Heat Conduction Paradox Involving Second-Sound Propagation in Moving Media

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In this Letter, we revisit the Maxwell-Cattaneo law of finite-speed heat conduction. We point out that the usual form of this law, which involves a partial time derivative, leads to a paradoxical result if the body is in motion. We then show that by using the material derivative of the thermal flux, in lieu of the local one, the paradox is completely resolved. Specifically, that using the material derivative yields a constitutive relation that is Galilean invariant. Finally, we show that under this invariant reformulation, the system of governing equations, while still hyperbolic, cannot be reduced to a single transport equation in the multidimensional case.

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Introduction.—According to classical continuum mechanics, in a homogeneous and isotropic thermally conducting medium, the thermal flux vector $\boldsymbol{q}$ obeys Fourier's law of heat conduction, namely,

$$\boldsymbol{q} = -K\nabla T, \tag{1}$$

where $T = T(\boldsymbol{x}, t)$ denotes the absolute temperature and the constant $K(>0)$ denotes the thermal conductivity. Equation (1), also known as Fick's law in the context of mass diffusion, is one of the basic constitutive relations in the physical sciences. Unfortunately, Fourier's law, which we note is *local in time,* predicts that thermal signals propagate with infinite speed, a drawback which appears to have first been noted by Nernst in 1917 (see Ref. [1]). Such behavior, which is most apparent under low temperature and/or high heat-flux conditions [2,3], clearly violates causality.

To show how this defect manifests itself, we begin by noting that the balance law for the internal (heat) energy can, in the absence of all thermal sources or sinks and neglecting internal dissipation, be expressed in terms of T as (see, e.g., Ref. [4], p. 202)

$$\rho c_p \frac{DT}{Dt} + \nabla \cdot \boldsymbol{q} = 0, \tag{2}$$

where c_p, the specific heat at constant pressure, is a constant. Here, $\boldsymbol{u}$ is the velocity vector of the material point, t is time, ρ is the mass density, and we note that the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0, \tag{3}$$

was employed in obtaining Eq. (2) from the balance equation for the internal energy. In addition,

$$\frac{D}{Dt} \overset{\text{def}}{=} \frac{\partial}{\partial t} + \boldsymbol{u} \cdot \nabla, \tag{4}$$

which denotes the material derivative, acknowledges the fact that the time rate of change of a material quantity at a certain geometrical point is the result of two processes: A change in the geometrical point, i.e., the partial time derivative, and a change due to the fact that a different quantity is transported from the neighboring points to the point under consideration.

When combined with Eq. (2), the conservation of energy law, Eq. (1), yields the heat transport equation

$$T_t + \boldsymbol{u} \cdot \nabla T - \kappa \nabla^2 T = 0, \tag{5}$$

where $\kappa = K/(\rho c_p)$ is the thermal diffusivity. Like the much better known heat (or diffusion) equation, which Eq. (5) reduces to when $\boldsymbol{u} = \boldsymbol{0}$, the latter is also a partial differential equation (PDE) of the parabolic type. Consequently, it is predicted that a thermal disturbance at any point in a material body will be felt instantly, but unequally, at all other points of the body. From this, it is obvious that Fourier's law does not fully describe the diffusion process.

To correct this unrealistic feature, which is known as the "paradox of heat conduction" (PHC), various modifications of Fourier's law have been proposed. Of these, the best known is the Maxwell-Cattaneo (MC) law [1,2,5]

$$(1 + \lambda_0 \partial_t)\boldsymbol{q} = -K\nabla T, \tag{6}$$

where ∂_t denotes $\partial/\partial t$. Here, the thermal relaxation constant $\lambda_0(>0)$ represents the time lag required to establish steady heat conduction in a volume element once a temperature gradient has been imposed across it [2]. This generalization of Fourier's law accounts for the finite speed of heat conduction by adding a term proportional to the time derivative of the flux vector, known as the "thermal inertia" term, to the left-hand side of Eq. (1). It should be mentioned that the value of λ_0 has been experimentally determined for a number of materials [1,6]. And although λ_0 turns out to be very small in many instances, e.g., λ_0 is

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of the order of picoseconds for most metals [2,7], there are several materials where this is not the case, most notably sand (21 s), H acid (25 s), $NaHCO_3$ (29 s), and biological tissue (1–100 s) [2,7].

