

---

## A multiunit ADI scheme for biharmonic equation with accelerated convergence

---

X-H. Tang and C.I. Christov\*

Department of Mathematics,  
University of Louisiana,  
Lafayette, LA 70504-1010, USA

E-mail: xxt6793@louisiana.edu      E-mail: christov@louisiana.edu

\*Corresponding author

**Abstract:** We consider the problem of acceleration of the Alternative Directions Implicit (ADI) scheme for Dirichlet problem for biharmonic equation. The second Douglas scheme is used as the main vehicle and two full time steps are organised in a single iteration unit in which the explicit operators are arranged differently for the second step. Using an *a priori* estimate for the spectral radius of the operator, we show that there exists an optimal value for the acceleration parameter. An algorithm is devised implementing the scheme and the optimal range of the parameter is verified through numerical experiments. One iteration unit speeds up the convergence from two to three times in comparison with the standard ADI scheme. To obtain more significant acceleration for the cases when the standard ADI scheme has extremely slow convergence, a generalised multiunit scheme is constructed by introducing another acceleration parameter and treating two consecutive iteration units as one basic element.

**Keywords:** accelerated convergence; biharmonic equation; difference schemes; Dirichlet problem; operator splitting.

**Reference** to this paper should be made as follows: Tang, X-H. and Christov, C.I. (2007) 'A multiunit ADI scheme for biharmonic equation with accelerated convergence', *Int. J. Computational Science and Engineering*, Vol. 3, No. 4, pp.295–304.

**Biographical notes:** Xioahua Tang is a Graduate Student in the Department of Mathematics at the University of Louisiana at Lafayette, USA.

Christo I. Christov is a Professor in the Department of Mathematics at the University of Louisiana at Lafayette, USA.

---

### 1 Introduction

Biharmonic boundary value problems arise in many different areas of mechanics of continua. The best known are the stream function formulation for stationary Navier-Stokes equations for flows of viscous liquids and equations for deformation of elastic plates. Constructing efficient numerical algorithms is of prime importance in these cases. A well established approach to the problem is to introduce an artificial time in the elliptic equation and to use operator splitting technique for the resulting parabolic equation. The technique is summarily known as Alternative Directions Implicit (ADI) method. To the authors' knowledge, the first work in which an ADI scheme was applied to biharmonic equation is Conte and Dames (1960) where the so-called second Douglas scheme (1955) for Laplace operator is reformulated for the biharmonic case. The scheme of Conte and Dames (CD) exhibits the best of the world of ADI schemes, as being absolutely stable and low cost per iteration, but its rate of convergence has been shown to be rather slow in some cases (see, Greenspan and Schultz, 1972; Ehrlich and Gupta, 1975). In order to accelerate the convergence, we consider an iteration unit consisting of a

CD scheme and a modified scheme, the latter dependent upon a parameter. The two schemes damp different parts of the spectrum differently and in the present we show paper that the combination of them yields a faster convergence than the original ingredients. This idea is generalised in the second part of this paper. Two consecutive iteration units are treated as one operator, which has all necessary properties for us to arrange another iteration unit consisted of four previous iteration units. Eventually, we show that more significant acceleration can be obtained by this generalised multiunit scheme in the cases when CD scheme has extremely slow convergence.

### 2 Conte–Dames ADI Scheme

One of the ways to construct an iterative scheme for the biharmonic equation is to consider the following two-dimensional higher-order parabolic equation for the Dirichlet boundary value problem

$$\frac{\partial u}{\partial t} = -\Delta^2 u(x, y) + F(x, y), \quad (x, y) \in D; \quad (1)$$

$$u(x, y) = f(x, y), \quad (x, y) \in \partial D; \tag{2}$$

$$\begin{aligned} u_x(0, y) &= g_1(y), & \text{for } x = 0 \text{ and } 0 < y < 1; \\ u_x(1, y) &= g_2(y), & \text{for } x = 1 \text{ and } 0 < y < 1; \\ u_y(x, 0) &= g_3(x), & \text{for } y = 0 \text{ and } 0 < x < 1; \\ u_y(x, 1) &= g_4(x), & \text{for } y = 1 \text{ and } 0 < x < 1; \end{aligned} \tag{3}$$

where  $D$  is a square region  $\{(x, y) | 0 < x < 1, 0 < y < 1\}$  and  $\bar{D} = D \cup \partial D$  is its closure.  $\Delta^2$  is the biharmonic operator

$$\Delta^2 u = \frac{\partial^4 u}{\partial x^4} + 2 \frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4}. \tag{4}$$

In order to obtain second order of approximation of the difference scheme, we assume that function  $u$  possesses derivatives up to sixth order.

For the purpose of the present work, we employ an uniformly spaced mesh in  $\bar{D}$  with spacings  $h_x = h_y = h = 1/N$ , where  $N$  is the grid size. Then, we replace  $D$  and  $\partial D$  by sets of grid points  $D_h$  and  $\partial D_h$  where  $D_h = \{(ih, jh) | i = 1, 2, \dots, N-1; j = 1, 2, \dots, N-1\}$  is the set of interior grid points. Respectively,  $u_{i,j}^n$  is the difference approximation to  $u$  at the grid point  $x = ih$ ,  $y = jh$  and time stage  $t = n\tau$ , where  $\tau$  is the time increment. To obtain a second-order approximation, the usual central difference operators  $\delta_x^4$ ,  $\delta_y^4$  and  $\delta_x^2 \delta_y^2$  are implemented, namely,

$$\begin{aligned} \delta_x^4 u_{i,j} &= \frac{1}{h^4} (u_{i-2,j} - 4u_{i-1,j} + 6u_{i,j} - 4u_{i+1,j} + u_{i+2,j}), \\ \delta_y^4 u_{i,j} &= \frac{1}{h^4} (u_{i,j-2} - 4u_{i,j-1} + 6u_{i,j} - 4u_{i,j+1} + u_{i,j+2}), \\ \delta_x^2 \delta_y^2 u_{i,j} &= \frac{1}{h^4} (u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} - 2u_{i-1,j} \\ &\quad + 4u_{i,j} - 2u_{i+1,j} + u_{i-1,j-1} - 2u_{i,j-1} + u_{i+1,j-1}). \end{aligned}$$

Now, if an explicit scheme is used for solving the boundary value problem Equation (1), then the stability condition limits the time step at  $\frac{1}{2}h^4$  which makes it practically unusable. In the case under consideration, it is even more important to construct unconditionally stable implicit scheme, than it is for the case of a second-order in space parabolic equation. The most viable approach in this situation is to generalise to the case one of the operator splitting scheme, such as the Douglas scheme. It was done first by Conte and Dames (1960) and we call their scheme in what follows ‘CD’. Just as the Douglas scheme (1955) does, the CD scheme consists of two sweeps in  $x$ - and  $y$ -directions over the interior grid points  $D_h$  and they are given by

$$\tilde{u}_{i,j} = u_{i,j}^n - \tau(\delta_x^4 \tilde{u} + 2\delta_x^2 \delta_y^2 u_{i,j}^n + \delta_y^4 u_{i,j}^n - F_{i,j}), \tag{5}$$

$$u_{i,j}^{n+1} = \tilde{u}_{i,j} - \tau(\delta_y^4 u_{i,j}^{n+1} - \delta_y^4 u_{i,j}^n), \tag{6}$$

where  $u_{i,j}^0 = 0$  is an (arbitrary) initial condition and the time increment  $\tau$  plays the role of an iteration parameter and can be chosen to accelerate convergence.

