when the 3D elastic continuum is assumed to be a hypersurface in a 4D geometrical space.

CAUCHY VS MAXWELL

For small velocities, the Lagrangian and Eulerian descriptions of a Hookean elastic medium coinside (see, e.g., [4])

$$\mu_0 \frac{\partial \vec{v}}{\partial t} \stackrel{\text{def}}{=} \mu_0 \frac{\partial^2 \vec{u}}{\partial t^2} = \nabla \cdot \tau, \qquad \tau = \eta\,(\nabla \vec{u} + \nabla \vec{u}^T) + \lambda (\nabla \cdot \vec{u}) I, \tag{1}$$

where $\vec{u}, \vec{v}$ are the displacement and velocity vectors; η, λ are Lamé elasticity coefficients, μ_0 is the density of the elastic continuum in material (Lagrangian) coordinates, τ is the stress tensor in the Hooke's law for elastic body, and I stands for the unit tensor. Here the elastic coefficients η, λ and the density μ_0 are constant.

Eqs.(1) govern both the shear and the compression/dilation motions. The phase speeds of propagation of the shear, c, and dilational, c_s, small disturbances are

$$c = \left(\frac{\eta}{\mu_0}\right)^{\frac{1}{2}}, \qquad c_s = \left(\frac{2\eta + \lambda}{\mu_0}\right)^{\frac{1}{2}}, \quad \delta = \frac{\eta}{2\eta + \lambda}. \tag{2}$$

The interpretation of Cauchy is that c is the speed of light because it is connected to the transverse (shear) waves the latter being the light waves according to Young and Fresnel.

To deal with the second Lamé coefficient, ant thus with the *speed of sound*, c_s, one has two options: to consider an extremely compressible ("volatile") continuum with $c_s = 0$, or an incompressible continuum with $c_s \to \infty$. Cauchy chose the first option and although he succeeded in obtaining quantitative model for the Fresnel observations, his choice left the model of the luminiferous continuum in an unsatisfactory state. The second option refers to an incompressible continuum whose speed of sound is much greater than the speed of light, i.e. $\delta \ll 1$.

Here we examine the limiting case of virtually incompressible continuum when $\lambda \gg \eta$ ($\delta \ll 1$), then eq.(1) can be recast as follows

$$\delta \left(c^{-2} \frac{\partial^2 \vec{u}}{\partial t^2} + \nabla \times \nabla \times \vec{u} \right) = \nabla (\nabla \cdot \vec{u}). \tag{3}$$

Displacement $\vec{u}$ can be developed into an asymptotic power series with respect to δ

$$\vec{u} = \vec{u}_0 + \delta \vec{u}_1 + \cdots \ . \tag{4}$$

Introducing eq.(4) into eq.(3) and combining the terms with like powers we obtain for the first two terms

$$\nabla(\nabla\cdot\vec{u}_0) \;=\; 0\,, \tag{5}$$

$$c^{-2}\frac{\partial^2\vec{u}_0}{\partial t^2}+\nabla\times\nabla\times\vec{u}_0 \;=\; \nabla(\nabla\cdot\vec{u}_1). \tag{6}$$

From (5) one can deduce

$$\nabla\cdot\vec{u}_0=\text{const},\qquad \text{or}\qquad \nabla\cdot\vec{v}_0=0\,. \tag{7}$$

The preceeding is also a linear approximation to incompressibility condition for an elastic continuum. In the general model of nonlinear elasticity with finite deformations the incompressibility condition is imposed on the Jacobian of transformation from the material to the geometrical variables, but in the first-order approximation in δ, eq.(7) holds true. Henceforth we omit the index "0" for the variable $\vec{u}$ without fear of confusion. We denote formally

$$\varphi \stackrel{\text{def}}{=} -(\lambda+2\eta)\nabla\cdot\vec{u}_1,\qquad \vec{A}\stackrel{\text{def}}{=}\vec{v}_0. \tag{8}$$