While the MC law corrects one paradox, i.e., the PHC, it also gives rise to another, namely, the one that occurs when the MC law is used to describe heat conduction in a moving solid. Hence, it is to the investigation and resolution of this second paradox that the present Letter is devoted. Specifically, we consider the problem wherein a Heaviside (i.e., step) change in temperature occurs at one end of a thermally conducting half-space that is moving at a constant speed. It is shown that under Eq. (6), the heat transport equation in the moving frame is different from the one in the rest frame. We then shown how, by replacing the partial time derivative in Eq. (6) with the material derivative, the paradox is completely resolved.

Before going on, it is appropriate to note that Fourier's law itself remains a topic of considerable interest and study. Two fundamental questions that have been investigated in recent years are the following: (i) Is Fourier's law valid in low dimensional (i.e., 1D or 2D) systems [8,9]? (ii) What are the necessary or sufficient dynamical conditions that form its basis [10,11]? Addressing these issues is becoming increasingly important because of the need, driven by new technological demands, to have a better understanding of nanoscale heat transport [12].

Heat transport equation under the MC law.—Let us begin by observing that $\boldsymbol{q}$ can be eliminated from Eq. (2) using Eq. (6). Consequently, the heat transport equation arising from the MC law is found to be

$$\lambda_0[T_{tt} + (\boldsymbol{u} \cdot \nabla T)_t] + T_t + \boldsymbol{u} \cdot \nabla T = \lambda_0 c^2 \nabla^2 T, \tag{7}$$

where $c = \sqrt{\kappa/\lambda_0}$ and again no thermal sources or sinks are present. In the case of a continuum at rest, Eq. (7) reduces to the well known damped wave equation (DWE) (see, e.g., Ref. [13] and those therein). Hence, when $\boldsymbol{u} = \boldsymbol{0}$, Eq. (7) predicts that heat conduction occurs via the propagation of damped thermal waves of finite speed c, a phenomenon known as "second sound" [1].

To simplify our analysis, we introduce the following nondimensional variables: $\boldsymbol{x}' = \boldsymbol{x}/l$, $t' = t/\lambda_0$, $T' = T/T_0$, and $\boldsymbol{u}' = \boldsymbol{u}/c$, where $l = \sqrt{\kappa\lambda_0}$ and $T_0(>0)$, respectively, denote a characteristic length and temperature. On making the indicated replacements, Eq. (7) is reduced to the dimensionless form

$$T_{tt} + (\boldsymbol{u} \cdot \nabla T)_t + T_t + \boldsymbol{u} \cdot \nabla T = \nabla^2 T, \tag{8}$$

where all primes have been omitted for convenience.

Exact solution for a medium at rest.—First, we consider second sound propagation in a solid that is at rest. For the sake of simplicity, we chose the 1D case of a planar wave front, or singular surface, that is propagating along the x axis of a Cartesian coordinate system, say, in the positive x direction, into a semi-infinite, homogeneous, and isotropic medium of (constant) thermal diffusivity κ. (Here, we note that the same mathematical problem arises for MC-based heat conduction in a very thin, semi-infinite 1D rod whose lateral surfaces are perfectly insulated.) Furthermore, let $\theta(x, t)$ denote the dimensionless departure from a constant initial temperature, T_i, and let us also assume for simplicity that $\theta_t(x, 0) = 0$.

