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NOVEL NUMERICAL APPROACH TO SOLITARY–WAVE SOLUTIONS IDENTIFICATION OF BOUSSINESQ AND KORTEWEG–DE VRIES EQUATIONS

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A special numerical technique has been developed for identification of solitary wave solutions of Boussinesq and Korteweg–de Vries equations. Stationary localized waves are considered in the frame moving to the right. The original ill-posed problem is transferred into a problem of the unknown coefficient from over-posed boundary data in which the trivial solution is excluded. The Method of Variational Imbedding is used for solving the inverse problem. The generalized sixth-order Boussinesq equation is considered for illustrations.

Keywords: Inverse problem; solitary wave solutions; Boussinesq equation; Korteweg–de Vries equation; method of variational imbedding; MVI; bifurcation; coefficient identification.

1. Introduction

Waves on the surface of an ideal fluid under the force of gravity are governed by the classical Euler equations. The famous Boussinesq and Korteweg–de Vries (KdV) equations represent an approximation of unidirectional wave motion on the surface of a thin layer of an inviscid and incompressible fluid when deformability of the surface is acknowledged. Equation of this kind was first derived by Boussinesq to describe the propagation of small amplitude, long wavelength, gravity waves on the surface of water in a channel. The Boussinesq and Korteweg–de Vries equations can also be used to model the propagation of long waves on chains of matrial points connected with nonlinear elastic strings. These equations possess special traveling wave solutions called solitary waves. Boussinesq's theory [Boussinesq, 1871] was the first to give a satisfactory and scientific explanation of the phenomenon of solitary waves discovered by Scott Russell [Russell, 1844].

Depending on the modeling of dispersion, the resulting system may or may not have a linearization about the rest state which is well posed. Even when well posed, the linearized system may exhibit a lack of conservation of energy that is

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not in consistency with Euler equations. Therefore, higher-order terms need to be introduced to make the predictions more physically plausible (see e.g. [Christov *et al.*, 1996; Boyd, 1999; Tzirtzilakis *et al.*, 2002]). These new models are expected to provide additional basis for the investigation of the pseudo-particle behavior of the localized surface waves. If one allows for the appearance of higher-order terms a more complicated equation is obtained, which is nonintegrable but still admits some special wave solutions.

The propagation of nonlinear waves in a three-component reaction–diffusion system is given in [Nekorkin & Kazantsev, 2002]. The problem of the existence of the stationary pulse-like solutions is reduced to the analysis of homoclinic trajectories of a fourth-order system of nonlinear ODEs. They have obtained the set of parameters corresponding to the homoclinic bifurcations that defines the velocity spectra of the traveling pulses (see also [Velarde *et al.*, 1995]).

Recent advances in computation have also made possible the development of more accurate numerical schemes for soliton equations. Classical engineering problems demand information about the spectral content of signals. The difficulties in numerical treatment of solitary wave solutions are connected with the unboundness of the integration domain. For example, when using finite differences or finite element methods, it is necessary to consider large enough domains. When reducing the infinite interval to a finite one we can introduce artificial eigenvalue problems which are irrelevant to the original infinite domain. The discrete problem can have only a trivial solution, while the original one possesses a nontrivial one and vice versa. Spectral methods are generally used to study the soliton equations and soliton solutions, with the combination of analytical and numerical approaches. The theory of nonlinear systems, especially that of strange attractors and its perspectives, is given in [Shilnikov, 1997].

Here we construct new algorithm to investigate numerically the solitary wave solutions of the Boussinesq and the KdV equations. In Sec. 2, the equation is transformed to an ordinary differential equation which possesses nonunique solution. In Sec. 3, we reformulate the bifurcation problem introducing a new parameter and in such a way we expel the nontrivial solution from the original problem. We illustrate the new approach using the generalized Boussinesq Equation (6GBE) (see [Christov, 1995; Steyt *et al.*, 1996]). The method for solving the problem is described in Secs. 3 and 4. In Sec. 5, we introduce the difference scheme. The numerical investigation is presented in Sec. 6. Our conclusions are given in Sec. 7.