It is also convenient to introduce the deviator tensor

$$\tau^0=\eta\,(\nabla\vec{u}+\nabla\vec{u}^T)-2\eta\,(\nabla\cdot\vec{u})I.$$

The divergence of the deviator part of the stress tensor τ^0 gives a body force to which the action of the internal shear stresses of the continuum are reduced. We call this body force the "electric field", namely

$$\vec{E}\stackrel{\text{def}}{=}\nabla\cdot\tau^0\equiv\eta\nabla^2\vec{u}=-\eta\nabla\times\nabla\times\vec{u}, \tag{9}$$

where the last equality is obtained by acknowledging the incompressibility condition $\nabla\cdot\vec{u}=const$. Now we can recast the linearized Cauchy balance eq.(6) in terms of $\vec{E}$, as

$$\vec{E}=-\mu_0\frac{\partial\vec{A}}{\partial t}-\nabla\varphi\,, \tag{10}$$

which involves $\vec{A}$ and φ. In the same vein we define a "magnetic induction" $\vec{B}$ and "magnetic field" $\vec{H}$ as follows

$$\vec{B}=\mu_0\nabla\times\vec{A}=\mu_0\vec{H},\qquad \vec{H}\stackrel{\text{def}}{=}\nabla\times\vec{A}. \tag{11}$$

The system of eqs.(10)–(11) is simply the equations of electrodynamics in terms of $\vec{A}$ and φ which play the role of the vector and scalar potentials of electromagnetic field (see, [5]). Note that the density μ_0 of the elastic continuum appears as the magnetic permeability of the Maxwellian field.

Now one can derive the original Maxwell's equations. Taking the operation *curl* of eq.(10) and acknowledging eq.(11) we obtain

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \tag{12}$$

which is the first of Maxwell's equations (Faraday's law). Respectively, from eqs.(9), (11), and (8) one obtains

$$\frac{1}{\eta}\frac{\partial \vec{E}}{\partial t} = \nabla\times(\nabla\times\frac{\partial \vec{u}}{\partial t}) \equiv \nabla\times\vec{H}\,. \tag{13}$$

The last equation is precisely the Maxwell's second equation provided that the shear elastic modulus of metacontinuum is interpreted as the inverse of electric permittivity $\eta = \varepsilon_0^{-1}$. Thus we have shown that the Maxwell's second equation is a corollary of the elastic constitutive relation for the luminiferous continuum and is responsible for the propagation of the shear stresses (action at a distance).

The condition $\nabla\cdot\vec{H} = 0$ (Maxwell's third equation) follows directly from the very definition of magnetic field. Similarly, taking *div* of eq.(9), one gets $\nabla\cdot\vec{E} = 0$.

Thus, we have shown that the linearized equations of elastic continuum admit what can be called *Maxwell's form*. In the framework of such a paradigm, each point of the elastic continuum experiences a body force $\vec{E}$, to which the action of the internal elastic stresses is reduced. We call it the "electric force". The angular momentum of the velocity of a material point is called the "magnetic field".

NONLINEARITY, GALILEAN INVARIANCE, AND LORENTZ FORCE

A far reaching consequence of the previous section is that it gives a clue of how to look for a Galilean invariance of the equations of the luminiferous field. In classical continua the Galilean invariance is connected to the advective nonlinearity of the governing equations. Then the pertinent question here is of what kind of effects are to be expected due to the convective nonlinearity.

Consider the governing equations of an elastic continuum in the Lamb's form [4]

$$\mu_0\left(\frac{\partial \vec{v}}{\partial t} + \frac{1}{2}\nabla|\vec{v}|^2 - \vec{v}\times\mathrm{rot}\vec{v}\right) + \nabla\varphi = -\vec{E}. \tag{14}$$

where we have already substituted the notations for the above defined scalar potential and electric field.

This form allows one to assess the forces acting at a given material point of the metacontinuum due to the convective accelerations of the latter. The gradient part of the convective acceleration can not be observed independently from the pressure gradient $\nabla\varphi$ in the continuum. In fact one can measure only the quantity $\varphi_1 \equiv \varphi + \frac{1}{2}\vec{v}^2$. Thus the only observable quantity is the last term of the acceleration. By virtue of our definition of magnetic induction (11) the term under consideration adopts the form

$$F_l = \mu_0\vec{v}\times\vec{H}. \tag{15}$$

Eq.(15) gives a force acting in each material point of the metacontinuum. This force is part of the inertial force which is lost when the equations are linearized and it is analogous to the so-called Lorentz force. To find the exact quantitative coefficient of the Lorentz force one has to integrate the above relation over the spatial extent of a test charge.

The above result suffice to claim that a *Galilean invariant* generalization of the electrodynamics is possible and it incorporates the Lorentz force as an integral part the latter being manifestation of the convective nonlinearity of this more general model.

