Available online at www.sciencedirect.com
SCIENCE DIRECT®
Mathematics and Computers in Simulation 69 (2005) 257–268
MATHEMATICS AND COMPUTERS IN SIMULATION
www.elsevier.com/locate/matcom

Interacting localized waves for the regularized long wave equation via a Galerkin spectral method

M.A. Christou[a], C.I. Christov[b,*]

[a] *Department of Mathematics, University of Louisiana at Monroe, Monroe, LA, USA*
[b] *Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA*

Available online 25 March 2005

Abstract

We develop a Fourier–Galerkin spectral technique for computing the solutions of type of interacting localized waves. To this end, a special complete orthonormal system of functions in $L^2(-\infty, \infty)$ is used and a time-stepping algorithm implementing the spectral method is developed. The rate of convergence of the coefficients is shown to be exponential. We consider the regularized long wave equation (RLWE) which is not fully integrable. We demonstrate the stability of the algorithm and find numerically the threshold for the existence of such interactions. We also calculate the phase shifts of the interactions and compare them to the finite-difference solution.

Keywords: Localized waves; Regularized long wave equation; Galerkin spectral method

1. Introduction

In recent years, a number of physical problems have led to boundary value problems in infinite domains. These are the cases when no boundary conditions are specified at given points, but rather the square of the solution (or some other energy norm) is required to be integrable over an infinite domain. Such solutions are said to belong to the $L^2(-\infty, \infty)$ space. A typical example is furnished by the problem for solitary wave (soliton) solutions of different nonlinear evolution equations or generalized wave equations. There are many difficulties on the way of application of difference and/or finite-element numerical methods to

* Corresponding author. Tel.: +1 337 482 5273; fax: +1 337 482 5346.
E-mail address: christov.@louisiana.edu (C.I. Christov).

0378-4754/$30.00
doi:10.1016/j.matcom.2005.01.004

the problems in $L^2(-\infty, \infty)$ especially when some more subtle characteristics are sought. Sometimes, the finite-domain problem has a solution only for some denumerable set of intervals of specific length. It can even happen that each of the finite-domain approximations has only a trivial solution, while the original problem possesses a nontrivial one, or vice versa.

This difficulty can be surmounted if instead of a difference method a spectral method is used with basis system that consists of localized functions. Such expansion will automatically acknowledge the requirement that the solution belongs to $L^2(-\infty, \infty)$ space. Here, we make use of a complete orthonormal (CON) system of functions proposed in [4]. These functions are orthogonal without weight and for them expressions are available relating the product of two members of the system into series with respect to the system. This property is crucial because it allows one to use a Galerkin type expansion, the latter being much simpler and faster in implementation than the pseudo-spectral algorithms.

2. The physical problem

2.1. Posing the problem

John Scott Russell was first to observe the solitary ("permanent", "great") wave. Boussinesq provided a theory where the effect of the higher-order linear dispersion was incorporated and shown to balance the nonlinear steepening of the wave. However, the equation he derived turned out to be improper, incorrect in the sense of Hadamard. A way to overcome this incorrectness as is shown in [1], is to approximate the second spatial derivative with a second time derivative, $\frac{\partial^2}{\partial x^2} \approx \frac{\partial^2}{\partial t^2}$. This idea is known as regularized long wave equation (RLWE), namely

$$u_{tt} = (u - \alpha u^2 + \beta u_{tt})_{xx}, \quad \beta > 0 \quad (\text{here } \alpha = \beta = 1). \tag{2.1}$$

Similarly to the way Boussinesq found the "sech" type solution of his equation, one can find that RLWE possesses an analytical "sech" solution, namely

$$u = -\frac{3}{2}(c^2 - 1)\text{sech}^2\left(\frac{x - ct}{2c}\sqrt{c^2 - 1}\right). \tag{2.2}$$

As we can see from (2.2) that the soliton exists for $c > 1$ (supercritical phase speeds).

Before proceeding further, we mention that re-scaling the spatial variable x does not change the nature of the asymptotic boundary value problem in $L^2(-\infty, \infty)$. Upon introducing $z = \zeta x$, we recast (2.1) to the following

$$u_{tt} = \zeta^2(u - u^2 - \zeta^2 u_{tt})_{zz} \quad \text{with a.b.c.} \quad u(t, z) \to 0, \quad \text{for} \quad z \to \pm\infty. \tag{2.3}$$

The scaling parameter ζ can be used to optimize the method in the sense that its introduction allows one to bring the typical length scales of the employed system of functions closer to the support of the sought localized solution. Naturally, such a coordination between the scales will result in a faster convergence of the Fourier–Galerkin series.

2.2. Hamiltonian formulation

The Hamiltonian form of the Boussinesq model was shown in many different works, e.g. [11,9]. See also, the special discussion in [6].