2. Localized Waves Problem

Let us consider the soliton equation

$$\mathcal{L}(u) = 0, \tag{1}$$

where

$$\mathcal{L}(u) = \mathcal{L}^{\mathrm{KdV}}(u) = u_t + \gamma u_x + 2\alpha u u_x + u_{xxx}, \tag{2}$$

when we deal with the KdV equation,

$$\mathcal{L}(u) = \mathcal{L}^{\mathrm{Bsq}}(u) = -u_{tt} + \gamma^2 u_{xx} + \alpha(u^2)_{xx} - u_{xxxx}, \tag{3}$$

with the classical Boussinesq equation. Or, the simplest version of 6GBE that is paradigmatically consistent reads

$$\begin{aligned} \mathcal{L}(u) &= \mathcal{L}^{\mathrm{6GBE}}(u) \\ &= -u_{tt} + \gamma^2 u_{xx} + \alpha(u^2)_{xx} \\ &\quad + \beta u_{xxxx} + \delta u_{xxxxxx}. \end{aligned} \tag{4}$$

where the sixth-order dispersion coefficient is always positive $\delta > 0$. Since this coefficient can be rescaled, we will take $\delta = 1$ in the following without loss of generality. The function $u(x,t)$ represents the amplitude of the fluid surface.

Let us consider the stationary waves in the moving frame $\xi = x - ct$. After integration (double for the operators (3) and (4)) with respect to ξ and taking into account the localized character of the investigated solutions, one obtains the following nonlinear ordinary differential equations

$$\mathcal{L}^{\mathrm{KdV}}(u) = \lambda u + \alpha u^2 + u_{\xi\xi} = 0, \quad \lambda \equiv \gamma - c, \tag{5}$$

$$\mathcal{L}^{\mathrm{Bsq}}(u) = \lambda u + \alpha u^2 + u_{\xi\xi} = 0, \quad \lambda \equiv \gamma^2 - c^2, \tag{6}$$

$$\begin{aligned} \mathcal{L}^{\mathrm{6GBE}}(u) &= \lambda u + \alpha u^2 + \beta u_{\xi\xi} \\ &\quad + u_{\xi\xi\xi\xi} = 0, \quad \lambda \equiv \gamma^2 - c^2. \end{aligned} \tag{7}$$

We are looking for solutions of Eq. (7) with $u \to 0$ when $\xi \to \infty$. Then $u^2 \ll u$ in the tails and the linearized version of (7) coincides with its linear part. It possesses harmonic solutions of the type $e^{k\xi}$. The corresponding dispersion relation reads

$$k^4 + \beta k^2 + \lambda = 0 \;\Rightarrow\; k^2 = \frac{1}{2}\left[-\beta \pm \sqrt{\beta^2 - 4\lambda}\right]. \tag{8}$$

Equation (8) shows that the natural classification of stationary solutions should be based on two criteria which define together the spatial asymptotic behavior of the tails. Firstly, a stationary solution can be subsonic ($\lambda > 0$) or supersonic ($\lambda < 0$); secondly, the asymptotic tails of the localized wave can be either monotonic, purely oscillatory or damped oscillatory, depending on whether k^2 is real positive, real negative or complex.

3. Introducing Unknown Coefficient

As mentioned above, we are looking for nontrivial solutions of Eqs. (5)–(7) with $u(\xi) \to 0$ when $\xi \to \pm\infty$. Since we have the trivial solution, the problem is ill-posed in the sense of Hadamard (see [Hadamard, 1932]) because there are two solutions for given boundary conditions. In addition, the trivial solution is a very strong attractor. For this reason, it is difficult to construct a scheme based on finite differences or finite elements method for solving the problem.

Let $u(\xi) \not\equiv 0$ be a solution of one of Eqs. (5)–(7) with $u(\xi) \to 0$ when $\xi \to \pm\infty$. Obviously, the function $u(-\xi)$ is also a solution, i.e. the solution is an even function (satisfying the condition $u(\xi) = u(-\xi)$) and only even function can be a solution of those equations. This fact allows us to consider the problem on the half-line. Then the boundary conditions for $\xi = 0$ reads:

$$u(0) = \chi, \tag{9}$$

$$u'(0) = 0, \tag{10}$$

$$u'''(0) = 0, \tag{11}$$

where $\chi \neq 0$ is an unknown constant. We use the boundary condition (11) only in the case when we consider the 6GBE (7).

