Available online at www.sciencedirect.com

SCIENCE DIRECT®

Mathematics and Computers in Simulation 69 (2005) 514–525

MATHEMATICS AND COMPUTERS IN SIMULATION

www.elsevier.com/locate/matcom

Strong coupling of Schrödinger equations: Conservative scheme approach

W.J. Sonnier*, C.I. Christov

Department of Mathematics, University of Louisiana at Lafayette, P.O. Box 1010, Lafayette, LA 70504, USA

Available online 5 July 2005

Abstract

The system of coupled nonlinear Schrödinger's equations (CNLSE) is considered and the physical meaning of the coupling terms is identified. The attention is focused on the case of real-valued parameter of linear cross-diffusion. A new analytical solution for the coupled case is found and used as initial condition for the interaction and evolution of two pulses.

Conservative numerical scheme and algorithm are devised for the time evolution of solitons in CNLSE. The results show that the coupling term brings into play localized solutions with rotating polarization which in many instances behave as breathers. Both elastic and inelastic collisions are uncovered numerically.

Keywords: Schrödinger equation; CNLSE; Elastic and inelastic collisions

1. Introduction

Numerous phenomena of physical significance are described by the various forms of the nonlinear Schrödinger equations. Topics in nonlinear optics, quantum fluids/condensed matter physics, gravitation, biological modeling, plasma physics, and many others are modeled with members of this class of equations.

$$\mathrm{i}\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0 \tag{1}$$

* Corresponding author.
E-mail address: wjs8684@louisiana.edu (W.J. Sonnier).

0378-4754/$30.00 © 2005 IMACS. Published by Elsevier B.V. All rights reserved.
doi:10.1016/j.matcom.2005.03.016

As far as the applications in nonlinear optics are concerned, the above equation describes the single-mode wave propagation in a fiber. However, these fibers also allow propagation of more than one orthogonally polarized propagating modes, which may be described by a multi-component version of (1) [1–3].

Depending on the sign of α, Eq. (1) admits single and multiple sech-solutions ("bright solitons"), as well as tanh-profile, or "dark soliton" solutions. In this paper, we concentrate on the case of bright solitons.

When the physical system considered includes a coupling, then a multi-component version of the NLSE is to be used. The model involving coupled nonlinear Schrödinger equations (CNLSE), is used to uncover a wealth of information about a wide variety of phenomena, such as the interaction between pulses in nonlinear optics, modeling the Bose–Einstein condensates, or signals in nonlinear acoustic media. The CNLSE model is

$$\begin{aligned} &\mathrm{i}\psi_t + \beta\psi_{xx} + [\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2]\psi + \gamma\psi + \Gamma\phi = 0, \\ &\mathrm{i}\phi_t + \beta\phi_{xx} + [\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2]\phi + \gamma\phi + \Gamma\psi = 0. \end{aligned} \tag{2}$$

Of particular interest for the present work is the linear coupling parameter Γ. It accounts for effects that arise from twisting of the fiber and elliptic deformation of the fiber. It is also referred to as linear birefringence [4] or relative propagation constant [5]. The term proportional to α_1, describes the self-focusing of a signal for pulses in birefringent media [2]. The parameter β describes the group velocity dispersion, and $(\alpha_1 + 2\alpha_2)$ is the cross-phase modulation, and defines the integrability of Eq. (2). Finally, the term γ appears as constant ambient potential called normalized birefringence [6].

Introducing the two new parameters in Eq. (2), γ and Γ makes the phenomenology of the system (2) much reachable. It is important here to identify the role of these new parameters, especially in the nonlinear-optics context. $\Re[\Gamma]$ describes additional dispersion in the system due to the coupling. A similar dispersive effect has $\Im[\Gamma]$ but in a sense it can be called "self-dispersion" and it accounts for the dispersion due to the internal properties of the media (the optical fiber). The interpretation of these parameters in the models of wave propagation in nonlinear wave guides remains to be better understood. When devising the numerical tool, we retain these parameters but the results of the present paper concern mostly the case $\gamma = 0$ and $\Im[\Gamma] = 0$ which leaves a sole new parameter $\Re[\Gamma]$ which accounts for the dispersion of the signals due to coupling between the components. From now onwards, we will think about Γ as of real-valued parameter.

For $\Gamma = 0$, Eq. (2) is alternatively called the Gross–Pitaevskii equation or an equation of Manakov-type. It was derived independently by Gross [7,8] and Pitaevskii [9], to describe the behaviors of Bose–Einstein condensates as well as optic pulse propagation. It was solved analytically for the case $\alpha_2 = 0$, $\beta = \frac{1}{2}$ by Manakov [1] via inverse scattering transform who generalized an earlier result by Zakharov and Shabat [10,11] for the scalar cubic NLSE (i.e. Eq. ((2)$-\psi$) with $(\phi(x, t) = 0)$. Recently, Chow et al. [12] studied periodic waves in optic fibers using a version of (2) with $\Gamma \neq 0$.