A note is in order here about the difference between the CD scheme and the Douglas scheme for second order equations. Due to the presence of the mixed fourth derivative in Equation (1), the CD scheme will always contain an explicit element because the operator of the mixed derivative cannot be effectively split. Yet, as shown in Conte and Dames (1960), the scheme is still unconditionally stable regardless of the explicit element present in it. In a sense, the reduction in the norm provided by the inversion of the fourth order operators is so large that it overcomes the possible increase due to the explicit operators. This is a good news or the generalisation of the scheme for anisotropic case, when the fourth order operators contain coefficients that are functions of the spatial variables. In such a case, more mixed derivative will appear in the right hand side of Equation (1). The recipe is to keep the additional mixed derivative on the old time stage.

All this makes the operator-splitting scheme an important tool for solving biharmonic problems and warrants a further work on speeding up its convergence. To this task is devoted the present paper.

Now, the second-order approximations for the boundary conditions read

$$\begin{aligned} \tilde{u}_{i,j} &= u_{i,j}^{n+1} = f_{i,j}, \quad (ih, jh) \in \partial D_h; \\ \tilde{u}_{-1,j} &= \tilde{u}_{1,j} - 2hg_1(jh), \\ \tilde{u}_{N+1,j} &= \tilde{u}_{N-1,j} + 2hg_2(jh), \\ u_{i,-1}^{n+1} &= u_{i,1}^{n+1} - 2hg_3(ih), \\ u_{i,N+1}^{n+1} &= u_{i,N-1}^{n+1} + 2hg_4(ih), \end{aligned} \tag{7}$$

for  $i = 1, 2, \dots, N-1$  and  $j = 1, 2, \dots, N-1$ . The boundary conditions for the ‘‘half-time-step’’ variable  $\tilde{u}$  can be obtained from the approximations for the boundary condition (7) by reversing the second half-time step at the boundary (see Yanenko, 1971; Strikwerda, 1989)

$$\tilde{u} = u^{n+1} + \tau \delta_y^4 (u^{n+1} - u^n). \tag{8}$$

Since we are solving a stationary problem, the boundary settings for  $u^n$  are invariant in the iterative process, that it is,

$$\begin{aligned} u_{i,j}^n &= u_{i,j}^{n+1} = f_{i,j}, \quad (ih, jh) \in \partial D_h; \\ u_{1,j}^{n+1} - u_{-1,j}^{n+1} &= u_{1,j}^n - u_{-1,j}^n = 2hg_1(jh), \\ u_{N+1,j}^{n+1} - u_{N-1,j}^{n+1} &= u_{N+1,j}^n - u_{N-1,j}^n = 2hg_2(jh) \end{aligned} \tag{9}$$

for  $j = 1, 2, \dots, N-1$ ; which are substituted in Equation (8) to derive the  $x$ -direction of Equation (7).

If the intermediate variable  $\tilde{u}_{i,j}$  is eliminated between the Equations (5) and (6) become

$$\begin{aligned} (E + \tau \delta_x^4 + \tau \delta_y^4 + \tau^2 \delta_x^2 \delta_y^2) u^{n+1} \\ = (E - 2\tau \delta_x^2 \delta_y^2 + \tau^2 \delta_x^4 \delta_y^4) u^n + \tau F, \end{aligned} \tag{10}$$

where  $F$  stands for the grid function of  $F(x, y)$  on  $D_h$ ,  $E$  denotes the identity matrix and the difference operators are considered as the corresponding matrices for grid function  $u^n$ . The subscripts  $i, j$  are omitted for brevity of notation. The transition matrix  $T$  from one time step to another is

$$T = (E + \tau\delta_x^4 + \tau\delta_y^4 + \tau^2\delta_x^4\delta_y^4)^{-1}(E - 2\tau\delta_x^2\delta_y^2 + \tau^2\delta_x^4\delta_y^4). \quad (11)$$

The convergence of CD scheme for arbitrary positive iteration parameter  $\tau$  is shown in Conte and Dames (1960) by demonstrating that  $\|T\| < 1$  for any  $\tau > 0$  for the case when the second-order difference approximations for the Dirichlet boundary conditions are implemented. However, the dependence of spectral radius of  $T$  on the iteration parameter  $\tau$  has not been investigated so far, because of the difficulty in obtaining appropriate eigenvectors of the difference operator for biharmonic problem with Dirichlet boundary conditions. If an arbitrary  $\tau$  is chosen, the convergence rate of CD scheme can be quite slow (see discussions in Greenspan and Schultz, 1972; Ehrlich and Gupta, 1975).

Therefore we have three essential objectives to achieve in this work:

- 1 to reformulate the CD scheme in a manner that allows one to accelerate its convergence rate depending upon an iteration parameter;
- 2 to prove that acceleration is possible and to find an estimate for the iteration parameter introduced;
- 3 to find the optimal choice of the iteration parameter through numerical experiment.

Suppose that  $w$  is the grid function that is the solution of the stationary difference problem with Dirichlet boundary conditions, namely,

$$\Delta_h^2 w = (\delta_x^4 + \delta_y^4 + 2\delta_x^2\delta_y^2)w = F \quad \text{on } D_h \quad (12)$$

$$w_{i,j}^n = f_{i,j} \quad \text{for } (ih, jh) \in \partial D_h, \quad (13)$$

$$w_{-1,j} = w_{1,j} - 2hg_1(jh), \quad (14)$$

$$w_{N+1,j} = w_{N-1,j} + 2hg_2(jh),$$

$$w_{i,-1} = w_{i,1} - 2hg_3(ih), \quad (15)$$

$$w_{i,N+1} = w_{i,N-1} + 2hg_4(ih),$$

for  $i = 1, 2, \dots, N-1$  and  $j = 1, 2, \dots, N-1$ ; where  $\Delta_h^2$  denotes the matrix for 13-point stencil approximation of the biharmonic operator.