It is convenient to scale the function u by the unknown constant χ introducing a new unknown function $w(\xi)$ as follows

$$u(\xi) = \chi w(\xi), \tag{12}$$

arriving thus at the problem of coefficient identification for the differential equations

$$\mathcal{A}^{\mathrm{KdV}}(u, \chi) \equiv (\gamma - c)w + \alpha\chi w^2 + w_{\xi\xi} = 0, \tag{13}$$

$$\mathcal{A}^{\mathrm{Bsq}}(u, \chi) \equiv \lambda w + \alpha\chi w^2 + w_{\xi\xi} = 0, \tag{14}$$

$$\mathcal{A}^{\mathrm{6GBE}}(u, \chi) \equiv \lambda w + \alpha\chi w^2 + \beta w_{\xi\xi} + w_{\xi\xi\xi\xi} = 0, \tag{15}$$

under boundary conditions:

$$w(0) = 1, \tag{16}$$

$$w'(0) = 0, \tag{17}$$

$$w(\xi) \to 0 \quad \text{when} \quad \xi \to \infty, \tag{18}$$

and

$$w'''(0) = 0, \tag{19}$$

$$w'(\xi) \to 0 \quad \text{when} \quad \xi \to \infty, \tag{20}$$

in the case of 6GBE.

Thus the difficulties connected with the unknown constant χ in the boundary condition (9) are circumvented. Yet, it is a problem of unknown coefficient χ from over-posed boundary data. Under certain natural conditions it is possible to find a constant χ such that Eq. (13), (14) or (15) has a solution $w(\xi)$ and this solution also satisfies the boundary conditions (16)–(18) (or (16)–(20) when we consider Eq. (15)). In such a case we say that the pair (w, χ) constitutes a solution to the problem (15)–(20). The problem is of an inverse nature and is similar to the problem of identification of heat-conduction coefficient from overdetermined data considered in [Christov & Marinov, 1997, 1998].

Equation (7) will be considered below. However, we emphasize that the proposed method is applicable to the lower order equations in a similar fashion.

4. Variational Imbedding

Following the method of variational imbedding (MVI) we replace the original problem by the problem of minimization of the functional

$$\mathcal{I}(w, \chi) = \int_0^\infty [\mathcal{A}(w, \chi)]^2 \mathrm{d}x = \int_0^\infty [\lambda w + \chi\alpha w^2 + \beta w_{\xi\xi} + w_{\xi\xi\xi\xi}]^2 \mathrm{d}x \to \min, \tag{21}$$

where u must satisfy the conditions (16)–(20) and $\chi \neq 0$ is an unknown constant. The functional $\mathcal{I}$ is a quadratic and homogeneous function of the $\mathcal{A}(w, \chi)$ and hence it attains its minimum if and only if $\mathcal{A}(w, \chi) \equiv 0$. In this sense there is one-to-one correspondence between the solution of the original problem (15)–(20) and the minimization problem (21).

Clearly, the Euler(-Lagrange) equation for this functional possesses cubic nonlinearity with respect to the function w. It means that for the numerical solution one has to linearize the said equation in order to solve it numerically. Alternatively, one can linearize the integrand in (21) considering the function w as known (say, from the previous iteration) when they appear as coefficients, i.e. $w(\xi) = q(\xi)$. Following the second approach for the linearization we consider the problem of minimization of the following functional

$$\mathcal{I}(u,\chi) = \int_0^\infty [F(\xi)]^2 \mathrm{d}x \to \min\,, \tag{22}$$

where

$$F(\xi) \equiv \lambda w + \chi\alpha q w + \beta w_{\xi\xi} + w_{\xi\xi\xi\xi} \tag{23}$$

is the residual of the given equation.