Now, at time $t = 0+$, the temperature at the boundary $x = 0$ jumps instantly from 0 to 1, and is maintained there for all $t > 0$. Assuming that heat conduction within the half-space is governed by the MC law, we wish to determine θ for all $x, t > 0$. Consequently, we are led to consider the following initial and boundary value problem (IBVP):

$$\begin{gathered}\theta_t + \theta_{tt} = \theta_{xx}, \qquad (x, t) \in (0, \infty) \times (0, \infty); \\ \theta(0, t) = H(t), \qquad \theta(\infty, t) = 0, \qquad t > 0; \\ \theta(x, 0) = 0, \qquad \theta_t(x, 0) = 0, \qquad x > 0;\end{gathered} \tag{9}$$

where $H(\cdot)$ is the Heaviside unit step function and T_0 denotes the initial jump amplitude at $x = 0$ in the dimensional variables. The solution of this IBVP is well known and is given by [14]

$$\theta(x, t) = H(t - x)\left\{e^{-x/2} + x\int_x^t e^{-\zeta/2}\frac{I_1[\frac{1}{2}\sqrt{\zeta^2 - x^2}]d\zeta}{2\sqrt{\zeta^2 - x^2}}\right\},$$

where $I_1[\cdot]$ denotes the modified Bessel function of the first kind of order one. From this, we see that θ suffers a propagating jump, of magnitude $e^{-x/2}$, across the wave front $x = t$.

A paradox in the moving frame.—Consider now the simplest possible motion of the half-space, namely, translation along the x axis with constant (dimensionless) velocity $u(x, t) \equiv U$. (Clearly, $|U|$ can also be regarded as a Mach number.) In this case, Eq. (8) reduces to

$$T_{tt} + UT_{xt} + T_t + UT_x = T_{xx}. \tag{10}$$

This PDE is a generalization of the DWE, and like the latter is hyperbolic. Yet, it exhibits a paradoxical feature in that the wave speeds for thermal disturbances are nonlinear functions of U, namely,

$$c_{1,2} = \tfrac{1}{2}[U \pm \sqrt{U^2 + 4}], \qquad \text{where } c_1 > \max[U, 0], \\ c_2 < \min[U, 0]. \tag{11}$$

Clearly, $c_{1,2}$ are not equal to $U \pm 1$, i.e., the sum or difference of the (dimensionless) frame velocity and thermal wave speed, and therefore they lack physical meaning.

If we further suppose that the boundary heat source is moving with the half-space, then the Heaviside boundary condition is imposed at the moving plane $x = Ut$. Now, according to Galileo's principle of relativity (or Galilean relativity), the fundamental postulate of classical mechanics that asserts the equivalence of all inertial frames (see Refs. [15,16]), the propagation of second sound in the moving body should be exactly the same as in the resting body. Mathematically, this means that one should get an identical IBVP in a frame attached to the moving half-space.

To see if this is true, we reconsider the problem in the moving frame, i.e., under the Galilean transformations

$$\chi = x - U\tau, \qquad \tau = t, \qquad \theta(\chi, t) = T(x, t) - T_i. \tag{12}$$

Then

$$T_t = \theta_t - U\theta_\chi, \qquad T_x = \theta_\chi \Rightarrow T_t + UT_x = \theta_t; \tag{13}$$

$$\begin{gathered} T_{xx} = \theta_{\chi\chi}, \qquad T_{xt} = \theta_{\chi t} - U\theta_{\chi\chi}, \\ T_{tt} = \theta_{tt} - 2U\theta_{\chi t} + U^2\theta_{\chi\chi}. \end{gathered} \tag{14}$$

Introducing these formulas in Eq. (10) and reformulating the IBVP in the moving frame yields

$$\begin{gathered} \theta_t + \theta_{tt} - U\theta_{\chi t} = \theta_{\chi\chi}, \qquad (\chi, t) \in (0, \infty) \times (0, \infty); \\ \theta(0, t) = H(t), \qquad \theta(\infty, t) = 0, \qquad t > 0; \\ \theta(\chi, 0) = 0, \qquad \theta_t(\chi, 0) = 0, \qquad \chi > 0. \end{gathered} \tag{15}$$

Note that because of our assumption that the boundary heat source is in the same (moving) frame as the half-space, we get *almost* the same IBVP as in (9), the single exception being that the heat transport equation now contains a mixed derivative term with coefficient U.