So we are looking for a solitary wave solution of (2.1) which approaches zero at $\pm\infty$ and hence all its derivatives decay automatically to zero. We introduce an auxiliary function q and show that (2.3) is a corollary of the following system

$$u_t = \zeta^2 q_{zz}, \tag{2.4}$$

$$q_t = u - u^2 - \zeta^2 u_{tt}. \tag{2.5}$$

Here, the auxiliary function q is also a localized function but it can assume nonzero values at infinities. It has the shape of a hydraulic jump (or *kink*). Then, the asymptotic boundary conditions for the system (2.4) and (2.5) have the form

$$u, q_x \to 0 \quad \text{for} \quad x \to -\infty, \infty.$$

Define the wave mass M, the wave momentum (*pseudomomentum*) P, and the energy E of the wave system as

$$M = \int_{-\infty}^{\infty} u \, \mathrm{d}x, \qquad P = \int_{-\infty}^{\infty} u(q_x - \beta q_{xxx}) \, \mathrm{d}x, \qquad E = \int_{-\infty}^{\infty} \frac{1}{2}[u^2 + q_x^2 + u^3 + \beta u_t^2] \, \mathrm{d}x, \tag{2.6}$$

then we show after [9] that the following conservation/balance laws hold

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = \frac{1}{2}(q_x^2 - \beta q_{xx}^2)\Big|_{-\infty}^{\infty} \equiv F,$$

where F is called sometimes “pseudo-force”.

The conservation laws require that the mass and the energy remain constant during the evolution of the solution. For the case of asymptotic boundary conditions, the balance law for P secures that the pseudomomentum remains constant as well.

An important requirement for a numerical scheme is that alongside with its good local approximation it can also secure the integral limitation such as the conservation of mass, energy and wave momentum.

One of the authors developed a variety of finite-difference schemes implementing the energy-conservation laws. A peculiar property of the initial boundary value problem for RLWE is that it is impossible to devise a finite-difference scheme that preserves the mass (see [9]) although the energy is conserved. The cause for this is the truncation of the infinite interval and the fact that now there is only one boundary condition at each end of the interval. The latter contradicts the nature of the asymptotic conditions. The way to surmount this difficulty is to devise a scheme and algorithm which do not require truncation of the interval. This means that one has to use a spectral series with respect to a system of functions which automatically satisfy the asymptotic b.c.

3. The Fourier–Galerkin method in $L^2(-\infty, \infty)$

From the known spectral techniques, we choose the Galerkin method because it has the advantage of simplicity in implementation in comparison with the spectral collocation method or tau-method (see the arguments in [2]). Moreover, the last two techniques exhibit the same difficulties for the asymptotic b.c. as the difference schemes do.

The only problem is that the Galerkin technique requires explicit formulas expressing the products of members of the CON system into series with respect to the system. This is the reason to choose a system of functions for which the product of two members of the system can be expanded into a series with respect to the system. A system with the desired property has been proposed as early as in 1982 in [4] and since then applied to Boussinesq equation and some other nonlinear wave problems (see [7,5,3]). The applications were limited to stationary in the moving frame waves. The time dependence is taken into account in [8], where the interacting solitary waves of Boussinesq equation are treated. In order not to abuse the space, we will outline here only the RLWE case and rely on derivations from [8] for the double-soliton case and on the earlier works for the single-soliton case.

3.1. The complete orthonormal (CON) system

The system

$$\rho_n = \frac{1}{\sqrt{\pi}} \frac{(\mathrm{i}x - 1)^n}{(\mathrm{i}\,x + 1)^{n+1}}, \quad n = 0, 1, 2, \ldots, \ \mathrm{i} = \sqrt{-1} \tag{3.1}$$

was introduced by Wiener (see [12]) as Fourier transform of the Laguerre functions (functions of parabolic cylinder). Higgins [10] defined it also for negative indices n and proved its completeness and orthogonality. The significance of (3.1) for nonlinear problems was demonstrated in [4], where the product formula was derived and the two real-valued subsequences of odd functions S_n and even functions C_n were introduced, namely

$$\rho_n \rho_k = \frac{\rho_{n+k} - \rho_{n-k}}{2\sqrt{\pi}}, \qquad S_n = \frac{\rho_n + \rho_{-n-1}}{i\sqrt{2}}, \qquad C_n = \frac{\rho_n - \rho_{-n-1}}{\sqrt{2}}. \tag{3.2}$$