The solutions are of the form

$$\begin{pmatrix} \psi(x,t) \\ \phi(x,t) \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} \Psi(x,t) \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ \Phi(x,t) \end{pmatrix},$$

where

$$\Psi(x,t),\Phi(x,t) = A\mathrm{sech}\left[\sqrt{\frac{\alpha_1}{2\beta}}A(x-u_{\mathrm{e}}t)\right]\exp\left[\mathrm{i}\frac{c}{2}(x-u_c t)\right] \tag{3}$$

are both solutions to the scalar NLSE with:

$$u_{\mathrm{e}} = \beta c \rightarrow \text{envelope velocity} \quad \text{and} \quad u_c = \beta c - \frac{2}{c}\left(\frac{\alpha}{2}A^2 + \frac{c^2}{4}\beta\right) \rightarrow \text{phase velocity.}$$

The amplitude, A, depends on the signal's envelope velocity, carrier frequency, and as we will see, the dispersion/focusing parameter γ. These sech-profile solutions, of special interest to optics, are referred to as bright solitons. However, depending on the sign of α_1, Eq. (1) admits single and multiple sech-solutions, as well as tanh-profile, or dark, soliton solutions. This parameter describes a self-interaction of the signal. For example, in the study of Bose–Einstein condensates, α_1 is the measure of interparticle interaction of the population in question. For pulses in birefringent media [2], it describes the self-focusing of a signal. The solitons suffer no deformation under collision but exhibit a phase shift, resulting in a change in the relative separation.

The functions ψ and ϕ have various interpretations in the context of optic pulses. These include the amplitudes of x and y polarizations in a birefringent nonlinear planar waveguide [6,13] and the amplitudes of x and y linear polarizations in a birefringent fiber [6,14], pulsed wave amplitudes of left and right circular polarizations [4,15,5], and pulsed wave amplitudes of symmetric and antisymmetric modes in a nonlinear coupler [16,17].

Eq. (1) have been shown to be completely integrable for the Manakov case [18]. However, this condition is unnecessarily restrictive for the physical systems under consideration. The nonlinear coupling constant parameter $\alpha_1 + 2\alpha_2$ represents the degree to which each component of the solution is influenced by the other component. In the context of Schrödinger operators, the respective α's may be seen as intensity dependent interaction potentials. In a birefringent medium, $\alpha_1 + 2\alpha_2$ is the cross-phase modulation of the signal, and distinguishes the description of elliptic, circular and linear polarizations. In this more interesting case, integrability is lost, and numerical methods have to be used to study the evolution of the system [19]. Of particular interest are the cases which are nearly integrable [20].

For $\alpha_2 \neq 0$, the solitons experience deformation after a collision, manifested in the creation of "shadow pulses" since these deformations often take the form of stable oscillations tied to the partner of a soliton in question. The oscillations are left behind the initial pulse, and they are also called "tail modes". In keeping with Ref. [19], we will refer to these as inelastic collisions because they give rise to excitation of tail modes.

2. Formulation of the problem

The purpose of this work is to find localized solutions (both analytic and numerical) of CNLSE (2) for $\gamma = 0$ and Γ real. The complex linear terms, γ_r and γ_i, can be shown to be focusing/dispersion and gain/dissipation, respectively. In some cases, the linear coupling term will be shown to correspond to a Rabi-like oscillation or breathing and will be treated elsewhere. Here, we focus our attention on the interacting solitary waves (collision of solitons).

The asymptotic boundary conditions (a, b, c) read

$$\psi, \psi_x, \phi, \phi_x \to 0 \quad \text{for} \quad |x| \to \infty \tag{4}$$

with initial conditions $\psi(x, 0) = \psi_o(x)$ and $\phi(0, t) = \phi_o(x)$ that are consistent with the a, b, c.

The so-called one-soliton solution allows one to unravel the physical mechanisms behind the sustained propagation of permanent waves and is of importance in nonlinear optics when the propagation of a simple pulse is considered.

Manakov [1], Zkharov and Shabat [10,11], among others, addressed the self-focusing NLSE and solved it analytically via inverse scattering transform. A generalization of this technique was used by Manakov to solve the CNLSE (with $\alpha_2 = 0$). Other investigators have used variations of the IST and bilinear transform to study various aspects of single soliton behavior and multi-soliton collisions. Both IST and bilinear transform may be applied to (2) with $\gamma_i = \Gamma = \alpha_2 = 0$, to obtain analytic solutions to what is termed the focusing case. Using these solutions as a starting point, we may construct analytic solutions to the case of linear, or strong, coupling ($\Gamma \neq 0$).

However, the integrable case is fairly limiting in terms of actual physical phenomena. Hence, we wish to investigate the system as it departs from integrability, i.e. when $\alpha_2 \neq 0$ which corresponds to the relative inelasticity of soliton collisions and is therefore of great importance for the understanding of the generic properties of soliton interaction.