LOCALIZED SHEAR WAVES — "TWISTONS"

The stationary version of the governing equations of the metacontinuum is the vectorial Laplace equation

$$\nabla^2 \vec{u} \equiv \frac{\partial^2 \vec{u}}{\partial x^2} + \frac{\partial^2 \vec{u}}{\partial y^2} + \frac{\partial^2 \vec{u}}{\partial z^2} = 0, \tag{16}$$

The following stationary vortex-like solution for the components of the displacements

$$u_x = \frac{y}{r^2}, \quad u_y = -\frac{x}{r^2}, \quad r = \sqrt{x^2 + y^2 + z^2}, \tag{17}$$

provides an example of a localized solution. Its divergence is easily proven to be zero, except the origin, where it is undefined, namely

$$\nabla \cdot \vec{u} \equiv \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \quad \text{for} \quad r \neq 0.$$

The solution (17) has a topological charge ("circulation" of the displacement field) for which a conservation law similar to Thomson's circulation theorem (see, e.g. [4]) holds,

$$\Gamma = \oint u_x \mathrm{d}x + u_y \mathrm{d}y, \qquad \frac{d\Gamma}{dt} = 0,$$

which means that once created, the torsional dislocation (17) cannot be destroyed unless dissipation is introduced in the medium.

In what follows, we call eq.(17) a "twiston". It could serve as a good analogy for the electrical charges. Note that it is only a part of a full fledged charge since there is no component in z-direction. Hence it is a polarized charge. Fig. 1-(a) shows the vector field of the displacement, except for the origin where there is a singularity.

Should the above described "dislocation" be allowed to move, it would not "plow" through the material points of the continuum. It will propagate as a *phase pattern*, much the same way a wave propagates over the water surface.

As already above mentioned, solution in eq.(17) is singular at the origin of the coordinate system. This is the same sort of singularity as the vortex solution in fluid dynamics. The improper behavior can be mitigated if a higher-order elasticity model is considered (see, e. g., the discussion in [6]), such as

$$\frac{\partial^2 \vec{u}}{\partial t^2} = \Delta \vec{u} - \nabla \varphi - \chi \Delta \Delta \vec{u}, \tag{18}$$

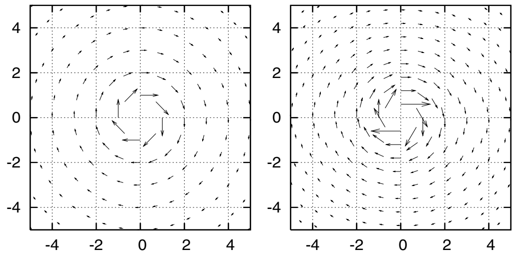

FIGURE 1. The localized torsional dislocation (twiston) in two dimensions for two different phase velocities of propagation. Left panel: $c_1 = 0, c_2 = 0$. Right panel: $c_1 = 0, c_2 = 0.8$

where χ is the coefficient of the higher-order elasticity. It is beyond the scope of present work to give a detailed description of higher-order elasticity. We just mention that upon introducing a "deformation function", $u_x = \frac{\partial\psi}{\partial y}$ and $u_y = -\frac{\partial\psi}{\partial x}$ in 2D, we can reduce eq.(18) for the model 2D case to a scalar equation for ψ; the preceeding admits

$$\chi\Delta\Delta\psi - \Delta\psi = 0, \qquad \psi = K_0(r/\sqrt{\chi}) + \ln(r/\sqrt{\chi}), \tag{19}$$

where $r = \sqrt{x^2+y^2}$ and has no singularity at the origin. For the displacement components, this gives

$$u_x = \frac{y}{r\sqrt{\chi}}\left[K_1\left(\frac{r}{\sqrt{\chi}}\right) + \ln\left(\frac{r}{\sqrt{\chi}}\right)\right], \quad u_y = -\frac{x}{r\sqrt{\chi}}\left[K_1\left(\frac{r}{\sqrt{\chi}}\right) + \ln\left(\frac{r}{\sqrt{\chi}}\right)\right]. \tag{20}$$

This form shows that there is a way to tackle the singularity and the vortex-like localized deformation can occur in a real continuum.

Finally, we also note that in the linearized model the amplitude of a twiston remains undetermined unless one considers the nonlinear model of finite elasticity.