Making use of (3.2), one easily shows that the product of members of the real-valued sequences, e.g. $C_n C_k$, is expanded in series with respect to the system as follows (see [4])

$$C_n C_k = \frac{1}{2\sqrt{2\pi}}[C_{n+k+1} - C_{n+k} - C_{n-k} + C_{n-k-1}] = \sum_{m=1}^{\infty} \beta_{nk,m} C_m, \tag{3.3}$$

$$S_n S_k = \frac{1}{2\sqrt{2\pi}}[C_{n+k+1} - C_{n+k} + C_{n-k} - C_{n-k-1}] = \sum_{m=1}^{\infty} \alpha_{nk,m} C_m, \tag{3.4}$$

$$S_n C_k = S_n C_k = \frac{1}{2\sqrt{2\pi}}[-S_{n+k+1} + S_{n+k} + S_{n-k} - S_{n-k-1}] = \sum_{m=1}^{\infty} \gamma_{nk,m} S_m, \tag{3.5}$$

$$\alpha_{nk,m} = \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k+1} + \delta_{m,|n-k|} - \delta_{m,n+k} - \mathrm{sgn}[|n-k| - 0.5]\delta_{m,[|n-k|-0.5]}\},$$

$$\beta_{nk,m} = \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k} + \delta_{m,|n-k|} - \delta_{m,n+k+1} - \mathrm{sgn}[|n-k| - 0.5]\delta_{m,[|n-k|-0.5]}\},$$

$$\gamma_{nk,m} = \frac{1}{2\sqrt{2\pi}}\{\delta_{m,n+k} + \mathrm{sgn}(n-k)\delta_{m,|n-k|} - \delta_{m,n+k+1} - \mathrm{sgn}(n-k)\delta_{m,|n-k|-1}\}.$$

Before proceeding further we discuss the way to increase the computational effectiveness of the product formulas. Consider an even and an odd function from the $L^2(-\infty, \infty)$ space, say

$$U(x) = \sum_{n=0}^{\infty} u_n C_n \quad \text{and} \quad V(x) = \sum_{m=0}^{\infty} v_m S_m.$$

Then, for U^2 we have the following.

Theorem 1. *For the nonlinear term $U^2(x)$, the following expansion holds*

$$U^2(x) = \sum_{n=0}^{N}\sum_{m=0}^{N} u_n u_m C_n C_m = \sum_{l=0}^{N}\left[\sum_{n=0}^{N}\sum_{m=0}^{N}\beta_{nm,l}u_n u_m\right] C_l \stackrel{\mathrm{def}}{=} \frac{1}{2\sqrt{2\pi}}\sum_{l=0}^{N} b_l C_l, \tag{3.6}$$

where

$$b_l = \sum_{n=0}^{l-1} u_n u_{l-1-n} - \sum_{n=0}^{l} u_n u_{l-n} - 2\sum_{n=l}^{N} u_n u_{n-l} + 2\sum_{n=l+1}^{N} u_n u_{n-l-1}. \tag{3.7}$$

Proof. The detailed proof can be found in [8]. Here, we note that to calculate the right-hand side for a specific index l, requires a number of $4N^2$ operations (multiplication). Now, we can count the needed number of arithmetic operations (multiplications) for a given node l in the new formula (3.7) as

$$N_{\mathrm{r}} = l + (l+1) + 2(N-l+1) + 2(N-l), \quad 2N - 1 \le N_{\mathrm{r}} \le 4N - 1 \le 4N. \tag{3.8}$$

This estimate is N times lower than for the original formula (3.5). In fact, through the rearranging of the terms according to the above theorem, one avoids the multiplications of terms which are anyway equal to zero. Thus, many parasitic multiplications are avoided. As a result, on top of the natural efficiency of the spectral method (connected with the small number of terms), one has also a significant acceleration of the calculations according to formula (3.8).

For the second derivative of the basis functions, one has (see [4])

$$C_n'' = \sum_{m=0}^{\infty}\chi_{m,n}C_m, \qquad S_n'' = \sum_{m=0}^{\infty}\chi_{m,n}S_m, \tag{3.9}$$

where

$$\chi_{m,n} = -\tfrac{1}{4}n(n-1)\delta_{m,n-2} + n^2\delta_{m,n-1} - \tfrac{1}{4}(n+1)(n+2)\delta_{m,n+2} - \tfrac{1}{4}n^2 + (2n+1)^2 + (n+1)^2\delta_{m,n} + (n+1)^2\delta_{m,n+1},$$

Here, χ is a diagonal matrix (pentadiagonal, more specifically), which can be inverted for $\mathrm{O}(N \ln N)$ operations. This gives another edge in the computational efficiency of the developed here technique.

The result of the above theorem can be generalized to the case of cubic nonlinearity through repeated application of (3.6), (3.7). The respective formula, however, contains 16 different sums while the quadratic nonlinearity needs only four sums (see (3.7)). □

3.2. Exponential convergence

The most important issue for a spectral method is its rate of convergence which here can be shown through the following relationship between the Fourier periodic functions and our system, namely

$$C_n(x) = (-1)^n \frac{\cos(n+1)\theta + \cos n\theta}{\sqrt{2}}, \qquad S_n(x) = (-1)^{n+1} \frac{\sin(n+1)\theta + \sin n\theta}{\sqrt{2}}, \tag{3.10}$$

where $x = \tan\frac{\theta}{2}$ or $\theta = 2\arctan x$ is a transformation of the independent variable. Note that any function $f(x)$ is a periodic function of θ with period 2π.