Here, we find a new analytical solution, which is constructed from previously obtained solutions for the CNLSE. Specifically, if Ψ and Φ are solutions to the CNLSE ,

$$\begin{aligned} &\mathrm{i}\Psi_t + \beta\Psi_{xx} + \alpha_1[|\Psi|^2 + |\Phi|^2]\Psi = 0, \\ &\mathrm{i}\Phi_t + \beta\Phi_{xx} + \alpha_1[|\Psi|^2 + |\Phi|^2]\Phi = 0. \end{aligned} \tag{5}$$

Then, the functions

$$\begin{aligned} \psi &= \Psi\ \cos(\Gamma t) + \mathrm{i}\Phi \sin(\Gamma t), \\ \phi &= \Phi\ \cos(\Gamma t) + \mathrm{i}\Psi \sin(\Gamma t). \end{aligned}$$

are solutions to the strongly coupled NLSE

$$\begin{aligned} &\mathrm{i}\psi_t + \beta\psi_{xx} + \alpha_1[|\psi|^2 + |\phi|^2]\psi + \Gamma\phi = 0, \\ &\mathrm{i}\phi_t + \beta\phi_{xx} + \alpha_1[|\phi|^2 + |\psi|^2]\phi + \Gamma\psi = 0. \end{aligned} \tag{6}$$

This new analytic solution is given by,

$$\begin{aligned} \psi(x,t) = {} & A_\Psi \mathrm{sech}\left[\sqrt{\tfrac{\alpha_1}{2\beta}} A_\Psi (x - u_{\mathrm{e}\Psi} t + x_\mathrm{o})\right] \exp\left[\mathrm{i}\tfrac{c_\Psi}{2}(x - u_{c\Psi} t)\right] \cos(\Gamma t) \\ & + \mathrm{i} A_\Phi \mathrm{sech}\left[\sqrt{\tfrac{\alpha_1}{2\beta}} A_\Phi (x - u_{\mathrm{e}\Phi} t)\right] \exp\left[\mathrm{i}\tfrac{c_\Phi}{2}(x - u_{c\Phi} t)\right] \sin(\Gamma t), \\ \phi(x,t) = {} & A_\Phi \mathrm{sech}\left[\sqrt{\tfrac{\alpha_1}{2\beta}} A_\Phi (x - u_{\mathrm{e}\Phi} t)\right] \exp\left[\mathrm{i}\tfrac{c_\Phi}{2}(x - u_{c\Phi} t)\right] \cos(\Gamma t) \\ & + \mathrm{i} A_\Psi \mathrm{sech}\left[\sqrt{\tfrac{\alpha_1}{2\beta}} A_\Psi (x - u_{\mathrm{e}\Psi} t + x_\mathrm{o})\right] \exp\left[\mathrm{i}\tfrac{c_\Psi}{2}(x - u_{c\Psi} t)\right] \sin(\Gamma t). \end{aligned} \tag{7}$$

These are pulses whose modulation amplitude is of the form of sech and their polarization rotates with time. This means that even without an interaction, the ψ and ϕ components “breathe” with time.

Apart of its importance as a bright-soliton solution for CNSE, the above solution will also serve as initial condition for the numerical investigation of the temporal evolution of system of interacting solitons.

3. Conservation laws

The CNLSE (2) has two standard conserved quantities: mass and energy. We can obtain the conservation law for the wave mass by multiplying the first and second parts of (2) by $\bar{\psi}$ and $\bar{\phi}$, respectively, collecting the imaginary terms, and then integrating over space. This yields

$$\frac{\partial}{\partial t}\boldsymbol{M} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}(|\psi|^2 + |\phi|^2)\,\mathrm{d}x = 0. \tag{8}$$

Notice that the "masses" of the functions ψ and ϕ are not conserved separately. This differs from the case $\Gamma = 0$, when separate conservation laws are found, namely

$$\begin{aligned}\frac{\partial}{\partial t}\boldsymbol{M}_\psi &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\psi|^2\,\mathrm{d}x = 0,\\ \frac{\partial}{\partial t}\boldsymbol{M}_\phi &= \frac{\partial}{\partial t}\int_{-\infty}^{\infty}|\phi|^2\,\mathrm{d}x = 0.\end{aligned} \tag{9}$$

The conservation of energy can be derived by multiplying the first and second parts of (2) by $\bar{\psi}_t$ and $\bar{\phi}_t$, respectively, collecting the real terms, and integrating over space. This yields for real Γ and γ-that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{E} = \frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{\infty} &-\beta[|\psi_x|^2 + |\phi_x|^2] + \frac{\alpha_1}{2}[|\psi|^4 + |\phi|^4] + (\alpha_1 + 2\alpha_2)[|\psi|^2|\phi|^2] + \gamma[|\psi|^2\\ &+ |\psi|^2] + 2\Gamma\Re[\bar{\phi}\psi]\,\mathrm{d}x = 0.\end{aligned} \tag{10}$$

Similarly to the case with the "wave mass", M, the energy is not conserved individually for ψ and ϕ which is a manifestation of the role of coupling in (2). For this reason, we call the case of purely nonlinear coupling a "weak coupling" while the coupling through the terms proportional to Γ (cross-dispersion) we address as "strong coupling".

4. Numerical scheme

Since the system we study is nonintegrable, numerical tools must be developed to find approximate solutions of (2). However, it is vital that the numerical methods reflect the system's dynamic properties; specifically, we construct a scheme which mimics the integral constants of the system. Conservation of energy is especially important in the context of long-time calculations, where non-conservative methods can produce spurious data for the energy.