Define the error vector  $\xi^n$  for the  $n$ -th iteration as

$$\xi^n = u^n - w. \quad (16)$$

By Equation (10) and boundary condition (7), we obtain the system of equations with homogeneous boundary conditions for  $\xi^n$

$$\xi^{n+1} = T\xi^n \quad \text{on } D_h, \quad (17)$$

$$\xi^{n+1} = 0 \quad \text{on } \partial D_h, \quad (18)$$

$$\xi_{-1,j}^{n+1} = \xi_{1,j}^{n+1} \quad \xi_{N+1,j}^{n+1} = \xi_{N-1,j}^{n+1}; \quad (19)$$

$$\xi_{i,-1}^{n+1} = \xi_{i,1}^{n+1} \quad \xi_{i,N+1}^{n+1} = \xi_{i,N-1}^{n+1}. \quad (20)$$

Using Courant's theorem, Bodewig (1956) and Conte and Dames (1960) showed that for the boundary conditions (18)–(20) the transition matrix  $T$  has a complete set of eigenvectors  $v_k$  for the vector space  $\Phi$  defined on  $D_h$  and the corresponding eigenvalues are  $0 < \alpha(T) = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_{m-1} \leq \lambda_m = \beta(T) < 1$ . This is the only *a priori* information about transition matrix  $T$  we need in order to be able to improve the original CD method. For convenience, we denote the matrices involved in the scheme as

$$B_x = E + \tau\delta_x^4,$$

$$B_y = E + \tau\delta_y^4,$$

$$R = E - 2\tau\delta_x^2\delta_y^2 + \tau^2\delta_x^4\delta_y^4,$$

Then  $T$  can be written as  $T = B_y^{-1}B_x^{-1}R$ .

### 3 The modified splitting scheme

In order to accelerate the convergence of the iterations for a given time increment  $\tau$ , we try to find a modification of the CD scheme in which the transition operator has smaller norm than the original scheme.

The gist of present paper is that we introduce an iteration unit consisting of two CD iterations with different arrangements of the explicit terms by means of an auxiliary parameter  $\theta$  as follows for the  $(n + 1)$ th unit

$$B_x B_y u^{2n+1} = R u^{2n} + \tau F \quad (a) \quad (21)$$

$$B_x B_y u^{2n+2} = R[(\theta + 1)u^{2n+1} - \theta u^{2n}] + \tau F \quad (b)$$

where the intermediate variable  $\tilde{u}$  is eliminated and  $(\theta + 1)u^{2n+1} - \theta u^{2n}$  is considered as the input vector to compute  $u^{2n+2}$  using CD scheme in (b). The boundary conditions are the same for the two steps and are omitted for the sake of brevity. Clearly, there is no extra cost for implementing the new scheme.

To study the convergence of the iteration units we observe that by definition, Equation (16), and some standard algebraic manipulations, one can recast the equations of the full iteration unit, Equation (21), as equations for the error  $\xi^n$ , namely,

$$\xi^{2n+1} = T\xi^{2n}, \quad \xi^{2n+2} = T((\theta + 1)\xi^{2n+1} - \theta\xi^{2n}), \quad (22)$$

and hence

$$\xi^{2n+2} = ((\theta + 1)T^2 - \theta T)\xi^{2n}. \quad (23)$$

The iteration units are convergent for  $0 < \theta < 2 + 2\sqrt{2}$ . Begin with a trivial initial condition  $u^0 = 0$ , which means

that the initial condition for the error is  $\xi^0 = -w$ . The Fourier expansion of  $\xi$  with respect to the complete set of eigenvectors  $\{v_k\}$  of  $T$  reads

$$\xi^0 = -w = \sum_{k=1}^m c_k v_k, \tag{24}$$

where  $C_k$  is the corresponding Fourier coefficient. Then by Equations (22) and (23), we represent  $\xi^n$  as

$$\xi^n = \begin{cases} \sum_{k=1}^m \lambda_k \rho_\theta^l(\lambda_k) c_k v_k & \text{if } n = 2l + 1 \\ \sum_{k=1}^m \rho_\theta^l(\lambda_k) c_k v_k & \text{if } n = 2l \end{cases} \tag{25}$$

for  $l = 0, 1, 2, \dots$ ; where we define the quadratic function  $\rho_\theta(\lambda_k) = (1 + \theta)\lambda_k^2 - \theta\lambda_k$  and  $\lambda_k$  is the corresponding eigenvalue for  $T$ . In order to show the convergence of the iteration units, it is sufficient for us to show that for each  $\lambda_k, |\rho_\theta(\lambda_k)| < 1$ , since  $0 < \lambda_k < 1$ . Indeed, we have

$$\max_{\lambda_k} |\rho_\theta(\lambda_k)| \leq \max_{0 < x < 1} |\rho_\theta(x)| \tag{26}$$

$$\leq \max \left\{ \left| \rho_\theta(0) \right|, \left| \rho_\theta(1) \right|, \left| \rho_\theta \left( \frac{\theta}{2(\theta+1)} \right) \right| \right\},$$

where we consider only  $\theta > 0$  because only positive choice of  $\theta$  can accelerate the iterations. Since  $\rho_\theta(0) = 0$  and  $\rho_\theta(1) = 1$ , Equation (26) will hold when

$$\left| \rho_\theta \left( \frac{\theta}{2(\theta+1)} \right) \right| = \frac{\theta^2}{4(\theta+1)} < 1, \tag{27}$$

whence it follows that  $0 < \theta < 2 + 2\sqrt{2}$  must hold in order to ensure that we have  $\max_{\lambda_k} |\rho_\theta(\lambda_k)| < 1$  which is needed for the convergence of the iteration unit Equation (21).

Now, we analyse the acceleration of the convergence for the iteration units. Consider the error  $\xi_{cd}^n$  for the CD scheme after  $n$  iterations. Using Equations (17) and (24), we can represent  $\xi_{cd}^n$  in terms of Fourier expansion with respect to the eigenvectors  $v_k$  of the transition matrix  $T$ , namely,

$$\xi_{cd}^n = \sum_{k=1}^m \lambda_k^n (c_k v_k). \tag{28}$$

The coefficients of  $v_k$  in the error vector  $\xi_{cd}^n$  decrease in absolute value by the multiplicative factor of  $\lambda_k$ . The least affected is the coefficient of  $v_m$ , which corresponds to the largest eigenvalue  $\lambda_m$ . Repeating CD iterations leads us to the asymptotic case, because for  $n \gg 1$  all other coefficients become negligibly small compared to the coefficient of  $v_m$ , and hence we have the error for CD scheme given by

$$\xi_{cd}^n = \lambda_m \xi_{cd}^{n-1} = \beta(T) \xi_{cd}^{n-1}. \tag{29}$$

From Equation (29) follows for the standard norm  $\|\xi_{cd}^n\|^2 = (\xi_{cd}^n, \xi_{cd}^n)$  that

$$\|\xi_{cd}^n\| = \beta(T) \|\xi_{cd}^{n-1}\|. \tag{30}$$

Although for small  $n$  the amplifier of the norm depends upon the iteration, the performance of the iterative process is usually judged by the asymptotic rate of convergence  $s$  ( $n \gg 1$ ) which is defined as

$$s = -\ln |\beta(T)| = -\ln \frac{\|\xi_{cd}^n\|}{\|\xi_{cd}^{n-1}\|}. \tag{31}$$