Now any localized function of $f(x)$ can be expanded into a series with respect to C_n, S_n

$$f(x) = \sum_{n=0}^{\infty} a_n C_n(x) + b_n S_n(x),$$

which in its turn can be rewritten as a Fourier series for the periodic function

$$f\left(\tan\frac{\theta}{2}\right) = \sum_{i=0}^{\infty} a_n (-1)^n \frac{\cos(n+1)\theta + \cos n\theta}{\sqrt{2}} + b_n (-1)^{n+1} \frac{\sin(n+1)\theta + \sin n\theta}{\sqrt{2}}, \tag{3.11}$$

which are known to have exponential convergence. From this fact, one infers the exponential convergence of the C_n, S_n series.

4. Algorithm and verification

When the initial condition is known from a two-soliton analytical solution, it is only a matter of expanding it into the same spectral series. This case is rather exceptional and has no practical meaning, since there seldom can be found reason to continue an analytical solution numerically after certain moment of time. The only information we have is the one-hump solution. Each one of the humps can be obtained analytically, or once again by means of the present spectral series (see the authors' papers cited earlier). Hence, the initial condition in this paper is the superposition of two "sech" solutions which are situated far enough from each other in order to neglect their intersection in the initial moment.

4.1. The initial condition

The boundary conditions on the auxiliary function q do not require that it decays to zero at $\pm\infty$. Then, it cannot be expanded into series with respect to a CON system of functions whose members vanish at infinities unless we sacrifice the exponential convergence. We use a function $r(z)$ which "absorbs" the undesired behavior of $q(z, t)$, namely $p(z, t) = q(z, t) + r(z)$. Thus, we replaced the auxiliary function q

with the function p from $L^2(-\infty,\infty]$ for which

$$r(z) = -\left[\frac{3c_1}{\zeta^2}\sqrt{1-c_1^2} + \frac{3c_2}{\zeta^2}\sqrt{1-c_2^2}\right]\tanh(z).$$

Here, c_1 and c_2 are the phase speeds of the two initial solitons. The left soliton is stationary in the frame moving to the right ($x - c_1 t$, $c_1 > 0$), while the right soliton is stationary in the frame moving to the left ($x - c_2 t$, $c_2 < 0$).

Note that the introduction of the new function $p(z,t)$ does not alter the equation (2.3) because $r(z)$ does not depend on t. In terms of the new auxiliary function, the system (2.4) and (2.5) recasts to the following

$$u_t = \zeta^2 p_{zz} + \zeta^2 r_{zz}, \tag{4.1}$$

$$p_t = u - u^2 - \zeta^2 u_{tt}. \tag{4.2}$$

4.2. The time-stepping algorithm

To solve the system (4.1), we choose the fully implicit time-stepping scheme

$$\frac{u^{l+1}-u^l}{\tau} = \zeta^2 p_{zz}^{l+(1/2)} + \zeta^2 r_{zz}, \tag{4.3}$$

$$\frac{p^{l+(1/2)}-p^{l-(1/2)}}{\tau} = \frac{u^{l+1}+u^{l-1}}{2} - (u^l)^2 - \frac{\zeta^2}{2}[u_{zz}^{l+1}+u_{zz}^{l-1}]. \tag{4.4}$$

We expand the sought solution u, p into series with respect to the subsequences C_n and S_n namely

$$u^l(z) = \sum_{n=0}^{\infty} a_n^l C_n(z) + b_n^l S_n(z), \qquad p^{l+(1/2)}(z) = \sum_{m=0}^{\infty} d_m^{l+(1/2)} C_m(z) + e_m^{l+(1/2)} S_m(z). \tag{4.5}$$

Now, we insert the spectral expansion (4.5) into Eq. (4.3) and (4.4) and if we make use of the orthogonality of the system of our functions, the system for the odd functions (for the even functions we consider a similar system but without the function $r(z)$) coefficients reads

$$\begin{aligned}
\frac{b_m^{l+1}-b_m^l}{\tau} &= \zeta^2 \sum_{k=0}^{\infty} b_k^{l+(1/2)} \chi_{m,k} + \zeta^2 \sum_{k=0}^{\infty} r_k \chi_{m,k},\\
\frac{e_m^{l+(1/2)}-e_m^{l-(1/2)}}{\tau} &= \frac{b_m^{l+1}+b_m^{l-1}}{2} - \frac{\zeta^2}{2}\sum_{k=0}^{\infty}(b_k^{l+1}+b_k^{l-1})\chi_{k,m}\\
&\quad - \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}(a_{k_1}^l b_{k_2}^l \gamma_{k_1k_2,m} + b_{k_1}^l a_{k_2}^l \gamma_{k_1k_2,m}),
\end{aligned} \tag{4.6}$$

where γ is given in (3.5) and r_k are the coefficients of the spectral expansion of function r_{zz}. In the numerical calculations, a truncated version of the above system is used in which the infinity is replaced by N.