FITZGERALD-LORENTZ CONTRACTION OF PATTERNS

Consider now a moving coordinate system, say

$$\xi = x - v_x t,\ \eta = y - v_y t,\ \zeta = z - v_z t, \quad \vec{v} = \frac{\partial \vec{\hat{u}}}{\partial t} + \vec{V}, \quad \vec{V} = (V_x, V_y, V_z).$$

where $\vec{\hat{u}}$ are the displacements referred to the Eulerian moving frame. The Galilean invariance means that the last term in the above expression cancels exactly the term in (14) arising from the time derivative in the moving frame. Consider also a stationary solution in the moving frame, when $\vec{\hat{u}}_t = 0$. Then eq.(16) transforms to the following

$$\left(1 - \frac{V_x^2}{c^2}\right)\frac{\partial^2 \vec{\hat{u}}}{\partial \xi^2} + \left(1 - \frac{V_y^2}{c^2}\right)\frac{\partial^2 \vec{\hat{u}}}{\partial \eta^2} + \left(1 - \frac{V_z^2}{c^2}\right)\frac{\partial^2 \vec{\hat{u}}}{\partial \zeta^2} = 0,$$

It is easily seen that upon scaling the independent variables as

$$\hat{x} = \xi \left(1 - \frac{V_x^2}{c^2}\right)^{-\frac{1}{2}}, \quad \hat{y} = \eta \left(1 - \frac{V_x^2}{c^2}\right)^{-\frac{1}{2}}, \quad \hat{z} = \zeta \left(1 - \frac{V_x^2}{c^2}\right)^{-\frac{1}{2}},$$

one arrives once again to a Laplace equation like eq.(16) in terms of the new variables. Then the solution in eq.(17) is valid for $\vec{\hat{x}}$, thus

$$u_x = \frac{y - c_y t}{r^2 \sqrt{1 - c_y^2}}, \quad u_y = -\frac{x - c_x t}{r^2 \sqrt{1 - c_x^2}}, \quad r = \sqrt{\frac{x^2}{1 - \frac{V_x^2}{c^2}} + \frac{y^2}{1 - \frac{V_y^2}{c^2}} + \frac{z^2}{1 - \frac{V_z^2}{c^2}}}. \qquad (21)$$

The lines of equal amplitude $|\vec{u}|$ of displacement for eq.(17) are concentric spheres while the same lines for eq.(21), are ellipsoids. as shown in Fig. 1-(b). The measures of twiston are scaled (shortened) exactly with the respective Lorentz factors. The amplitude of the deformations is increased (proportionally to $[1 - V^2/c^2]^{-\frac{1}{2}}$) in comparison with the twiston at rest, but the total circulation (the charge) remains the same. The increased amplitude means that it is harder to accelerate the steady propagating localized pattern in eq.(21) rather than the pattern at rest in eq.(17). This effect is well known as the increase of mass of moving bodies with the inverse of the Lorentz factor.

The conclusion of this section is that the localized solutions *must* experience contraction in the direction of propagation. The contraction of the localized waves is of the same nature as the Doppler effect for the harmonic waves [7].

CONCLUDING REMARKS

We consider propagation of shear waves in an incompressible elastic medium. We show that the linearized model admits Maxwell's form analogous to the classical electrodynamics. We introduce the notion of a localized shear wave which is a vortex-like structure of the displacement field. The vortex-like localized patterns of displacement field (called twiston) possess topological charges, which are related to the notion of electical charges. The twistons propagating with constant phase speed are shown to experience Lorentz contraction in the direction of motion. We found also that due to the advective nonlinearity of the material time derivative a force arises, which is an exact analogy of the Lorentz force acting on moving charges.

REFERENCES

1. E. Whittaker, *A History of the Theories of Aether & Electricity vol. 1 and 2*, Dover, New York, 1989.
2. C. I. Christov, "Discrete out of continuous: Dynamics of Phase patterns in Continua," in *Continuum Models and Discrete Systems – Proceedings of CMDS8*, edited by K. Markov, World Scientific, Singapore, 1996, pp. 370–394.
3. C. I. Christov, *Annuaire de L'Universite de Sofia*, **95**, 109–121 (2001).
4. L. I. Sedov, *A Course in Continuum Mechanics, vol. I and II*, Walters–Nordhoff, Groningen, 1981.
5. J. D. Jackson, *Classical Electrodynamics*, John Wiley Sons, 1998.
6. C. I. Christov, G. A. Maugin, and M. G. Velarde, *Phys. Rev. E*, **54**, 3621–3638 (1996).
7. T. P. Gill, *The Doppler Effect*, Logos Press, 1965.