Solving IBVP (15) using the Laplace transform yields the exact χ, t-domain solution

$$\theta(\chi, t) = H(t - \chi/C) \times \left\{ e^{-ad\chi} + ad\chi \int_{\chi/C}^{t - U\chi/2} e^{-a\varsigma} \frac{I_1[a\sqrt{\varsigma^2 - (d\chi)^2}]d\varsigma}{\sqrt{\varsigma^2 - (d\chi)^2}} \right\}, \tag{16}$$

where $a = 2(4 + U^2)^{-1}$, $d = (2a)^{-1/2}$, $C^{-1} = d + U/2$, and we note that C is strictly positive.

Clearly, the transport equations in IBVPs (9) and (15), and therefore their solutions, are *not* the same. This means that according to the MC law, the process of heat conduction in the moving body is different from what it is in the resting body, thus violating Galilean relativity.

In Fig. 1, we have graphed Eq. (16) for three values of U. There, we see that the propagation speed C of the wave front in the moving frame depends on U, a physically unrealistic prediction. In particular, C is a decreasing function of U, where we note that $C = 1$ correctly corresponds to the wave front speed in the resting frame when $U = 0$. Hence, we now proceed to alleviate this deficiency by appropriately modifying the MC law so as to ensure its Galilean invariance; in other words, that it satisfies Galileo's principle of relativity.

The MC law in the material framework.—The standard form of the MC law involves only a partial time derivative. However, a more physically justified approach is to assume

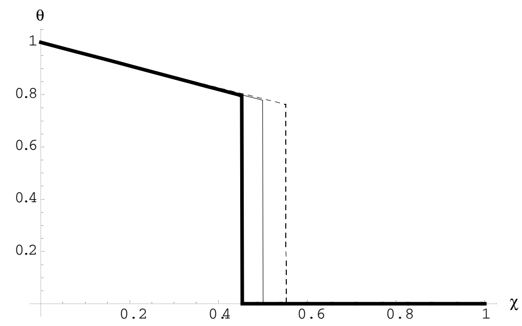

FIG. 1. θ vs χ for $t = 0.5$. Bold line: $U = 0.2$. Thin solid line: $U = 0$. Dashed line: $U = -0.2$.

that the thermal inertia is a property of the material point, rather than of the geometrical point as it is in Eq. (6). Mathematically, this means that the partial time derivative is replaced by the material derivative.

Consider again the 1D case where the flux, velocity, and temperature are scalar functions of x and t. To simplify the mathematics, we assume that the density $\rho =$ const. Consequently, Eq. (3) reduces to $u_x = 0$, or $u = u(t)$. If we also assume that the motion occurs at a constant velocity U along the x axis, then $u = U$. Now, if we replace ∂_t in Eq. (6) with the material derivative operator D/Dt, we get the following system governing second sound-based heat conduction in 1D:

$$T_t + UT_x = -(\rho c_p)^{-1} q_x, \tag{17}$$

$$q + \lambda_0(q_t + Uq_x) = -KT_x. \tag{18}$$

Once again eliminating $q(x, t)$, we obtain the dimensionless heat transport equation

$$T_t + UT_x + T_{tt} + 2UT_{xt} - (1 - U^2)T_{xx} = 0, \tag{19}$$

which is strictly hyperbolic. Note that unlike Eq. (11), the wave speeds of thermal disturbances with respect to the resting frame are now $c_{1,2} = U \pm 1$, exactly as we would expect for a body moving with velocity U.

Returning to the moving frame [Eqs. (12)], and again making use of Eqs. (13) and (14), we find that Eq. (19) reduces exactly to the transport equation in IBVP (9).

This means that in a body in uniform motion, the heat conduction process as described by the "material" form of the MC law [Eq. (18)] is identical to that in the resting body; in other words, the material form of the MC law is strictly Galilean invariant.

This relatively simple example involving a solid body illustrates the importance of using the material derivative in the MC law. Let us now extend this kind of invariance to second sound-based heat conduction occurring in general continua.