We implement a scheme which extends the work of [19], making use of internal iterations to treat the nonlinear coupling. A scheme with internal iterations was first proposed for the single NLS in the extensive numerical treatise [21] and named Crank–Nicolson implicit scheme. The CNLSE ($\Gamma = 0$) is investigated numerically in [22] by means of difference schemes. Here, we use the concept of the internal iterations to secure the implementation of the conservation laws on difference level within the round-off error of the calculations. Hence, we have a strict conservation of discrete approximations of the energy and the mass of the wave system.

Consider a uniform mesh in the interval $[-L_1, L_2]$, namely $x_i = (i-1)h$, where $h = (L_1 + L_2)/(N-1)$ and N is the total number of grid points in the interval.

Let τ be the time discretization. Respectively, ψ_i^n and ϕ_i^n denote the values of ψ and ϕ at the ith spatial point and at the time $n\tau$. Consider the scheme

$$\begin{aligned} \mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} &= \frac{\beta}{2h^2}[\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n] + \frac{\psi_i^{n+1,k}+\psi_i^n}{4} \\ &\quad \times \{\alpha_1[|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2] + (\alpha_1+2\alpha_2)[|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2]\} \\ &\quad + \frac{\gamma_r}{2}(\psi_i^n + \psi_i^{n+1,k+1}) + \frac{\Gamma}{2}(\phi_i^n + \phi_i^{n+1,k+1}), \end{aligned} \tag{11}$$

$$\begin{aligned} \mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} &= \frac{\beta}{2h^2}[\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n] + \frac{\phi_i^{n+1,k}+\phi_i^n}{4} \\ &\quad \times \{\alpha_1[|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2] + (\alpha_1+2\alpha_2)[|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2]\} \\ &\quad + \frac{\gamma_r}{2}(\phi_i^n + \phi_i^{n+1,k+1}) + \frac{\Gamma}{2}(\psi_i^n + \psi_i^{n+1,k+1}). \end{aligned} \tag{12}$$

We conduct the internal iterations (repeating time steps) until convergence, i.e. $\|\psi^{n+1,k+1} - \psi^{n+1,k+1}\|/\|\psi^{n+1,k+1}\| \leq 10^{-12}$. Suppose that the internal iterations converge. Then, at each time step we have the solution of the following nonlinear scheme

$$\begin{aligned} \mathrm{i}\frac{\psi_i^{n+1}-\psi_i^n}{\tau} &= \frac{\beta}{2h^2}[\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n] + \frac{\psi_i^{n+1}+\psi_i^n}{4} \\ &\quad \times \{\alpha_1[|\psi_i^{n+1}|^2 + |\psi_i^n|^2] + (\alpha_1+2\alpha_2)[|\phi_i^{n+1}|^2 + |\phi_i^n|^2]\} \\ &\quad + \frac{\gamma_r}{2}(\psi_i^n + \psi_i^{n+1}) + \frac{\Gamma}{2}(\phi_i^n + \phi_i^{n+1}), \end{aligned} \tag{13}$$

$$\begin{aligned} \mathrm{i}\frac{\phi_i^{n+1}-\phi_i^n}{\tau} &= \frac{\beta}{2h^2}[\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n] + \frac{\phi_i^{n+1}+\phi_i^n}{4} \times \{\alpha_1[|\phi_i^{n+1}|^2 + |\phi_i^n|^2] \\ &\quad + (\alpha_1+2\alpha_2)[|\psi_i^{n+1}|^2 + |\psi_i^n|^2]\}\frac{\gamma_r}{2}(\phi_i^n + \phi_i^{n+1}) + \frac{\Gamma}{2}(\psi_i^n + \psi_i^{n+1}). \end{aligned} \tag{14}$$

Following [19] (see also [23] for similar derivations for the Boussinesq equation), we show that the scheme (13) conserves mass and energy, i.e. there exist discrete analogs of (8) and (10).

For all $n \geq 0$,

$$\boldsymbol{M}^n = \sum_i |\psi_i^n|^2 + |\phi_i^n|^2 = \text{const},$$

$$\begin{aligned} \boldsymbol{E}^n &= \sum_i \frac{-\beta}{2h^2}[|\psi_{i+1}^n - \psi_i^n|^2 + |\phi_{i+1}^n - \phi_i^n|^2] + \frac{\alpha_1}{4}[|\psi_i^n|^4 + |\phi_i^n|^4] + \frac{\alpha_2}{2}[|\psi_i^n|^2|\psi_i^n|^2] + \frac{\gamma_r}{2}[|\phi_i^n|^2 \\ &\quad + |\psi_i^n|^2] + \Gamma \mathrm{Re}[\overline{\phi}_\mathrm{i}^n \psi_i^n] = \text{const}. \end{aligned}$$

5. Results and discussion

We discuss two types of collisions—elastic and inelastic. Elastic collisions are those from which the original shapes re-emerge undeformed. If the collision results in additional oscillations in the initial wave-forms, it is referred to as inelastic.

As a test of the algorithm and scheme proposed, we treated the cases without linear (strong) coupling form [19] and recovered the results obtained there. Without overloading the present paper with details, we can report that we have very good quantitative agreement for both elastic and inelastic collisions. Note that in [19] the values of Γ is zero.