By the recursive relation, Equation (30), we obtain the asymptotic equation

$$\|\xi_{cd}^n\| = e^{n \ln \beta(T)} \|\xi_{cd}^0\| = e^{-ns} \|\xi_{cd}^0\|. \tag{32}$$

In this way, the asymptotic rate of convergence  $s$  characterises the rate of the exponential error decrease. The cause of slow convergence of CD scheme is that  $\beta(T)$  is close to unity. In such a case, we use the notation  $\beta = 1 - p$  where  $0 < p \ll 1$ . Using the Taylor expansion, the asymptotic rate of convergence  $s$  of CD scheme is given by

$$s = -\ln(1 - p) \approx p \ll 1. \tag{33}$$

In the same manner, we investigate the asymptotic rate of convergence  $s$  of our iteration units. In order not to obscure the main idea, we limit the discussion here to some typical values of  $\theta$ , as  $\theta = 3$ , which is in the range  $0 < \theta < 2 + 2\sqrt{2}$ . The coefficient of eigenvector  $v_k$  in Equation (25) decreases in absolute value by the multiplicative factor of  $|\rho_3(\lambda_k)|$ . Since,  $\rho_3(x) = 4x^2 - 3x$  is a quadratic function, it is easy to show that its minimum is at  $x = 3/8$  and has the magnitude of  $-9/16$ . Then

$$\max_{0 < \lambda_k \leq 1-p} |\rho_3(\lambda_k)| \leq \max \left\{ \frac{9}{16}, |\rho_3(1-p)| \right\}, \tag{34}$$

which means that if  $9/16 > |\rho_3(1-p)|$ , we have already obtained a very fast convergence of the iteration units which will reduce the error to  $10^{-5}$  within 20 iteration units.

On the other hand, for  $p \ll 1$  we have  $|\rho_3(1-p)| = |(1-p)(1-4p)| \approx (1-5p) > 9/16$  and using Equation (23) asymptotically for  $n \gg 1$ , we can write

$$\|\xi^{2n+2}\| \approx (1-5p) \|\xi^{2n}\|. \tag{35}$$

Therefore, the corresponding asymptotic rate of convergence of our iteration units is

$$s = -\frac{1}{2} \ln(1-5p) \approx 2.5p, \tag{36}$$

where we compare one iteration unit consisted of two iterations with two original CD iterations for which the reduction factor would be  $(1-p)^2 \approx 1-2p$ . All this means that the introduction of the iteration unit can speed up the convergence rate at least 2.5 times. Note that the actual

factor of acceleration depends upon the value of time increment  $\tau$  and is discussed in Section 4. Similarly, we can verify that for  $0 < \theta < 2 + \sqrt{2}$ , the asymptotic rate of convergence is

$$\begin{aligned} s &= -\frac{1}{2} \ln \rho_\theta(1-p) \\ &= -\frac{1}{2} \ln \{(1-p)[(\theta+1)(1-p)-\theta]\} \\ &\approx \frac{1}{2}(\theta+2)p. \end{aligned} \quad (37)$$

But, this conclusion depends upon the assumption that  $1 - \beta(T) = p \ll 1$  and

$$\left| \rho_\theta \left( \frac{\theta}{2(1+\theta)} \right) \right| = \frac{\theta^2}{4(1+\theta)} \leq \rho_\theta(1-p). \quad (38)$$

Actually, the maximum eigenvalue  $\beta(T)$  of the transition matrix can be estimated in the numerical experiment, based on which we can choose a proper value of the auxiliary parameter  $\theta$  to maximise the acceleration of the devised iteration units. By Equations (10) and (11), we have the recursive relation

$$(u^{n+1} - u^n) = T(u^n - u^{n-1}). \quad (39)$$

Since  $\|T\| = \beta(T) < 1$ , after a few CD iterations the largest eigenvalue  $\beta(T)$  becomes the dominant multiplicative factor in the iterations while other smaller eigenvalues become negligible. Therefore, we can compute the numerical quantity

$$q = \frac{\|u^{n+1} - u^n\|}{\|u^n - u^{n-1}\|} \quad (40)$$

in each CD iteration and when  $q$  varies little between two consecutive CD iterations, an estimate of  $\beta(T)$  is obtained. Based on the *a posteriori* numerical estimate  $q \approx \beta(T)$ , we can determine the optimal choice of  $\theta$  to maximise the acceleration. At first, we notice that  $\rho_\theta(q)$  is a linear function of  $\theta$  when  $q$  is fixed

$$h_1(\theta) = \rho_\theta(q) = (q^2 - q)\theta + q^2. \quad (41)$$

since  $q^2 - q < 0$ ,  $h_1(\theta)$  is a monotone decreasing function. Next, we consider another function  $h_2$  of  $\theta$

$$h_2(\theta) = \left| \rho_\theta \left( \frac{\theta}{2(1+\theta)} \right) \right| = \frac{\theta^2}{4(1+\theta)}, \quad (42)$$

which is a monotone increasing function on the interval  $(0, 2\sqrt{2})$ , since for  $\theta \in (0, 2\sqrt{2})$  we have

$$h_2'(\theta) = \frac{\theta^2 + 2\theta}{4(\theta+1)} > 0. \quad (43)$$

To maximise the acceleration, by Equation (26) we need to find  $\theta$  from

$$\min_{0 < \theta < 2 + \sqrt{2}} \{ \max[|h_1(\theta)|, h_2(\theta)] \}. \quad (44)$$

By the monotonicity of  $h_1(\theta)$  and  $h_2(\theta)$ , it is easy to verify that the optimal  $\theta$  must be the positive solution of the following equation and automatically  $\theta \in (0, 2 + 2\sqrt{2})$ ,

$$(q^2 - q)\theta + q^2 = \frac{\theta^2}{4(1+\theta)}. \quad (45)$$

Therefore, we obtain the optimal  $\theta$  by solving (45)

$$\theta_{\text{opt}}(q) = \frac{2q^2 + (\sqrt{2} - 1)q}{-2q^2 + 2q + \frac{1}{2}}. \quad (46)$$

Since  $h_1(\theta) = q^2$  and  $h_1(\theta)$  is a decreasing function, then we have

$$0 < h_1(\theta_{\text{opt}}) = \frac{\theta_{\text{opt}}^2}{4(1+\theta_{\text{opt}})} < q^2 \quad (47)$$

where  $q^2$  stands for the convergent effect of two CD iterations. Hence, we have shown that for different choices of the iteration parameter  $\tau$  in the CD method, which give us different transition matrices  $T$  (i.e. different  $q$ ), we are always able to select a  $\theta_{\text{opt}}$  to accelerate the CD scheme.