4.1. *Imbedded boundary-value problem*

Necessary conditions for minimization of (22) are the Euler–Lagrange equations for the functions $w(\xi)$ and χ. The equation for w reads

$$(\alpha\chi q + \lambda)F(\xi) + \beta\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}F(\xi) + \frac{\mathrm{d}^4}{\mathrm{d}\xi^4}F(\xi) = 0 \tag{24}$$

This equation is of eight order and it is convenient to reduce it to a system of two Eqs. (23) and (24), each of them of fourth order. The solution of the system (23), (24) can satisfy four conditions at each boundary point. For $\xi \to \infty$ we have enough boundary conditions since from $w \to 0$ follows

$$F(\xi) \to 0 \quad \text{and} \quad F'(\xi) \to 0, \quad \text{for} \quad \xi \to \infty. \tag{25}$$

We make use of the so-called natural boundary condition for minimization of the functional at $\xi = 0$, which is nothing else than the original Eq. (15), i.e.

$$F(0) = 0\,. \tag{26}$$

The problem is coupled with the equation for χ. We rewrite the functional (21) in the form

$$\mathcal{I} = \chi^2 \int_0^\infty (\alpha w^2)^2 \mathrm{d}x + 2\chi \int_0^\infty (\alpha w^2)(\lambda w + \beta w_{\xi\xi} + w_{\xi\xi\xi\xi})\mathrm{d}x + \int_0^\infty (\lambda w + \beta w_{\xi\xi} + w_{\xi\xi\xi\xi})^2 \mathrm{d}x \tag{27}$$

and after some obvious manipulations the equation for χ adopts the form:

$$\chi = -\frac{\displaystyle\int_0^\infty (\alpha w^2)\,(\lambda w + \beta w_{\xi\xi} + w_{\xi\xi\xi\xi})\mathrm{d}x}{\displaystyle\int_0^\infty \left(\alpha w^2\right)^2 \mathrm{d}x}\,. \tag{28}$$

5. Difference Scheme

We solve the above boundary value problem using finite differences. The mesh (see Fig. 1) is regular and allows us to approximate all operators with standard central differences.

5.1. *Grid pattern and approximations*

For the grid spacing we have

$$h \equiv \frac{\xi_\infty}{n-3}\,,$$

where n is the total number of grid points and ξ_∞ is an sufficiently large number, called "*numerical infinity*". Then the gird points are defined as follows:

$$\xi_i = (i - 2.5)h \quad \text{for} \quad i = 1, \ldots, n. \tag{29}$$

It is an important fact that the point $\xi = 0$ is the mid-point $\xi_{2\frac{1}{2}}$.

Let us introduce the notation

$$w_i = w(\xi_i), \quad F_i = F(\xi_i), \quad \text{for} \quad i = 1, \ldots, n. \tag{30}$$

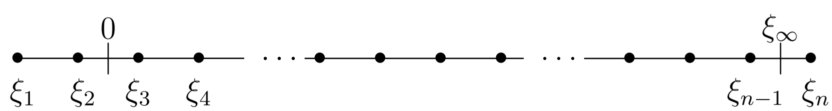

Fig. 1. The mesh.

We employ symmetric central differences for approximating the differential operators as follows:

$$G_{\xi\xi\xi\xi}(\xi_i) = \Lambda_{\xi\xi\xi\xi}G_i + O(h^2) = \frac{1}{h^4}(G_{i-2} - 4G_{i-1} + 6G_i - 4G_{i+1} + G_{i+2}) + O(h^2) \tag{31}$$

and

$$G_{\xi\xi}(\xi_i) = \Lambda_{\xi\xi}G_i + O(h^2) = \frac{1}{h^2}(G_{i-1} - 2G_i + G_{i+1}) + O(h^2) \tag{32}$$

for $i = 3, \ldots, n$ and G denotes one of the functions w or F.

The grid allows us to approximate the boundary conditions with second order as well. At $\xi = 0$ we have:

$$w(0) = 1 \Rightarrow w_2 + w_3 = 2, \tag{33}$$

$$w'(0) = 0 \Rightarrow w_2 - w_3 = 0, \tag{34}$$

$$w'''(0) = 0 \Rightarrow w_1 - 3w_2 + 3w_3 - w_4 = 0, \tag{35}$$

$$F(0) = 1 \Rightarrow F_2 + F_3 = 0. \tag{36}$$

For $\xi = \xi_\infty$ we have:

$$w(\xi) \to 0, \quad \xi \to \infty \Rightarrow w_n + w_{n-1} = 0, \tag{37}$$

$$w'(\xi) \to 0, \quad \xi \to \infty \Rightarrow w_n - w_{n-1} = 0, \tag{38}$$

$$F(\xi) \to 0, \quad \xi \to \infty \Rightarrow F_n + F_{n-1} = 0, \tag{39}$$

$$F'(\xi) \to 0, \quad \xi \to \infty \Rightarrow F_n - F_{n-1} = 0. \tag{40}$$

The gist of the developed numerical scheme is that it allows a special treatment of the algebraic systems. The systems for the respective functions w and F is conjugated by the boundary conditions for $\xi = 0$ — we have three boundary conditions for the functions w and only one for F. Let us define a vector

$$\boldsymbol{\theta}^T = (\theta_1, \theta_2, \ldots, \theta_{2n}) \tag{41}$$

as

$$\theta_{2k-1} = w_k, \quad \theta_{2k} = F_k \tag{42}$$

$k = 1, 2, \ldots, n$, i.e.

$$\boldsymbol{\theta}^T = (w_1, F_1, w_2, F_2, \ldots, w_n, F_n). \tag{43}$$

We obtain the components θ_k from the nine-diagonal system

$$\boldsymbol{A\theta} = \boldsymbol{\Upsilon}, \tag{44}$$

where

$$\boldsymbol{\Upsilon}^T = (2, 0, 0, 0, \ldots, 0, 0), \tag{45}$$

$$\boldsymbol{A} = \begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & -3 & 0 & 3 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\
1 & 0 & a & 0 & b_3 & h^4 & a & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & a & 0 & b_3 & 0 & a & 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1
\end{pmatrix} \tag{46}$$

and $a = -4 + \beta h^2$, $b_k = 6 - 2\beta h^2 + (\lambda + \chi q_k)h^4$, $k = 3, \ldots, n-2$. The nine-diagonal system is solved by means of a specialized solver, which is a generalization of what is called Thomas algorithm.

5.2. *Estimation of χ*

We approximate the integrals in Eq. (28) for χ using so-called "extended midpoint rule"

$$\chi = -\frac{\sum_{k=3}^{n-2} (\alpha w_k^2)(\lambda w_k + \beta\Lambda_{\xi\xi}w_k + \Lambda_{\xi\xi\xi\xi}w_k)}{\sum_{k=3}^{n-2} (\alpha w_k^2)^2} + O(h^2). \tag{47}$$

Here, the error term is again of the second order.

5.3. *Algorithm*

(I) The eight-order boundary value problem (15), (23) is solved for the function w with given χ and q.

(II) If the deviation of the calculated w from q is smaller than ε then we proceed to (III), otherwise q is replaced by w and then go to (I);

(III) With the newly computed w, the coefficient χ is evaluated from (28). If the following criterion is satisfied

$$\frac{\max|w^{n+1} - w^n|}{\tau \max|w^{n+1}|} < \varepsilon \quad \text{and} \quad \frac{\max|\chi^{n+1} - \chi^n|}{\tau \max|\chi^{n+1}|} < \varepsilon, \tag{48}$$

then the calculations are terminated. Otherwise the index of iterations is stepped up $n := n + 1$ and the algorithm is returned to step (I).

The admissible tolerance is chosen to be $\varepsilon \leq 10^{-10}$ for the criterion of convergence (48).

6. Numerical Results

6.1. *Scheme validation*

In order to verify the performance of the described numerical scheme we started the numerical experiments with the following particular values of the parameters: $\lambda = 36/169$, $\beta = -1.0$ and $\alpha = 1.0$. Then an analytical solution exists

$$u_{\text{an}}(\xi) = -\frac{105}{338}\,\text{sech}^4\left(\frac{\xi}{2\sqrt{13}}\right). \tag{49}$$

From the analytical solution (49) we obtain the exact value of the coefficient

$$\chi_{\text{an}} = -\frac{105}{338} \approx -0.3106508876.$$

The discretization error term is $O(h^2)$, and the total error is $O(h^2)$. Our numerical results demonstrate clearly these error orders. The differences

$$r(\xi) = u_h(\xi) - u_{\text{an}}(\xi) \tag{50}$$

between four solutions with mesh-steps $h = 1/10$, $h = 1/20$, $h = 1/40$, $h = 1/80$ and the analytical solution (49), respectively, are shown in Fig. 2.