As mentioned above, we have obtained analytic solutions (7) with $\alpha_2 = 0$. This is essentially the Manakov case with the addition of self-dispersion and linear coupling. As mentioned above, the polarization of these new analytic solutions rotates with time. We use them as initial conditions in the numerical studies. Owing to the conservative properties of our scheme, we were able to follow the evolution of the wave systems for very long times. A typical interaction is presented in Fig. 1 where the initial polarization is chosen so that each pulse lies entirely in one of the planes. One sees that the rotation of the solutions begins immediately. In addition, the interaction is strictly elastic because each of them recovers its exact initial shape after they pass through each other.

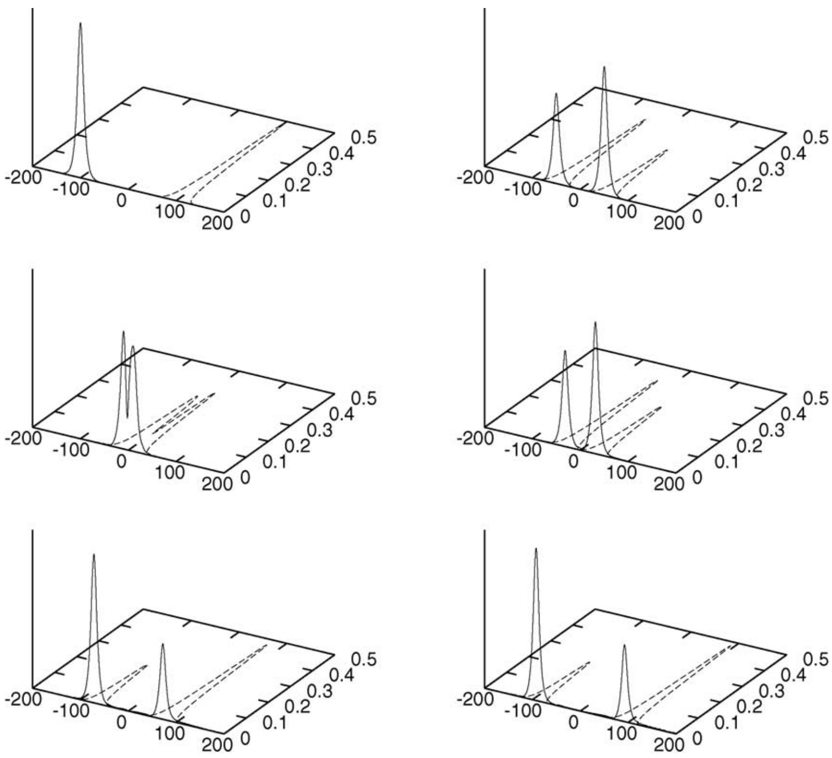

Fig. 1. The interaction of two pulses with $\Gamma = 0.0175$, $\gamma = 0$, $\alpha_1 = 0.5$, $\alpha_2 = 0$ at $t = 0$. The time increases from the top-left panel to bottom-right panel.

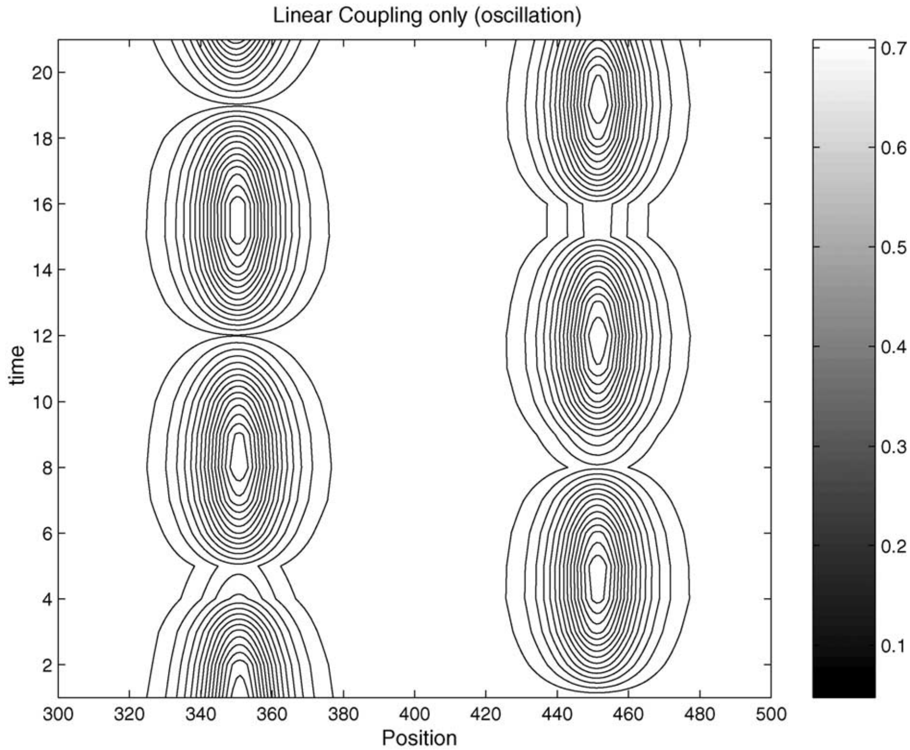

Fig. 2. Breathing of left soliton with envelope speed equal to zero.