By Equations (37) and (46), we can see that when  $\beta(T)$  is closer to unity, that is to say, the original CD method has a slower convergence, larger auxiliary parameter  $\theta$  can be chosen in the iteration units which leads us to a more significant acceleration over the CD method. Since the value of  $\theta$  is bounded by  $2 + 2\sqrt{2} \approx 4.8$ , by Equation (37) the best acceleration that our method can reach is 3.4 times faster than CD method.

In implementation of the algorithm, we follow Christov and Ridha (1994) where a splitting scheme of type of CD was applied to lid-driven cavity flow of viscous liquid. Later on a similar algorithm was used in Christov and Pontes (2002) for another kind of higher-order diffusion equation.

## 4 Results for the 2-unit scheme

We begin with numerical verification of the performance of a unit consisting of two iterations as introduced above. We call it '2-unit iteration'. It is possible to construct 8- and 32-unit iterations with increased rate of convergence, which will be discussed in a later section.

In constructing the test cases, we first selected a possible solution of the non-homogeneous biharmonic equation which satisfies the boundary conditions. Then, we introduced the solution-to-be in the biharmonic equation and found what was the non-homogeneous term that had produced the chosen solution. The first test case is given by

$$\hat{u}(x, y) = \sin^2(\pi x) \sinh^2 y, \quad (48)$$

$$f(x, y) = 8(\sinh^2 y + \cosh^2 y)[\pi^2 \cos(2\pi x) + \sin^2(\pi x)] - 8\pi^4 \cos(2\pi x) \sinh^2 y, \quad (49)$$

where  $\hat{u}$  is the solution, and  $f$  is the right hand side of the biharmonic equation. As a second test case, we use the analytical solution form Arad, Yakhot and BenDor (1996), namely,

$$\hat{u}(x, y) = 2350x^4(x-1)^2y^4(y-1)^2, \tag{50}$$

$$f(x, y) = 56,400(1-10x+15x^2)(1-y)^2y^4 + 18,800x^2(6-20x+15x^2)y^2(6-20y+15y^2) + 56,400(1-x)^2x^4(1-10y+15y^2). \tag{51}$$

The operator to be inverted is the same in both cases and does not depend upon the actual solution, and hence on the right hand side. Yet, the convergence rate does depend upon the specific solution because of the fact that different eigenfunctions decay differently with the iterations. Since different analytical solutions have different content of eigenfunctions, the convergence rate can vary significantly from case to case. This is the rationale to check the performance for two radically different analytical solutions.

First of all, we demonstrate the second order approximation of the scheme. It is relevant to do that, because one needs to be assured that an iteration unit does give the same truncation error as a single CD iteration. For this particular test, we fix  $\theta = 1, 2$  and  $\tau = \frac{1}{4}h$  and calculate the solution with three different grids. Here, we fix the value of  $\theta$  and vary the number of grid points both along  $x$  and  $y$  directions simultaneously. We define the computed order of approximation as

$$R = \log_2 \frac{\|u_N - \hat{u}\|}{\|u_{2N} - \hat{u}\|} \tag{52}.$$

For schemes whose theoretical truncation error is proportional to  $h^2$  the value of  $R=2$ . Table 1 shows the results for  $R$ , as calculated from the obtained numerical solutions for the two test cases. It is seen that in both cases, the computed convergence rate is very close to two. Therefore, our method is indeed second-order accurate for biharmonic equation.

**Table 1**  $L_\infty$  norms of the difference between numerical and analytical solutions and computed order of approximation  $R$  for different grids

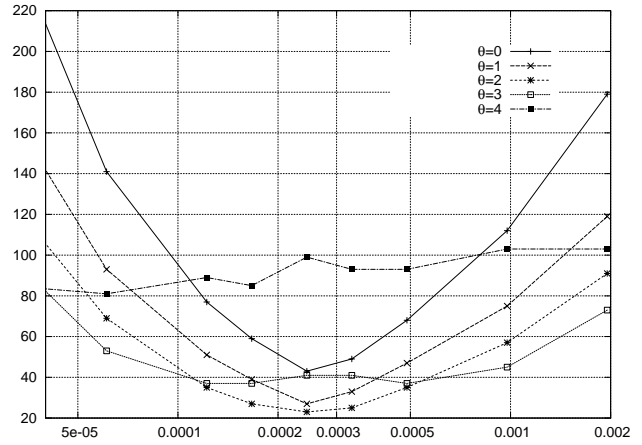
$N$	$\theta = 1$	$R$	$\theta = 2$	$R$
<i>case Equation (48)</i>				
256	$1.578 \times 10^{-5}$	–	$1.582 \times 10^{-5}$	–
512	$3.797 \times 10^{-6}$	2.05	$3.810 \times 10^{-6}$	2.05
1024	$9.687 \times 10^{-7}$	1.97	$9.747 \times 10^{-7}$	1.967
<i>case Equation (50)</i>				
256	$3.093 \times 10^{-4}$	–	$3.080 \times 10^{-4}$	–
512	$7.981 \times 10^{-5}$	1.95	$8.063 \times 10^{-5}$	1.93
1024	$1.823 \times 10^{-5}$	2.13	$1.884 \times 10^{-5}$	2.09

Having confirmed the second-order accuracy of the scheme, we can address the issue of computational efficiency. The pertinent parameter here is the number of iterations, say

$N_{\text{iter}}$ , needed to reduce the norm of the difference between two iterations to  $10^{-6}$ .

Clearly, the rate of convergence is a function of the time increment,  $\tau$ , and the optimisation parameter  $\theta$ . For different values of  $\theta$ , we have performed calculations with several different  $\tau$ . In Figure 1, we present the number of iterations  $N_{\text{iter}}$  needed for the norm between two iterations to go down to  $10^{-6}$  for the solution Equation (50). Note that  $\theta=0$  corresponds to the original CD scheme.