The initial conditions $\{a_n^0\}$, $\{b_n^0\}$ and $\{a_n^1\}$, $\{b_n^1\}$ for the Fourier coefficients are calculated for $t = 0$ and $t = \tau$ by means of numerical quadrature of the analytic solution formulas after multiplied by C_n or S_n. In its turn, the initial conditions for the coefficients $\{d_n^{1/2}\}$, $\{e_n^{1/2}\}$ of p are computed via numerical

quadrature of its analytic expression after multiplied by C_n or S_n. Note that the initial conditions have to be calculated every time when the value ζ of the scaling parameter is changed. Having specified the initial conditions, we can begin the time stepping. Let us assume that the variables $\{a_n^{l-1}\}$, $\{b_n^{l-1}\}$, $\{a_n^l\}$, $\{b_n^l\}$, $\{d_n^{l-(1/2)}\}$, $\{e_n^{l-(1/2)}\}$ are known. Then, our systems for the coefficients give two coupled nine-diagonal algebraic systems for $\{b_n^{l+1}\}$, $\{e_n^{l+(1/2)}\}$ and for $\{a_n^{l+1}\}$, $\{d_n^{l+(1/2)}\}$, respectively.

After these systems are solved, one time step is completed, the time index l is reset, and the process is repeated.

4.3. Numerical experiments

The importance of scaling parameter ζ is understood by computing the solution for different values of ζ. Naturally, the optimal ζ is different for different initial configurations of the system of solitons. When the solitons are situated far from each other, one is faced with a system which is not tightly localized. Then, the optimal value of ζ is smaller. Conversely, if the initial configuration is tight enough, the value of ζ which brings the scales of the sought function and the CON system in a closest rapport, tends to be smaller. In the present work, we consider initial configurations of solitons that are well separated (in order not to overlap significantly), but not excessively far from each other (not to loose the localization). After extensive numerical experiments, we found for these cases that the optimal value of the scaling parameter is close to $\zeta = 0.06$, for which the convergence was faster and more accurate.

As far as the time increment is concerned, we found that the calculations are perfectly stable for τ as large as 0.1 even for $c = 1.64$, the latter being on the border of existence of the soliton solution for RLWE equation.

As already mentioned earlier, the rate of convergence of the Galerkin series is exponential and our calculations comply with this analytical result. In Fig. 1, we present the Galerkin coefficients from two different systems of solitons. In Fig. 1a, we consider the interaction of two solitons with phase speeds $c_1 = -1.2$ and $c_2 = 1.5$. As we can see the convergence remains exponential after 300 time steps even though we used a rather large time increment $\tau = 0.1$. In Fig. 1b, we present the coefficients when $|c_1| = c_2 = 1.05$ for the initial moment and after 1400 time steps. Although we are very close to the trivial solution (when $c = 1$) and the number of time steps is large, the exponential convergence is always present.

As mentioned in the end of Section 2, the proposed Galerkin method is supposed to conserve the mass of the wave since the functions automatically satisfy the asymptotic boundary conditions. The mass of the wave can be defined as

$$\mathcal{M} = \int_{-\infty}^{\infty} U(x,t)\,\mathrm{d}x + \mathrm{O}(\mathrm{e}^{-N}) = \sum_{n=0}^{N} u_n \overline{c}_n, \quad \overline{c}_n \stackrel{\mathrm{def}}{=} \int_{-\infty}^{\infty} C_n(x)\,\mathrm{d}x. \tag{4.7}$$

For a typical case treated in this section, we found that during 300 time steps fully enclosing the cross-section of the interaction the numerical value of $\mathcal{M}$ varied from 0.3551762071910549 to 0.3551762071911115, i.e. $0.3551762071910832 \pm 0.566 \times 10^{-13}$ which is of order of the *round-off* error of calculations with double precision. A straightforward discrete approximation of the energy is not conserved in this scheme. It was shown in [9] that it can be achieved only by means of internal iterations within a given time step. We did not employ iterations here in order to keep it simple when illustrating the new spectral technique. So the discrete approximation of the energy varies in our calculations up to

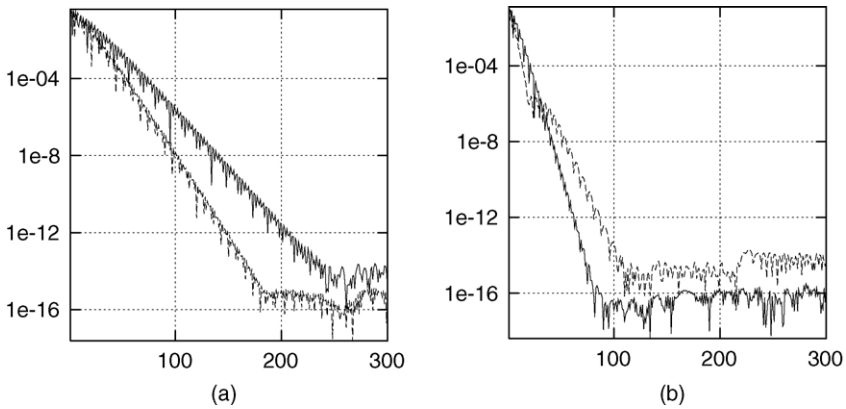

Fig. 1. Exponential decay of the computed coefficients of the spectral expansion for $\zeta = 0.06$ and $N = 300$ at $t = 0$ and $t = 300$, (a) $c_1 = 1.2$, $c_2 = -1.5$; (b) $c_1 = -c_2 = 1.05$.