We obtained results for variety of values of the governing parameters. In Fig. 2, we show two standing solitons with zero envelope speeds. One sees the "breathing" of the pulses even without any interaction. As already mentioned, this is the manifestation of the rotation of the polarization. Clearly this is an effect of the linear coupling which is accounted for through Γ. Thus, Γ is responsible for the exchange of wave "mass" between the modes and we will call it "cross" dispersion of the signals. In contrast, we call the parameter γ "self" dispersion and it is present even when a single mode is investigated. The interplay between the self-dispersion and cross-dispersion will be investigated in a separate treatise.

The cross-dispersion (parameterized by Γ) affects the results considerably. Our numerical investigations are managed to show its significance. The fact that we have analytic initial conditions, facilities much this task. Recall, however, that these analytic solutions apply specifically to the case $\alpha_2 = 0$. The numerical solutions we seek are for the more general case $\alpha_2 \neq 0$. It is readily seen that when the initial separation of the two pulses (solitons) is large enough, the term proportional to α_2 multiplies functions that are very small. So there is no crime in using our coupled solution (7) as initial condition for $\alpha_2 \neq 0$ to determine the accuracy of our solver. Using the two-soliton solution of the Manakov system will produce a mismatch between the initial conditions and the equations themselves since Manakov's solution is valid only for $\Gamma = 0$.

Now we can proceed to discussing the core results of our paper and we begin with the elastic collision for $\Gamma = 0.0175$. The evolution is depicted in Fig. 3. Despite the oscillation due to the linear coupling, the final waveforms do not experience deformation after the collision.

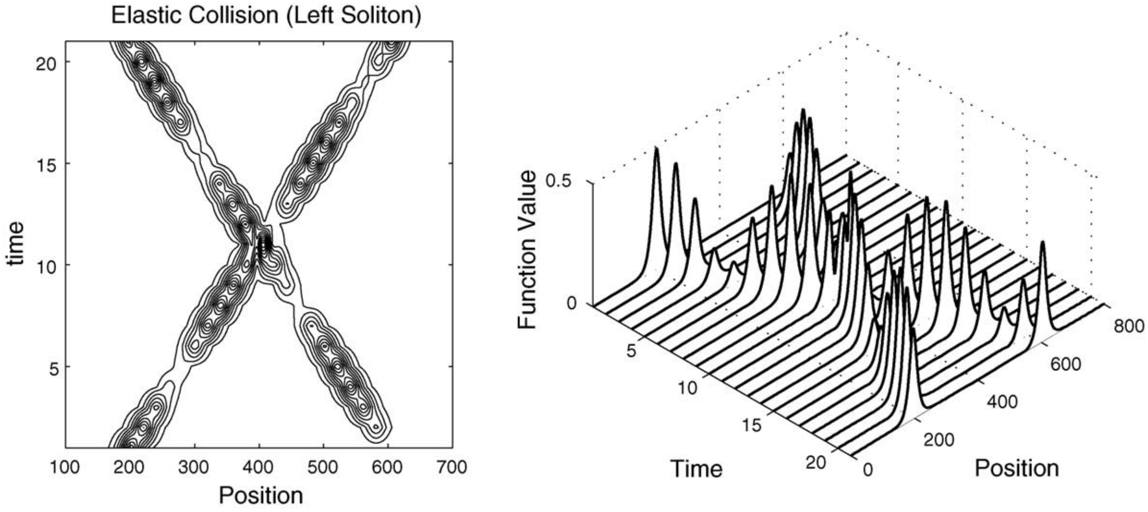

Fig. 3. Elastic collision with $\Gamma = 0.0175$.

There exists a class of soliton solutions called breathers, due to the oscillatory nature of their behavior. The solutions to (2) presented in Fig. 3 are therefore referred to as breather solutions, with one difference that here a solution oscillates, or breathes, due to the sharing of mass modes between the two wave functions.

An inelastic collision shown in Fig. 4. The waveforms in this case not only experience the oscillation from linear coupling (“cross” dispersion), but also a deformation typical for a system with coupling. After [19], we call the solution “inelastic collision”.

