On the Beam Functions Spectral Expansions for Fourth-Order Boundary Value Problems

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Abstract. In this paper we develop further the Galerkin technique based on the so-called beam functions with application to nonlinear problems. We make use of the formulas expressing a product of two beam functions into a series with respect to the system. First we prove that the overall convergence rate for a fourth-order linear b.v.p is algebraic fifth order, provided that the derivatives of the sought function up to fifth order exist. It is then shown that the inclusion of a quadratic nonlinear term in the equation does not degrade the fifth-order convergence. We validate our findings on a model problem which possesses analytical solution in the linear case. The agreement between the *beam*-Galerkin solution and the analytical solution for the linear problem is better than 10^{-12} for 200 terms. We also show that the error introduced by the expansion of the nonlinear term is lesser than 10^{-9}. The *beam*-Galerkin method outperforms finite differences due to its superior accuracy whilst its advantage over the Chebyshev-tau method is attributed to the smaller condition number of the matrices involved in the former.

Keywords: beam functions, Galerkin spectral method, Fourth-order boundary value problems
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INTRODUCTION

Fourth-order boundary value problems (b.v.p.) are very common in the study of elastic beams and viscous fluid flow. In their vast majority, these problems cannot be treated analytically and thus a numerical approach is required. The demand for increased accuracy and reliability of the numerical technique is especially pertinent in bifurcation problems. Here we develop a fast and accurate Galerkin spectral method using the set of beam functions as basis. These functions were first introduced by Lord Rayleigh in his book "Theory of Sound" [1] to describe the vibrations of elastic beams clamped on both sides.

The method has already been applied to Poiseuille flow [2, 3] and to the one-dimensional g-jitter thermoconvective flow [4, 5]. The complete orthonormal set of beam functions was chosen, because they automatically satisfy all the boundary conditions, avoiding Gibbs effects and guaranteeing a very good rate of convergence.

The applications mentioned above involve linear boundary value problems. It is important to examine the performance of the *beam*-Galerkin technique when nonlinear terms are present. In this case, the technique requires formulas for the products of beam functions which were derived in [2] but were never put to practical use. The aim of this paper is to verify the different ingredients of the *beam*-Galerkin technique for solving fourth-order nonlinear b.v.p's.

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THE BEAM-GALERKIN METHOD

In order to make the paper selfcontained we repeat here some basic formulas and compile the necessary new ones. Consider the fourth order Sturm-Liouville eigenvalue problem

$$\frac{d^4u}{dy^4} = \lambda^4 u\,, \quad u = \frac{du}{dy} = 0\,, \quad \text{for} \quad x = \pm 1\,. \tag{1}$$

The nontrivial solutions (eigenfunctions) are given by

$$s_m = \frac{1}{\sqrt{2}}\left[\frac{\sinh \lambda_m x}{\sinh \lambda_m} - \frac{\sin \lambda_m x}{\sin \lambda_m}\right], \quad \coth \lambda_m - \cot \lambda_m = 0, \tag{2}$$

$$c_m = \frac{1}{\sqrt{2}}\left[\frac{\cosh \kappa_m x}{\cosh \kappa_m} - \frac{\cos \kappa_m x}{\cos \kappa_m}\right], \quad \tanh \kappa_m + \tan \kappa_m = 0. \tag{3}$$

The eigenvalues can be calculated numerically and first couple of them are given in Table 1. For $m \geq 6$ the asymptotic formulas $\kappa_m \to (m - \frac{1}{4})\pi$ and $\lambda_m \to (m + \frac{1}{4})\pi$ are correct up to 16 decimals.

TABLE 1. Magnitude of eigenvalues

m	λ_m	κ_m	$\coth \lambda_m$	$\tanh \kappa_m$
1	3.926602	2.365020	1.000773	0.982502
2	7.068583	5.497804	1.000015	0.999966
3	10.210176	8.639380	1.000000	0.999999
4	13.351768	11.780972	1.000000	1.000000
5	16.493361	14.922565	1.000000	1.000000
6	19.634954	18.064157	1.000000	1.000000

Note that the s_m functions are odd whereas the c_m functions are even, resembling trigonometric *sines* and *cosines*. We can see the graphs of a few members of our complete orthonormal (CON) system in Figure 1.

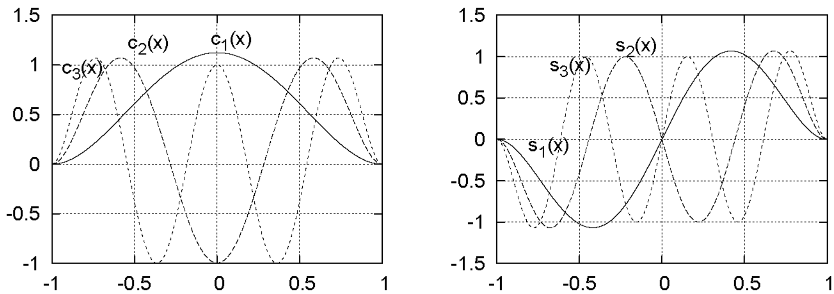

FIGURE 1. Left panel: the first three even members of the sequence, $c_n(x)$; right panel: the first three odd members of the sequence s_n.