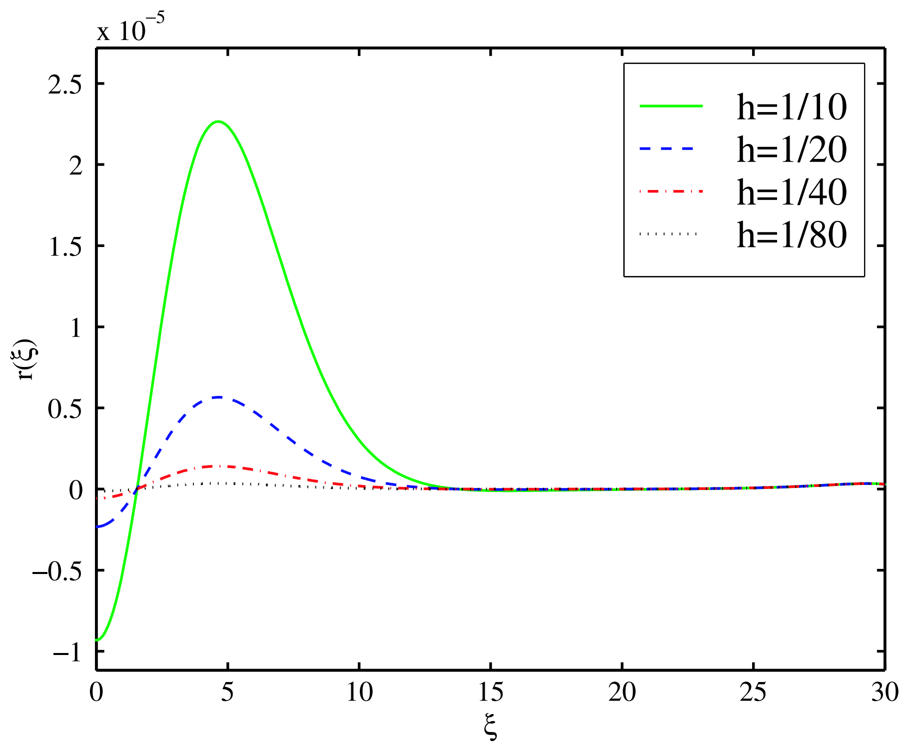

Fig. 2. Difference $r(\xi) = u_h - u_{\text{an.}}$ using four different steps h.

Table 1. Obtained values of the amplitude χ of the peak for five different values of the mesh spacing and for the value of actual infinity $\xi_\infty = 60$.

h	χ	Number of Iterations for w	Number of Iterations for χ	Rate of Convergence
Analytical	−0.3106508876	—	—	—
1/5	−0.3105686012	55	18	—
1/10	−0.3106303263	54	18	2.000720771
1/20	−0.3106457447	54	18	1.999305528
1/40	−0.3106495943	53	17	1.991497081
1/80	−0.3106505634	53	18	1.996100972

The values of the identified coefficient χ using five different steps are given in Table 1. The rate of convergence, calculated as

$$\text{rate} = \log_2 \left| \frac{\chi_h - \chi_{\text{an.}}}{\chi_{2h} - \chi_{\text{an.}}} \right| \tag{51}$$

is also shown in Table 1.

The domain of the problem is unbounded. This introduces an additional parameter in the numerical scheme, namely the numerical ("actual") infinity ξ_∞. The scheme is tested also for the magnitude of this parameter. The shapes of the difference $r(\xi) = u_h - u_{\text{an.}}$ for $\xi_\infty = 30$, $\xi_\infty = 45$, $\xi_\infty = 60$ and $\xi_\infty = 75$ are shown in Fig. 3. Obviously, the values for the residuals are indistinguishable when $\xi_\infty \geq 45$.