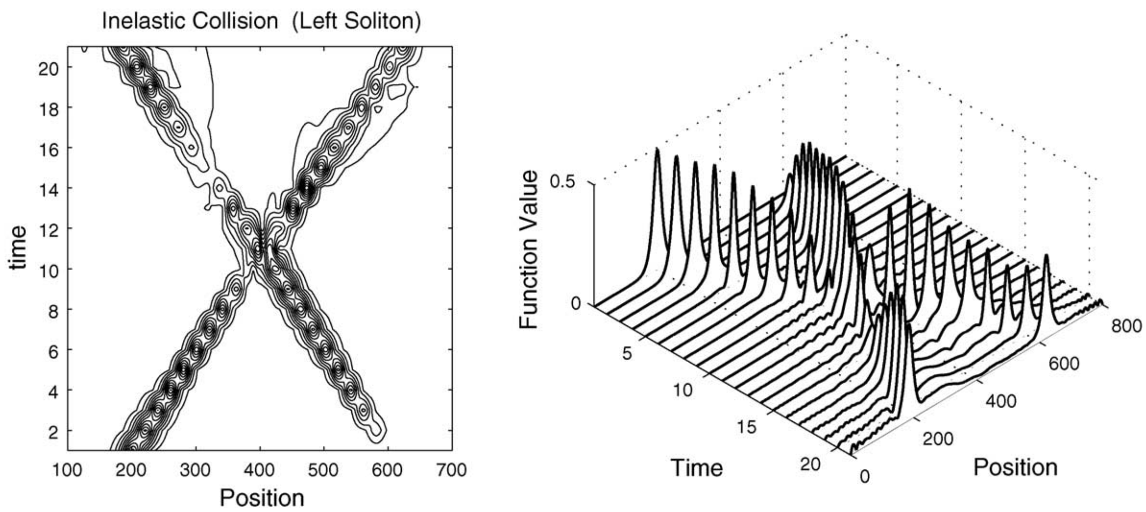

Fig. 4. Inelastic collision with $\Gamma = 0.01$.

6. Conclusions

In the present paper, we deal with system of two Schrödinger equations coupled through nonlinear and linear terms. We call the linear coupling "cross" dispersion. For the latter case, we obtain a new analytic solution when the nonlinear coupling is absent ($\alpha_2 = 0$). We use this solution as initial condition for the general case.

On the basis of a energy-conserving scheme, we have developed a numerical tool to address the most general case of coupling and implemented it in an interactive simulation environment which is user-friendly and provides immediate access to multi-parameter investigations of the system.

The soliton waveforms are seen to survive the collision undeformed despite the fact that the system cannot be proven to be full integrable. For $\alpha_2 \neq 0$ (cases with nonlinear coupling), the solitons experience excitation of tail-modes (inelastic collision) while the total energy is strictly conserved. Thus, our numerical results show that previously described elastic [1,10] and inelastic collisions [19] may be extended to include cross-dispersion. We obtain different characteristics for this case, including collision and phase shift. This extends the degree to which one may use the nonintegrable cases to predict the physical behavior of the systems under consideration.

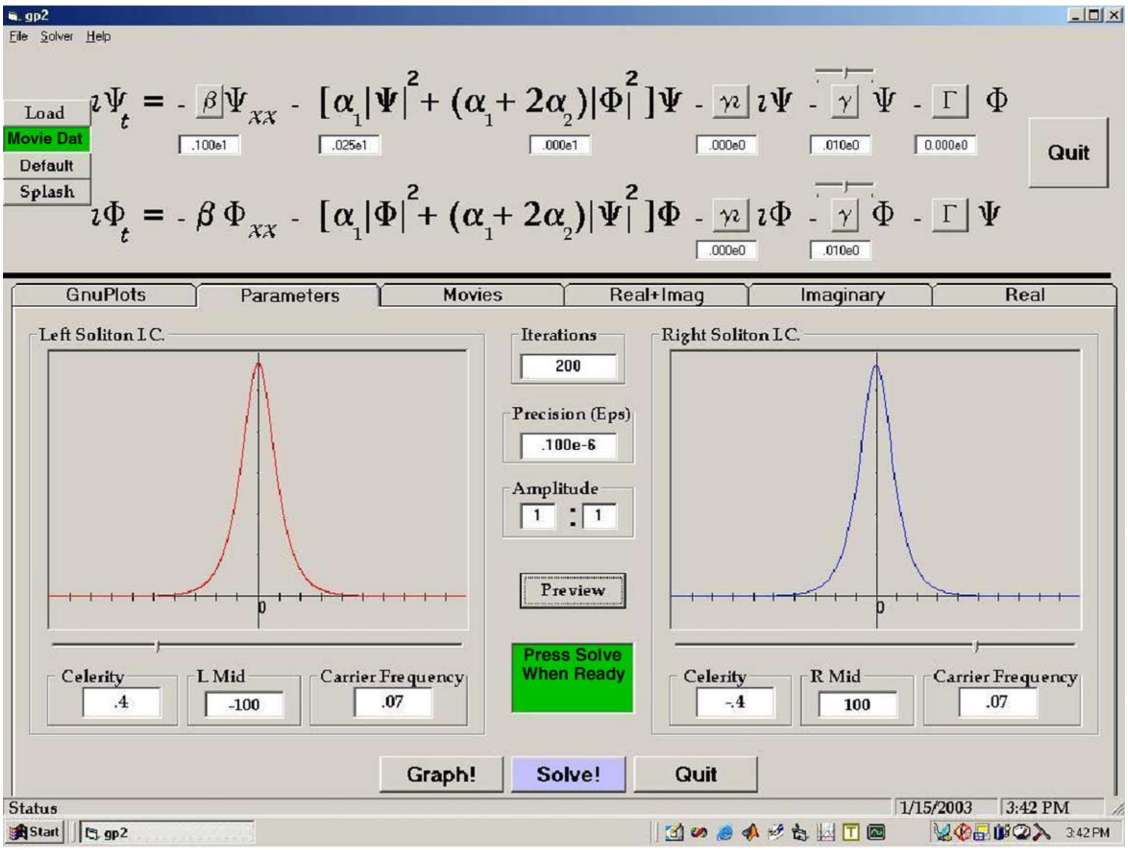

Fig. 5. Screen shot of simulation environment.