General 3D formulation.—In the general 3D case, the MC law in the material framework is

$$\boldsymbol{q} + \lambda_0(\partial_t + \boldsymbol{u}\cdot\nabla)\boldsymbol{q} = -K\nabla T. \tag{20}$$

From this, it is clear that in more than one dimension, the material MC law is inextricably coupled to Eq. (2) because it cannot be resolved with respect to $\boldsymbol{q}$. Hence, Eqs. (2), (3), and (20) compose a coupled system. We prove now that this system is Galilean invariant.

Returning to dimensional variables, we consider the general motion of a material continuum with velocity $\boldsymbol{u}(x, t)$. In a frame moving with constant velocity V, one has

$$\chi = \boldsymbol{x} - \boldsymbol{V}t, \qquad t = t, \qquad \boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{V} + \boldsymbol{v}(\chi, t),$$
$$\boldsymbol{q}(\boldsymbol{x}, t) = \boldsymbol{f}(\chi, t), \qquad T(\boldsymbol{x}, t) - T_i = \theta(\chi, t). \tag{21}$$

Note that, in general, $\boldsymbol{V}$ could be different from $\boldsymbol{u}$, and $\boldsymbol{v}(\chi, t)$ denotes the velocity relative to the moving frame.

Now we are in a position to derive the following vector formulas relating the moving frame to the resting one:

$$\boldsymbol{q}_t = \boldsymbol{f}_t - \boldsymbol{V}\cdot\nabla^{\chi}\boldsymbol{f}, \qquad T_t = \theta_t - \boldsymbol{V}\cdot\nabla^{\chi}\theta,$$
$$\boldsymbol{u}\cdot\nabla^{x} = [\boldsymbol{V} + \boldsymbol{v}(\chi, t)]\cdot\nabla^{\chi}, \tag{22}$$

where the superscripts on the gradient operators refer to the coordinates with respect to which they are taken. Then it is easy to verify that

$$\boldsymbol{q}_t + \boldsymbol{u}\cdot\nabla^{x}\boldsymbol{q} = \boldsymbol{f}_t + \boldsymbol{v}(\chi, t)\cdot\nabla^{\chi}\boldsymbol{f}, \tag{23a}$$
$$T_t + \boldsymbol{u}\cdot\nabla^{x}T = \theta_t + \boldsymbol{v}(\chi, t)\cdot\nabla^{\chi}\theta. \tag{23b}$$

Introducing the last formulas in Eqs. (2), (3), and (20), we get the following system describing second sound-based heat conduction in the moving frame:

$$\rho_t + \nabla\cdot(\rho\boldsymbol{v}) = 0, \tag{24a}$$
$$\rho c_p(\partial_t + \boldsymbol{v}\cdot\nabla)\theta = -\nabla\cdot f, \tag{24b}$$
$$\boldsymbol{f} + \lambda_0(\partial_t + \boldsymbol{v}\cdot\nabla)\boldsymbol{f} = -K\nabla\theta, \tag{24c}$$

where the superscripts are omitted without fear of confusion. Clearly, the last system demonstrates that the improved MC law is "material invariant" in the sense that the system of equations is Galilean invariant, thus assuring us that the nature of heat conduction is the same in any inertial frame. Also, it must be mentioned that system (24) is valid for any kind of material continuum; one has only to close the system by including the appropriate form of the (vector) momentum equation and constitutive relation for the stress.

Conclusions.—In this Letter, we have shown that the usual form of the MC thermal flux law leads to a paradoxical result if a moving continuum is considered. We then showed that by replacing the partial time derivative in this constitutive relation with the material derivative operator, the paradox is removed. This modification to the MC law that we have proposed yields material invariance; i.e., changing to a different inertial coordinate system leaves the system of governing equations [i.e., (24)] unchanged. The significance of this, of course, is that the material version of the MC law satisfies Galileo's principle of relativity. What is more, it is interesting to note that, while it does suffer from the PHC, Fourier's law, in itself, *does not* contradict Galilean relativity.

Finally, it should be mentioned that very often the issue of Galilean invariance of the models used in continuum mechanics is underestimated, especially when approximate amplitude equations are derived. For a case in point, we refer the reader to the Boussinesq type dispersive models of shallow water flows [17,18].

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