6.2. *Monotone shapes for different celerities*

Monotone shapes appear for $\beta < -2\sqrt{\lambda}$ with λ positive. In particular, for $\beta = -1$, the condition for having monotone tails is $c > \sqrt{1 - 0.25\beta^2} \approx 0.866$. In this case an analytical solution of Eq. (15) is available (see e.g. [Yamamoto, 1981])

$$u_{an}(\xi) = -\frac{105}{169}\frac{\beta^2}{2\alpha}\operatorname{sech}^4\left(\frac{\xi}{2}\sqrt{-\frac{\beta}{13}}\right), \qquad |c| = \sqrt{\gamma^2 - \frac{36}{169}\beta^2}\,. \tag{52}$$

We calculate the monotone shapes numerically for $0.866 < c < 1$. The present results are in very good quantitative agreement with the difference solution of [Christov *et al.*, 1996] for monotone shapes. Similar to the case of the fourth-order Boussinesq equation, subsonic humps have a larger amplitude when they are slower. The obtained monotone solutions for different celerities are presented in Fig. 4.

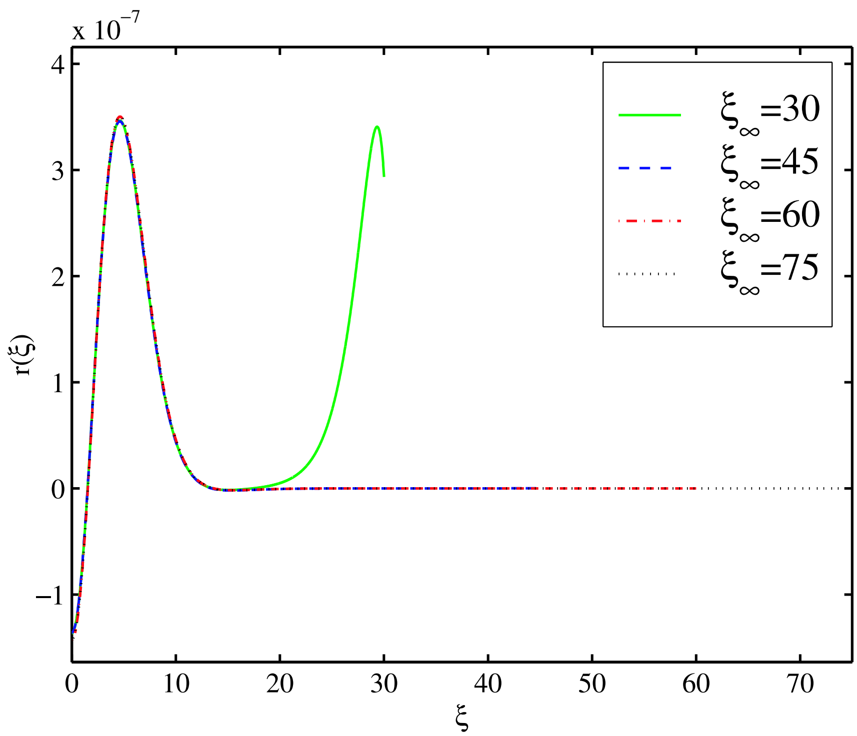

Fig. 3. Difference $r(\xi) = u_h - u_{\text{an.}}$ using four different values of the numerical infinity ξ_∞ ($h = 1/80$).