Appendix A. Simulation environment

We may implement the mathematical tools developed to solve the physical systems described into a rapid, robust, interactive simulation environment. The applications developed during this research are likewise industrially applicable. They also make the interrogation of a problem with vast, multi-parameter phase space easier.

Using the internal iterations of [19], we construct conservative numerical routines in the form of dynamic link libraries which are accessed by the front end at runtime. The Graphic User Interface is built using Visual Basic, and handles all the user data, preferences, static graphics and animations. The solutions may be viewed as simple magnitudes, separate real and imaginary parts, or in phase-space. The user is able to store specific case data externally and recall it via the GUI (Fig. 5). The speed of the numerics, coupled with the ease of the testing environment, makes investigation of the system's behaviors exceedingly convenient.

Although the numerics are tailored for this problem specifically, the techniques developed for the project's interactive nature will prove applicable far beyond the scope of the CNLSE. Given the great potential of the CNLSE for industrial applications, this sort of interactive environment seems a natural extension of the research. The application is graphic oriented, traces and analyzes various dynamic features of (2) and is imminently user friendly.

References

[1] S.V. Manakov, On the theory of two-dimensional stationary self-focusing of electromagnetic waves, Soviet Phys. JETP 38 (1974) 248–253.

[2] C.R. Menyuk, Pulse propogation in an ellipticaly birefringent Kerr medium, IEEE J. Quant. Electron. 25 (12) (1989) 267482.

[3] J. Hietarinta, A search for integrable two-dimensional Hamiltonian systems with polynomial potential, Phys. Lett. A 96 (1983) 273–278.

[4] Y. Chen, J. Atai, Polarization instabilities in birefringent fibers: a comparison between continuous waves and solitons, Phys. Rev. E 52 (September (3)) (1995) 3102–3105.

[5] H.G. Winful, Self-induced polarization changes in birefringent optical fibers, Appl. Phys. Lett. 47 (August (3)) (1985) 213–215.

[6] Y. Chen, Stability criterion of coupled soliton states, Phys Rev E 57 (March (3)) (1998) 3542–3550.

[7] E.P. Gross, Hydrodynamics of a superfluid condensate, J. Math. Phys. 4 (February (2)) (1963) 195–207.

[8] E.P. Gross, Structure of a quantized vortex in boson systems, Il Nuovo Cimiento 20 (1961) 454.

[9] L.P. Pitaevskii, Vortex lines in an imperfect bose gas, Soviet Phys. JETP 13 (August (2)) (1961) 451–454.

[10] V.E. Zakharov, A.B. Shabat, Exact theory of two-dimensional self focusing and one-dimensional self-modulation of waves in nonlinear media, Soviet Phys. JETP 34 (1972) 62–69.

[11] V.E. Zakharov, A.B. Shabat, Interaction between solitons in a stable medium, Soviet Phys. JETP 37 (1973) 823.

[12] K.W. Chow, K. Nakkeeran, B. Malomed, Periodic waves in bimodal optical fibers, Opt. Commun. 219 (March) (2003) 251–259.

[13] E.A. Orstrovskaya, N.N. Akhmediev, G.I. Stegeman, J.U. Kang, J.S. Aichison, Mixed-mode spatial solitons in emiconductor waveguides, J. Opt. Soc. Am. B 14 (1997) 880.

[14] K.J. Blow, N.J. Doran, D. Wood, Polarization instabilities for solitons in birefringent fibers, Opt. Lett. 12 (3) (1986) 202–204.

[15] N.N. Akhmediev, V. Buryak, J.M. Soto-Crespo, D.R. Anderson, Phase-locked stationary soltion states in birefringent nonlinear optical fibers, J. Opt. Soc. Am. B 12 (March (3)) (1995) 434–439.

[16] A.W. Snyder, Y. Chen, D. Rowland, D.J. Mitchell, Unification of nonlinear-optical fiber devices, Opt. Lett. 15 (February (3)) (1990) 171–173.
[17] S.M. Jensen, The nonlinear coherent coupler, IEEE J. Quant. Electron. 18 (1982) 1580.
[18] E.I. Shulman, V.E. Zakharov, To the integrability of the system of two coupled nonlinear Schrödinger equations, Phys. D 4 (1982) 270.
[19] C.I. Christov, S. Dost, G.A. Maugin, Inelasticity of soliton collisions in systems of coupled NLS equations, Phys. Scr. 50 (1994) 449–454.
[20] B.A. Malomed, S. Kivshar, Dynamics of solitons in nearly ingrable systems, Rev. Mod. Phys. 61 (1989) 763.
[21] T.R. Taha, M.J. Ablovitz, Analytical and numerical aspects of certain nonlinear evolution equations. II. Numerical, Schroödinger equation, J. Comput. Phys. 55 (1984) 203–230.
[22] M.S. Ismail, T.R. Taha, Numerical simulation of coupled nonlinear Schroödinger equation, Math. Comput. Simul. 56 (2001) 547–562.
[23] C.I. Christov, An energy-consistent dispersive shallow-water model, Wave Motion 34 (2001) 161–174.