The derivatives of the beam functions can be expressed as series with respect to the CON system as follows:

$$c_n' = \sum_{m=1}^{\infty} a_{nm} s_m\,, \quad s_n' = \sum_{m=1}^{\infty} \bar{a}_{nm} c_m\,, \qquad a_{nm} = -\bar{a}_{mn} = \frac{4\kappa_n^2 \lambda_m^2}{\kappa_n^4 - \lambda_m^4}, \tag{4}$$

$$c_n''' = \sum_{m=1}^{\infty} d_{nm} s_m\,, \quad s_n''' = \sum_{m=1}^{\infty} \bar{d}_{nm} c_m\,, \quad d_{nm} = -\bar{d}_{mn} = \frac{4\kappa_n^3 \lambda_m^3}{-\kappa_n^4 + \lambda_m^4} \tanh \kappa_n \coth \lambda_m, \tag{5}$$

$$c_n'' = \sum_{m=1}^{\infty} \beta_{nm} c_m, \quad \beta_{nm} = \begin{cases} \dfrac{4\kappa_n^2 \kappa_m^2}{\kappa_m^4 - \kappa_n^4} \left(\kappa_m \tanh \kappa_m - \kappa_n \tanh \kappa_n\right), & m \neq n, \\ \kappa_n \tanh \kappa_n - \left(\kappa_n \tanh \kappa_n\right)^2, & m = n, \end{cases} \tag{6}$$

$$s_n'' = \sum_{m=1}^{\infty} \bar{\beta}_{nm} s_m, \quad \bar{\beta}_{nm} = \begin{cases} \dfrac{4\lambda_n^2 \lambda_m^2}{\lambda_n^4 - \lambda_m^4} \left(\lambda_n \coth \lambda_n - \lambda_m \coth \lambda_m\right), & m \neq n, \\ \lambda_n \coth \lambda_n - \left(\lambda_n \coth \lambda_n\right)^2, & m = n, \end{cases} \tag{7}$$

For more details regarding the accuracy of formulas Eq. (4) see [5, 4, 6].

A very important reason for insisting on this technique is the convergence rate of the spectral coefficients. Implementing a technique described in [7] we arrive at the following theorem:

Theorem 1. *Suppose $u(x) \in \mathscr{C}^5([-1,1])$. Consider the fourth-order boundary value problem*

$$\mathscr{L}u = au^{(iv)}(x) + bu''(x) + cu(x) = f(x), \quad u = u' = 0\,, \quad \textit{for} \quad x = \pm 1\,, \tag{8}$$

where $a, b, c \in \mathbf{R}$, $a \neq 0$. Then, the convergence rate of the series $u(x) = \sum_{k=1}^{\infty} u_k c_k(x)$ for the spectral solution of (8) is fifth order algebraic.

Proof. The spectral series for the solution of (8) is given by

$$u(x) = \sum_{k=1}^{\infty} u_k c_k(x), \qquad u_k = \int_{-1}^{1} c_k(x) u(x)\, \mathrm{d}x. \tag{9}$$

After successive integrations by parts, acknowledging the boundary conditions, the characteristic equation $\tanh \kappa_k + \tan \kappa_k = 0$, and making use of the fact that $c_k = c_k' = 0$ at $x = \pm 1$, we get

$$u_k = \left[\frac{u^{(iv)}(x)}{\sqrt{2}\kappa_k^5} \left(\frac{\sinh \kappa_k x}{\cosh \kappa_k} - \frac{\sin \kappa_k x}{\cos \kappa_k} \right) \right]_{x=-1}^{x=1} - \frac{1}{\sqrt{2}\kappa_k^5} \int_{-1}^{1} \left(\frac{\sinh \kappa_k x}{\cosh \kappa_k} - \frac{\sin \kappa_k x}{\cos \kappa_k} \right) u^{(v)}(x)\, \mathrm{d}x. \tag{10}$$

Due to the lack of differentiability of u beyond the fifth order, the last term will contribute upon integration by parts quantities, which are not trivially equal to zero. Hence, continuing the process will not cancel the terms of order κ_m^5. □

This rate of convergence means that with only 100 terms we expect accuracy of order 10^{-10}.