10^{-4}. Yet, a more elaborated energy norm (some kind of pseudo-energy), it was once again conserved to the 13th digit as it was the case with mass. The energy norm employed here differs from the physical energy by quantity of order of $O(\tau^2)$.

5. Results and discussion

An important characteristics of the RLWE is that the localized waves can be computed numerically as initial value problem only for supercritical phase velocities. In the proper Boussinesq equation, the faster solitons have smaller amplitudes, while in the case of the RLWE, as in the KdV equations, the faster solitons are taller. When $c = 1$ then $u(z, t) = 0$, the amplitude of the wave is getting smaller as we approach 1 from the above.

In Fig. 2, we present the collision of two equal solitons for 10 different times $t = 0, 50, 100, 150, 200, 250, 300, 350, 400, 450$ with $\zeta = 0.06$ and $\tau = 0.1$. Since this is a highly nonlinear case and just as in the calculations with the finite-difference scheme [9] on the site of the bygone collision some oscillations develop. The residual signal becomes more significant as we are getting closer to the threshold. Note that even the slightest but persistent "leakage" of energy during the calculations would have led to eventual linear dispersion of the solution and disappearance of the permanent (sech) shapes. The quantitative agreement with the finite-difference scheme is very good which means that the appearance of oscillations is a property of the physical problem rather than numerical artifact. With the increase of the phase velocity of the solitons (therefore their amplitude), the effect rapidly develops. For large enough phase speed c, a blow-up occurs. The largest c for which we were able to find a solution was $c = 1.64$ which compares quantitatively very well with blow-up threshold $c \leq 1.64$ as established in [9].

The case of two unequal solitons is an important test of the numerical algorithm. The cases of equal solitons need only even functions in the expansion while the other cases need also the odd functions. Hence, the latter reveal the effectiveness of the method developed here. In Fig. 3, we show the collisions of two nonequal solitons. The second case is at the threshold of the blow-up. It is seen that if we keep $c_1 = -1.64$ and $c_2 = 1.71$ (Fig. 3c), then the solution blows up. For $c_2 = -1.71$, the largest value reached

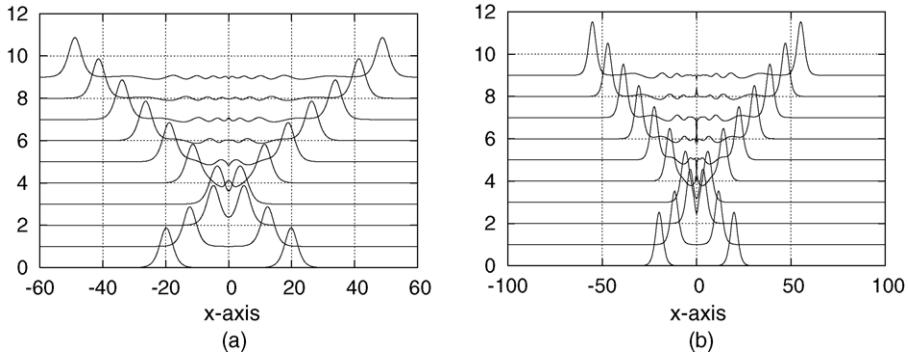

Fig. 2. Collision of two equal solitons: for $t = 0, 50, 100, 150, 200, 250, 300, 350, 400, 450$ with $\zeta = 0.06$ and $\tau = 0.1$, (a) $c_1 = -c_2 = 1.5$, (b) $c_1 = -c_2 = 1.64$.