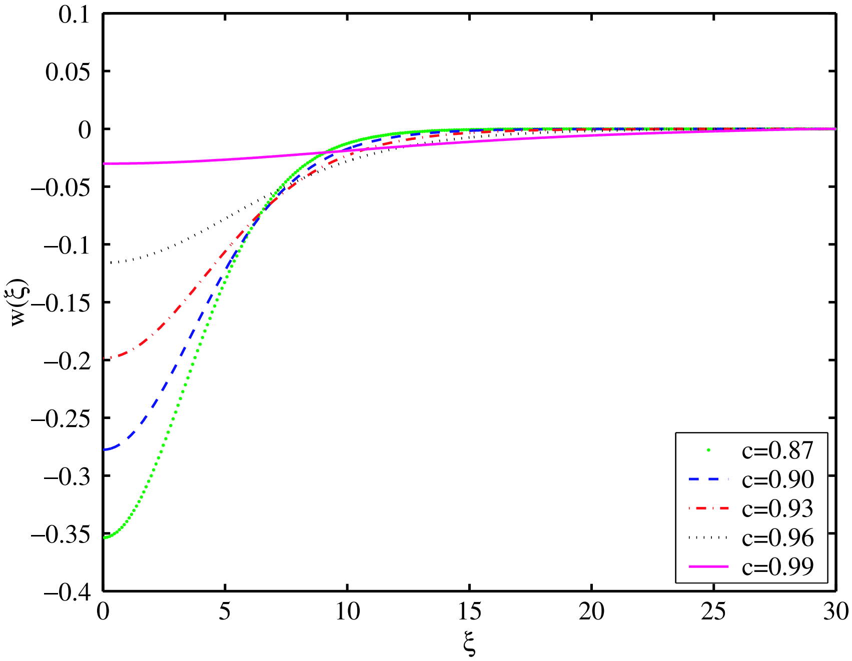

Fig. 4. The monotone shapes for different celerities $c = 0.87$, 0.90, 0.93, 0.96 and 0.99. Slower waves have a larger hump.

6.3. *Damped oscillatory shapes*

This class of subsonic waves corresponds to $-2\sqrt{\lambda} < \beta < 2\sqrt{\lambda}$. As β is increased from the lowest limit to the upper one, the soliton develops small-amplitude damped periodic tails and the damping is lesser with the increase of β. The damped oscillatory solutions are named after Kawahara [1972] who discovered them numerically

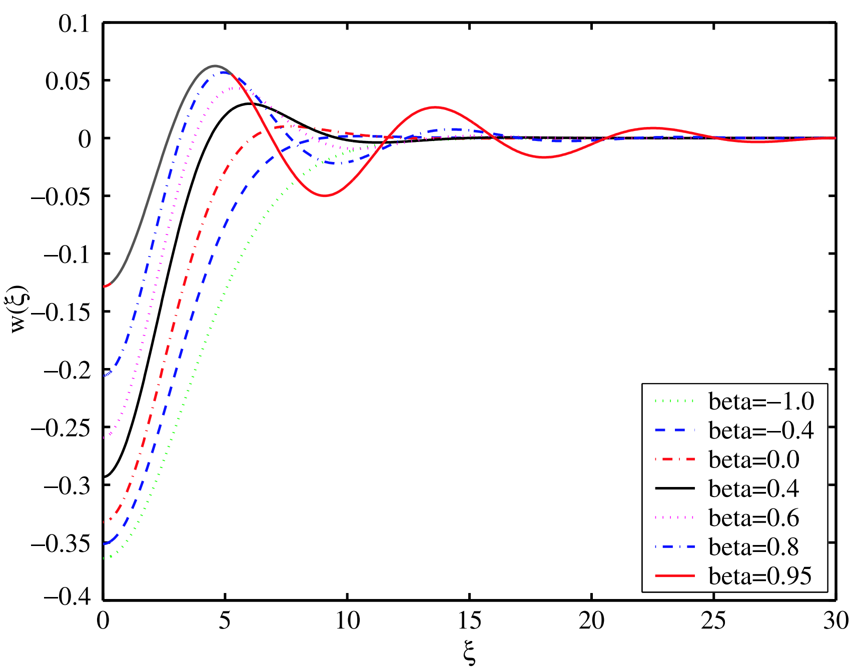

Fig. 5. Kawahara shapes $c = 0.866, \ldots$, $\beta = -1.00$, -0.40, 0.00, 0.40, 0.60, 0.80, 0.95. The oscillatory aspect is more and more visible as β is increased.

in the fifth-order KdV equation. The Kawahara solitons are an intermediate case between the monotone shapes and the above mentioned nonlocal waves. The transfer from one extreme case to the other is made through the progressive selection of a frequency. Fixing $\gamma = 1$ and $c \approx 0.866$ the "β-interval" corresponding to Kawahara's solitons is $[-1, 1]$. In order to show the progressive transformation from monotone shapes to persisting oscillatory shapes, we present in Fig. 5 different Kawahara's solitons computed for some values of β comprised between -1 and 1 and for $c \approx 0.866$. As for the case of monotone shapes, the following results are in full quantitative agreement with the corresponding ones obtained in [Christov *et al.*, 1996] by means of finite differences and [Steyt *et al.*, 1996] by spectral method.

7. Conclusions

In this work, we have presented an algorithm for identification of soliton solutions of Boussinesq and the Korteweg–de Vries equations. We have shown "experimentally" that it allows the investigation of the stationary solutions of 6GBE and consequently, represents an alternative tool for studying physical properties of "6GBE quasi-particles".

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