**Figure 1** Number of iterations, as function of  $\tau$  for different Values of  $\theta$  and grid size  $1024 \times 1024$



For  $\theta \leq 2$  there is a well pronounced minimum of the number of iterations for original CD scheme, as well as the scheme with one iteration unit developed here. In this range of  $\theta$ , our number of iterations is consistently smaller than CD scheme, the least ratio being two, i.e. our scheme is at least twice as fast as CD scheme. In the range of non-optimal  $\tau$ , it overperforms CD scheme with an even larger ratio. The result in Figure 1 is in a very good agreement with the theoretical estimate Equation (46) which gives us  $\theta_{\text{optimal}} \approx 1.8$  when the *a posteriori* estimate for  $q$  is approximately 0.77, which is the fastest one can get for a splitting scheme of the type of CD by choosing iteration parameter  $\tau = h/4$ .

It is interesting to mention here the non-monotone behaviour of  $N_{\text{iter}}$  with the increase of  $\theta$ , which means that there is an optimum for  $\theta$ , but only in the vicinity of optimal  $\tau$ . When  $\tau$  cannot be *a priori* chosen to be the optimal one, then the scheme proposed here will be better than CD scheme even for a wider range of  $\theta$ .

It is to be mentioned here that we have a similar result for grid size  $512 \times 512$ , for which the values of  $N_{\text{iter}}$  consistently lower by 10% from the presented case. This is completely natural for iterative algorithms because the eigenvalues of the difference operators depend upon  $h$ .

### 5 More complex iteration units

In Section 4, we discussed the acceleration of the 2-unit iteration in detail and showed that the acceleration can be

achieved only in the specified range for  $\theta$ . Since, we are dealing with fourth order derivatives in a biharmonic equation with Dirichlet boundary conditions, the matrix to be inverted has a very large condition number (see Ehrlich and Gupta, 1975). Therefore, for some particular solution, with specific content of eigenfunctions, CD scheme exhibits very slow convergence of the iterations due to the fact that  $\|T\|$  is very close to unity. In such cases, even our 2-unit scheme does not provide the desired acceleration. Hence, for the case  $p \ll 1$  (say,  $p \leq 0.01$ , which means  $\|T\| = 1 - p \geq 0.99$ ), we need to come up with an idea for a further acceleration. In this section, we show how to apply the same idea of acceleration, but with the 2-unit scheme playing the role of a single element. Thus, we generate a larger iteration unit, whose acceleration over the 2-unit scheme (and as a result over the CD scheme) is significant.

For convenience, we denote the iteration unit, Equation (21), using the auxiliary parameter  $\theta$  as

$$u^{2n+2} = K_{\theta}^{(0)} u^{2n}, \quad (53)$$

then for the error vector,  $\xi^n$ , we get by Equation (23) the following

$$\xi^{2n+2} = L_{\theta}^{(0)} \xi^{2n}, \quad (54)$$

where  $L_{\theta}^{(0)} = (\theta + 1)T^2 - \theta T$ . As shown in Section 4, the convergence and acceleration of the iteration unit stem from the fact that the eigenvalues of  $T$  are in the interval (0,1). Yet, now the eigenvalues of full transition operator for the 2-unit scheme,  $L_{\theta}^{(0)}$ , can be negative, since

$$\lambda_k(L_{\theta}^{(0)}) = (\theta + 1)\lambda_k^2 - \theta\lambda_k = \rho_{\theta}(\lambda_k). \quad (55)$$

Therefore, we cannot construct new iteration units when using the operator  $L_{\theta}^{(0)}$  in lieu of  $T$ . Instead, we can consider a block of two consecutive iteration units of type of Equation (53), as a basic iteration element which computes four full time steps

$$u^{4n+4} = K_{\theta}^{(1)} u^{4n} = \begin{cases} u^{4n+2} = K_{\theta}^{(0)} u^{4n} \\ u^{4n+4} = K_{\theta}^{(0)} u^{4n+2} \end{cases} \quad (56)$$

and the corresponding transition matrix between the error vectors  $\xi^{4n}$  and  $\xi^{4n+4}$  is

$$L_{\theta}^{(1)} = [L_{\theta}^{(0)}]^2. \quad (57)$$

Since  $|\rho_{\theta}(\lambda_k)| < 1$  for  $0 < \theta < 2 + 2\sqrt{2}$ , we get

$$0 < \lambda_k(L_{\theta}^{(1)}) = \rho_{\theta}^2(\lambda_k) < 1,$$

Hence, we can construct a new accelerated iteration unit containing two units Equation (56) in the same manner as in Section 4, namely,

$$\begin{aligned} u^{8n+4} &= K_{\theta}^{(1)} u^{8n} \\ u^{8n+8} &= K_{\theta}^{(1)} [(\theta_1 + 1)u^{8n+4} - \theta_1 u^{8n}], \end{aligned} \quad (58)$$

where the auxiliary parameter  $\theta_1$  can be chosen independently from  $\theta$ . We can denote the full unit as

$$u^{8n+8} = K_{\theta_1}^{(2)} u^{8n}. \quad (59)$$

So, we obtain an iteration unit  $K_{\theta_1}^{(2)}$  containing eight CD iterations, and the corresponding transition matrix between the error vectors  $\xi^{8n}$  and  $\xi^{8n+8}$  is written as

$$L_{\theta_1}^{(2)} = (\theta_1 + 1)[L_{\theta}^{(1)}]^2 - \theta_1 L_{\theta}^{(1)}. \quad (60)$$

For the sake of convenience we call Equation (60) the ‘8-unit scheme’. Its convergence and acceleration are ensured automatically by choosing  $0 < \theta_1 < 2 + 2\sqrt{2}$ , but the rate of convergence varies with  $\mu$  and  $p$ . We can derive an estimate for the asymptotic acceleration rate of the 8-unit scheme in a similar manner as in Equation (37) by observing that the acceleration rate of the 2-unit scheme itself figures into the 8-unit scheme cumulatively. We focus on the asymptotic case  $p \ll 1$ , when CD scheme is very slow. If  $p$  is larger, then the 2-unit scheme does not require too much acceleration and this case is beyond the scope of this section. Thus, the relevant case here is  $p \ll 1$  which is reflected in the numerical example chosen in Section 6. For very small  $p$ , the acceleration of the 8-unit scheme over the CD scheme is given by

$$s \approx \frac{1}{4}(\theta_1 + 2)(\theta + 2)p. \quad (61)$$

This estimate holds only under a similar constraint as Equation (38) for the 2-unit scheme, namely,

$$\left| \rho_{\theta_1} \left( \frac{\theta_1}{2(1+\theta_1)} \right) \right| \leq \rho_{\theta_1} [\rho_{\theta}(1-p)]. \quad (62)$$

Note that, if one should encounter even slower convergence, a 32-unit scheme can be developed based on the 8-unit scheme such that the further acceleration can be obtained.