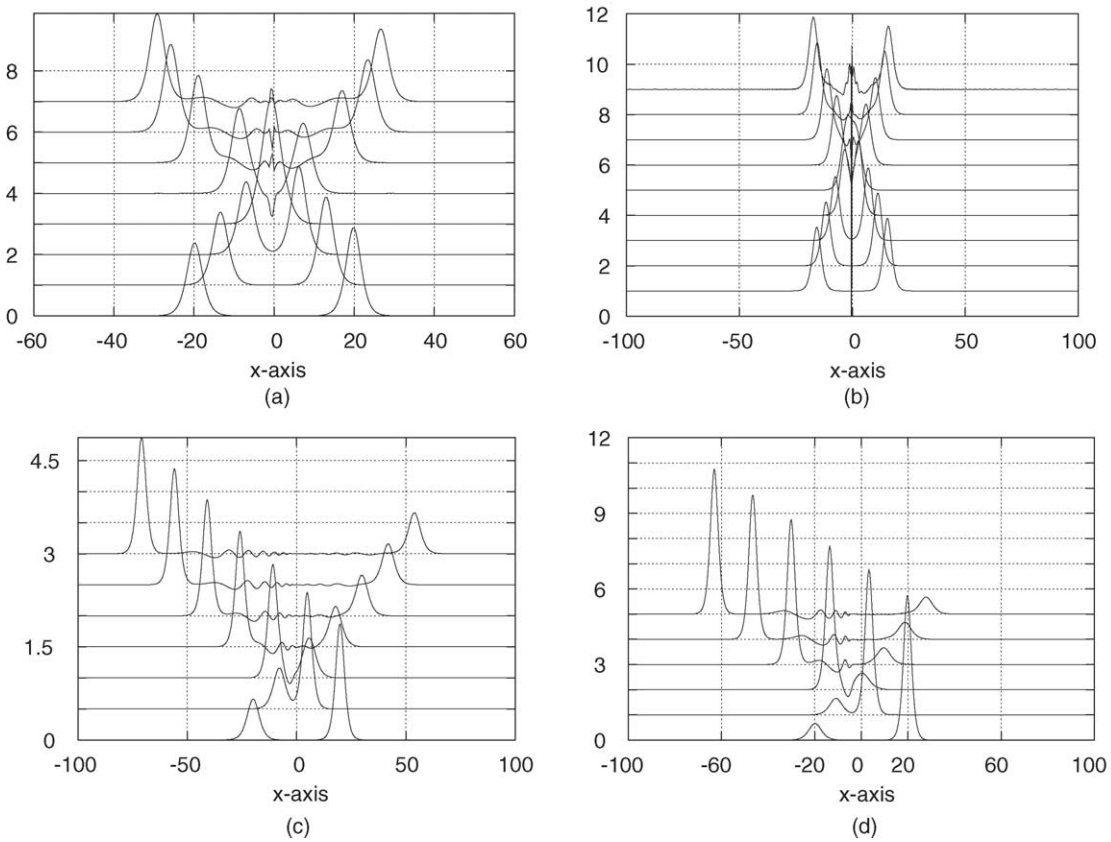

Fig. 3. Collision of two nonequal solitons: (a) $c_1 = -1.2$, $c_2 = 1.5$ for $t = 0$–600 with $\zeta = 0.06$ and $\tau = 0.1$; (b) $c_1 = -1.61$, $c_2 = 1.71$ for $t = 0$–280 with $\zeta = 0.06$ and $\tau = 0.1$; (c) $c_1 = -1.64$, $c_2 = 1.71$ for $t = 0$–210 with $\zeta = 0.06$ and $\tau = 0.1$; (d) $c_1 = -1.2$, $c_2 = 2.2$ for $t = 0$–375 with $\zeta = 0.06$ and $\tau = 0.1$.

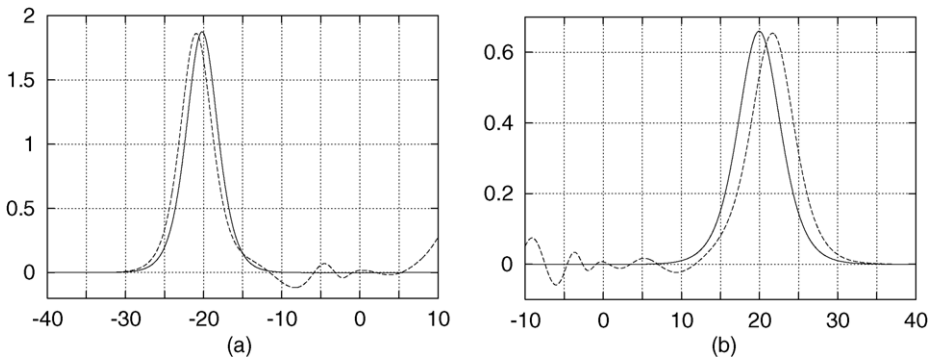

Fig. 4. The phase shifts for a system of solitons with: (a) $c_l = 1.5$, shift 267 steps; (b) $c_r = -1.2$, shift 332 steps.

for c is $c_1 = 1.61$. One can conjecture that the threshold to the blow-up is, in fact, a condition on the average value of the phase speeds. In Fig. 3d, we show that the same tendency is preserved for a case with one large supercritical phase speed. When both phase speeds are large enough (say, $c_1 = -3$, $c_2 = 5$), the blow-up occurs immediately after the collision of the solitons.

In the end, we focus our attention on the phase shift experienced by the solitons upon collisions. The idea of phase shift is illustrated in Fig. 4. In Fig. 4a is presented the faster soliton $c_1 = 1.5$ (larger amplitude), while in Fig. 4b is the slower one with $c_2 = -1.2$ (smaller amplitude).