## 6 Results for 8-unit scheme

In this section, we treat a case with very slow convergence with analytical solution given in Ehrlich and Gupta (1975),

$$\hat{u}(x, y) = x^2 - y^2 + xe^x \cos y, \quad (63)$$

$$f(x, y) = 0. \quad (64)$$

For this test solution, it takes approximately 3832 CD iterations to reach an error  $1.6 \times 10^{-6}$  and we determine *a posteriori* that  $q = 1 - p = \|T\| = 0.997$ . To assess the acceleration of the 8-unit scheme we choose  $\theta = 4$  and  $\theta_1 = 4$ . Since

$$\begin{aligned} \left| \rho_4 \left( \frac{4}{2(1+4)} \right) \right| &= \rho_4(0.4) = 0.8 \\ &\leq \rho_4(\rho_4(1 - 0.003)) = 0.8939, \end{aligned} \quad (65)$$

the constraint Equation (62) is satisfied. Respectively, the acceleration rate of our 8-unit scheme can be estimated on the base of Equation (61) as about nine times faster than CD scheme.

Turning to the implementation, one needs to establish a criterion to terminate the iterations. There are different ways to terminate the iterations. The standard way is to stop after the difference between two consecutive iterations becomes smaller than certain number. This not the best way to compare the efficiency of different schemes, because the speed of convergence may be different for different schemes and after the termination of the iterations the obtained solution differs from the analytical by different amount, etc. in order to be consistent with the adopted practice, we begin with the assessment of the schemes when the termination criterion is

$$\|u^{n+1} - u^n\| < \varepsilon. \tag{66}$$

In the left panel of Figure 2, we show the number of iterations needed to satisfy Equation (66) for  $\varepsilon = 10^{-6}$ . Note that the solid line,  $\theta = 0$ , corresponds to the CD scheme. We can see that our scheme overperforms CD for this criterion, except for  $\theta = 4$  and  $\theta_1 = 4$ . A similar comparison is presented in the left panel of Figure 3, but for  $\varepsilon = 10^{-7}$

Figures 2 and 3 show that increasing the requirement for the norm, makes our scheme much more efficient than CD. This means that if one requires better precision, the proposed here scheme becomes radically better than CD.

Although the criterion Equation (66) is the most widely accepted, it is not the best indicator when assessing the efficiency because it is not directly related to the real difference between the calculated and analytical solution. Naturally, in many practical situations, there is no analytical solution available to compare, but in a theoretical investigation of the type of the present paper, one should also compare the efficiency of the schemes in obtaining the same error. This point is illustrated by the right panels of Figures 2 and 3 which show the actual error at the moment of termination of the iterations. Because of the faster convergence of our scheme, the error at the moment of termination is much less than the error for CD scheme. Our error is from 2.5 to 6 times smaller that CD error.

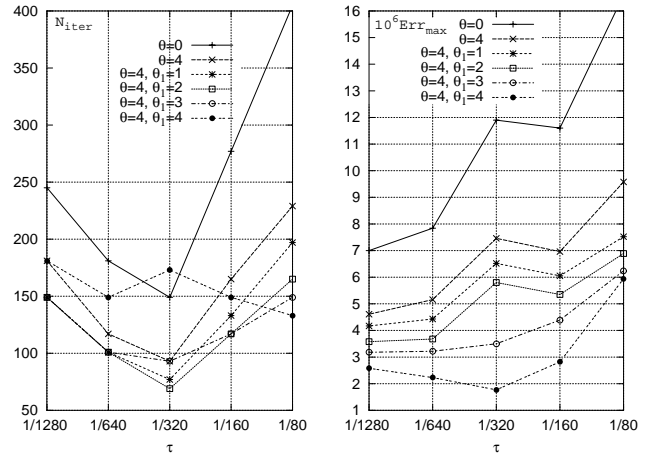
To corroborate this point, we performed a set of numerical experiments (possible only for cases with analytical solution) in which we terminated the calculations when the norm of the difference between the numerical and analytical solutions reached  $1.58 \times 10^{-6}$ , namely

$$\|u^N - \hat{u}\| \leq 1.58 \times 10^{-6},$$

where  $\hat{u}$  is defined in Equation (63). The above ‘magic’ number depends upon the grid parameters (320×320, in this particular case) and reflects the order of approximation of the CD scheme, the latter defined by the difference operators inverted. Clearly, one cannot approach an analytical solution colser than the truncation error. Using this new criterion, we secure that the calculations with the different schemes arrive at the same distance from the

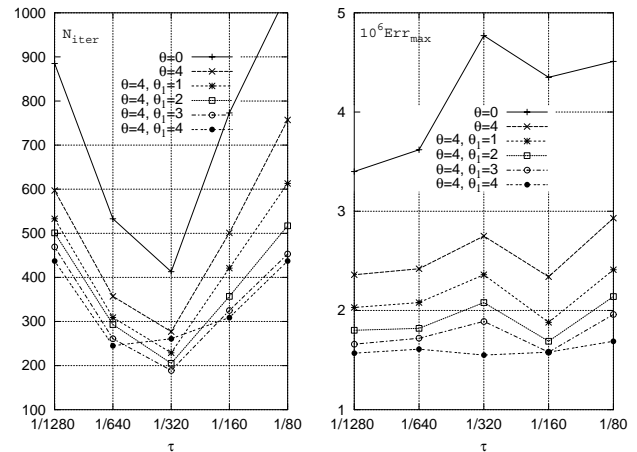
analytical solution. Then the number of iterations needed to reach the same distance is indeed the proper criterion for efficiency.

**Figure 2** Convergence results with  $\varepsilon = 10^{-6}$ y, Equation (66)



Left: the number of iterations  $N_{iter}$  for convergence. Right: difference from analytical solution after  $N_{iter}$  time steps.

**Figure 3** Convergence results with  $\varepsilon = 10^{-7}$ , Equation (66)

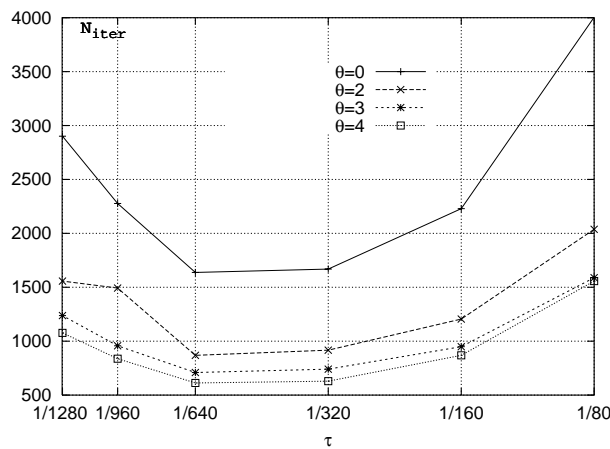


Left: the number of iterations  $N_{iter}$  for convergence. Right: difference from analytical solution after  $N_{iter}$  time steps.

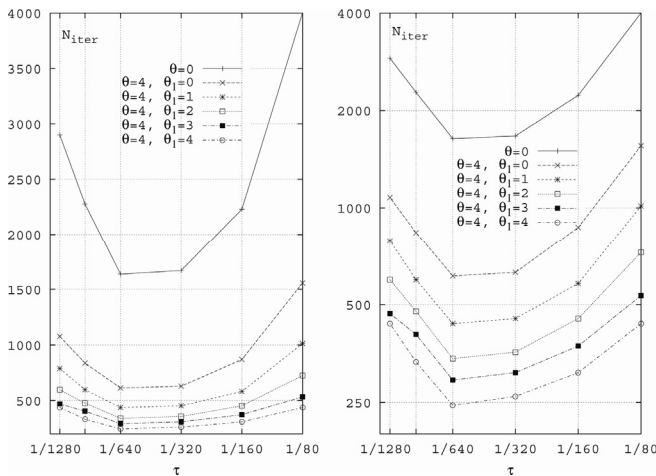
First, we show the acceleration archived by the 2-unit scheme when the above criterion is satisfied. Figure 4 shows the number of iterations needed by the 2-unit scheme to reach this criterion of convergence for different values of the parameter  $\theta$ . One can see that with this more pertinent criterion, the advantage of our scheme is even more conspicuous. Clearly, in this case  $\theta = 4$  provides the best acceleration. For this reason, we will use always  $\theta = 4$  in the 8-unit scheme, and vary only  $\theta_1$ . We demonstrate the acceleration of the 8-unit scheme in the left panel of Figure 5. Now, the overall acceleration achieved is better than nine times. For the convenience of the reader, the right panel in Figure 5 shows the the same result in logarithmic sale for the vertical coordinate. Note that the solid line of Figures 5 is

the same result as the solid line in Figure 4. When assessing the efficiency of the scheme, one should be aware that all the calculations initially start with the CD scheme. We found that optimal choice is to conduct about 30 iterations with the original CD scheme before switching to the accelerated 8-unit schemes. This was done for two reasons. First, to gather information to calculate the *a-posteriori* value for the norm of transition operator  $T$  (the factor  $1 - p$ ). Second, during the first 30 iterations, the small eigenfunctions are damped out and only the most die-hard one remains. Only then, the CD scheme becomes really slow. Having said that, we can see that the ratio between number of needed iterations,  $N_{iter}$  of the 8-unit scheme and the CD scheme is even better than the value one can infer from Figure 5.

**Figure 4** Acceleration in terms of error for 2-unit scheme achieved with different  $\theta$  for variety of time increments  $\tau$



**Figure 5** Acceleration in terms of error for  $\theta = 4$



Left: linear scale for vertical coordinate.  
Right: logarithmic scale for vertical coordinate

## 7 Conclusion

This paper considers an implementation of the operator splitting method for Dirichlet problem for the two-dimensional biharmonic equation in a square domain. A generalisation of Douglas-type ADI scheme proposed by

Conte and Dames (CD scheme) is used as the main vehicle for the splitting technique.

Since the spectral radius for the CD scheme is known to be less than one, we were able to show that it is possible to construct a scheme containing two CD-type iterations, with different arrangements for the explicit approximation of the mixed fourth derivative. The two components of the iteration unit damp the different eigenfunctions differently allowing significant acceleration of the iterative process. Our error analysis show that the improved scheme is about two to three times faster than the original CD scheme. An algorithm is devised implementing the new scheme and the numerical results are compared to the exact solutions for two different Dirichlet problems. The numerical experiments conducted on different grids with up to 1024 points in each direction, confirm the second order of spatial approximation. The theoretical finding about the existence of optimal value for the main scheme parameter is confirmed by the numerical experiments. The result is presented graphically.

For cases with norm of the transition operator very close to unity (very slow convergence of the CD scheme), we proposed one more scheme which provides a further acceleration over the 2-unit scheme. It makes use of eight CD iterations and involves two acceleration parameters. We showed that in the slow convergence cases, the 8-unit scheme converges up to nine times faster than the original CD scheme. We substantiated this theoretical finding with numerical experiment considering an analytical solution for which the CD scheme is known to be very slow. Our numerical experiments confirmed that, indeed, an acceleration of approximately nine times is possible.

The fact that we were able to accelerate the convergence makes the efficiency of the ADI schemes for fourth order operators similar to the celebrated efficiency for second-order operators (e.g. Laplace equation). The methodology of acceleration proposed here can be successfully applied to any other cases when the norm of the transition operator is very close to unity.

## References

- Arad, M., Yakhot, A., and BenDor, G. (1996) 'High-order-accurate discretization stencil for an elliptic equation', *Int. J. Num. Meth. Fluids*, Vol. 23, pp.367–377.
- Bodewig, E. (1956) *Matrix Calculus* 4. New York, NY: Interscience Publishers Inc.
- Christov, C.I. and Pontes, J. (2002) 'Numerical scheme for Swift–Hohenberg equation with strict implementation of Lyapunov functional', *Mathematical and Computer Modelling*, Vol. 35, pp.87–99.
- Christov, C.I. and Ridha, A. (1994) 'Splitting scheme for iterative solution of biharmonic equation. application to 2D Navier-Stokes problems', in I. Dimov, Bl. Sendov and P. Vasilevskii (Eds), *Advances in Numerical Methods and Applications* (pp.341–352). Singapore, Singapore: World Scientific.
- Conte, S.D. and Dames, R.T (1960) 'On an alternating direction method for solving the plate problem with mixed boundary conditions', *J. ACM*, Vol. 7, pp.264–273.

- Douglas, J. (1955) 'On the numerical integration of  $\partial^2 u / \partial x^2 + \partial^2 u / \partial y^2$  by implicit methods', *SIAM Journal*, Vol. 3, pp.42–65.
- Ehrlich, L.W. and Gupta, M.M. (1975) 'Some difference schemes for the biharmonic equation', *SIAM Journal of Numerical Analysis*, Vol. 12, pp.773–790.
- Greenspan, D. and Schultz, D. (1972) 'Fast finite-difference solution of biharmonic problems', *Communications of ACM*, Vol. 15, pp.:347–350.
- Strikwerda, J.C. (1989) *Finite Difference Schemes and Partial Differential Equations 4*. New York, NY: Chapman and Hall.
- Yanenko, N.N. (1971) *Method of Fractional Steps*. New York, NY: Gordon and Breach.