In each of the panels of Fig. 4, the profile of the soliton in its actual position is juxtaposed to the respective soliton for $t = 20$ (200 time steps). Since both solitons are initially at distance of 20 units from the origin of the coordinate system, then the slower soliton would need $t = 40/1.2 \approx 33.33$ (333 time steps) to reach the position $x = 20$ starting from $x = -20$. Respectively, the faster soliton would need time $t = 40/1.5 \approx 26.67$ time units (or 267 time steps) to clear up distance from $x = 20$ to $x = -20$. As it is clearly seen the larger soliton is shifted less than the smaller. The respective values are 0.96 time units for the larger soliton and 1.68 for the smaller.

We have conducted extensive numerical experiments and obtained results for phase shifts for a large number of initial configurations of the solitons. A selection of phase shifts for different phase velocities of the solitons which they appear to be in very good agreement with the finite-difference results of [9] are compiled in Table 1.

Table 1
Comparison for the phase shift of the RLWE, δ_i^* (spectral) and δ_i (finite differences) [9]

c_1	δ_1^*	δ_1	c_2	δ_2^*	δ_2
1.2	1.12	1.09	−1.2	1.12	1.09
1.5	1.24	1.18	−1.5	1.24	1.18
2.0	1.28	1.24	−2.0	1.28	1.24
5.0	1.16	1.27	−5.0	1.16	1.27
1.6	0.64	0.60	−1.2	1.88	1.84
4.0	0.92	0.97	−3.0	1.68	1.66
5.0	0.32	0.31	−2.0	2.44	2.44

6. Conclusions

In the present paper, a Fourier–Galerkin spectral technique is developed for calculating the interaction of localized waves. The interaction of solitary waves for the regularized long wave equation is considered as a featuring example. It is shown that the solitons recover their exact shapes after the collision, in some cases a residual signal is observed, when their phase velocities are in the threshold, but experience phase shift. The numerically obtained signs and magnitudes of the phase shifts are in very good quantitative agreement with the numerical results of [9]. We also calculate the phase shift of the solitons which are not in the limit of the threshold just before the blow-up.

A complete orthonormal basis system in $L^2(-\infty, \infty)$ is used to expand the solution into Fourier series with Galerkin identification of the coefficients. This is an important development of the technique in comparison with our previous works where the shape of a single stationary solitary wave was the object of investigation.

In the two-soliton case under consideration, the localization of the solution is much less tight and the number of terms needed for good approximation is larger. The treatment of the problem required very efficient implementation of the Fourier–Galerkin scheme.

A scaling parameter, ζ, is introduced which allows fine-tuning and optimization of the technique proposed. Numerical experiments with different number of terms are conducted for different values of the scaling parameter. These experiments establish the practical convergence of the method and indicate an exponential decay of the coefficients with the increase of their number (exponential convergence). Highly accurate results are obtained for the time dependent problem with as few as $N = 40$ terms. This demonstrates the efficiency of the proposed technique and encourages the future use of the CON system of functions for more applications.

References

[1] T.B. Benjamin, J.L. Bona, J.J. Mahony, Model equation for long waves in nonlinear dispersive systems, Philos. Trans. R. Soc. Lond. A 272 (1972) 47–78.
[2] J.P. Boyd, Chebyshev and Fourier Spectral Methods, Dover Press, 2000.
[3] M. Christou, C.I. Christov, Fourier–Galerkin method for localized solutions of equations with cubic nonlinearity, J. Comput. Anal. Appl. 4 (2002) 63–77.
[4] C.I. Christov, A complete orthonormal sequence of functions in $L^2(-\infty, \infty)$ space, SIAM J. Appl. Math. 42 (1982) 1337–1344.
[5] C.I. Christov, Fourier–Galerkin algorithm for 2D localized solutions, Annuaire de l'Univ. Sofia, Livre 2—Mathématiques Appliqee et Informatique 89 (1995) 169–179.
[6] C.I. Christov, An energy-consistent Galilean-invariant dispersive shallow-water model, Wave Motion 34 (2001) 161–174.
[7] C.I. Christov, K.L. Bekyarov, A Fourier-series method for solving soliton problems, SIAM J. Sci. Stat. Comput. 11 (1990) 631–647.
[8] C.I. Christov, M.A. Christou, Fourier–Galerkin method for interacting localized waves, in prepration.
[9] C.I. Christov, M.G. Velarde, Inelastic interaction of Boussinesq solitons, J. Bifurcation Chaos 4 (1994) 1095–1112.
[10] J.R. Higgins, Completeness and Basis Properties of Sets of Special Functions, Cambridge University Press, London, 1977.
[11] J.M. Sanz-Serna, S.V. Manoranjan, T. Ortega, Soliton and antisoliton interactions in the good Boussinesq equation, SIAM J. Sci. Stat. Comput. 5 (1984) 946–953.
[12] N. Wiener, Extrapolation, Interpolation and Smoothing of Stationary Time Series, Technology Press MIT/Wiley, New York, 1949.