PRODUCTS OF BEAM FUNCTIONS

To treat power-type nonlinear terms, we must have a formula for expressing products of beam functions into series of beam functions. The product of two even beam functions is expressed as a series of even beam functions as follows [2]:

$$c_n(x)c_m(x) = \sum_{k=1}^{\infty} h_k^{nm} c_k(x), \qquad \sqrt{2}h_k^{nm} = \sqrt{2}\int_{-1}^{1} c_n(x)c_m(x)c_k(x)\mathrm{d}x \tag{11}$$

$$= \frac{\kappa_n \tanh\kappa_n - (\kappa_m+\kappa_k)(\tanh\kappa_m+\tanh\kappa_k)}{-(\kappa_m+\kappa_k)^2+\kappa_n^2} + \frac{\kappa_n \tanh\kappa_n - (\kappa_m-\kappa_k)(\tanh\kappa_m-\tanh\kappa_k)}{-(\kappa_m-\kappa_k)^2+\kappa_n^2}$$
$$+ \frac{\kappa_n \tanh\kappa_n - (\kappa_m+\kappa_k)(\tanh\kappa_m+\tanh\kappa_k)}{(\kappa_m+\kappa_k)^2+\kappa_n^2} + \frac{\kappa_n \tanh\kappa_n - (\kappa_m-\kappa_k)(\tanh\kappa_m-\tanh\kappa_k)}{(\kappa_m-\kappa_k)^2+\kappa_n^2}$$
$$+ \frac{\kappa_m \tanh\kappa_m - (\kappa_n+\kappa_k)(\tanh\kappa_n+\tanh\kappa_k)}{(\kappa_n+\kappa_k)^2+\kappa_m^2} + \frac{\kappa_m \tanh\kappa_m - (\kappa_n-\kappa_k)(\tanh\kappa_n-\tanh\kappa_k)}{(\kappa_n-\kappa_k)^2+\kappa_m^2}$$
$$+ \frac{\kappa_k \tanh\kappa_k - (\kappa_n+\kappa_m)(\tanh\kappa_n+\tanh\kappa_m)}{(\kappa_n+\kappa_m)^2+\kappa_k^2} + \frac{\kappa_k \tanh\kappa_k - (\kappa_n-\kappa_m)(\tanh\kappa_n-\tanh\kappa_m)}{(\kappa_n-\kappa_m)^2+\kappa_k^2}.$$

The respective formulas for $c_n s_m$ and $s_n s_m$ are similar and can be found in [2]. It is important to point out that the complicated nature of this formula is not a problem since the entries h_k^{nm} need only be calculated once and stored in an array. Then, they can be retrieved whenever needed.

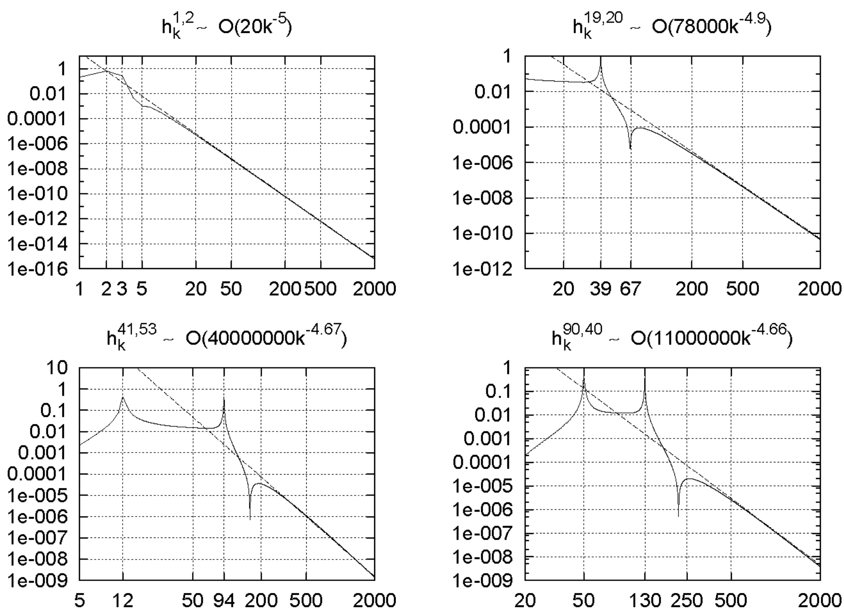

FIGURE 2. The convergence rates of the series for the products $c_n c_m$.

We verify the performance of formula (11) numerically, i.e., we compare the product of some particular functions c_n and c_m, $c_n c_m$, with their Galerkin expansions (see Figure 2 for couple of combinations of functions) . We have examined the convergence rate of the spectral coefficients $h_k^{n,m}$ for various values of n and m, and have observed peaks due to resonances at $k = n - m$ and $k = n + m$. For small values of n and m, i.e., less than 10, we obtain convergence rates of order $1/k^5$, as expected. However, the situation starts deteriorating after that. For instance, $h_k^{19,20} \sim O(780000/k^{4.9})$. Then, $h_k^{41,53} \sim O(4000000/k^{4.67})$ and $h_k^{90,40} \sim O(11000000/k^{4.66})$. We attribute this to the stronger effect of the resonances.

It is important to note though that the effect of this on the overall behavior of the scheme is limited because the slower convergence of $h_k^{n,m}$ corresponds to high k values. If u_k are the spectral coefficients of the solution of a fourth-order b.v.p. then a product $u_m u_n$ contributes $k^{-(m+n)}$ to a term of order k^{-l}. Recall, $u_m \sim O(1/m^5)$, $u_n \sim O(1/n^5)$ and so $u_m u_n \sim O(1/(m+n)^5)$.

FEATURING EXAMPLE

Consider the following generic nonlinear fourth-order b.v.p.:

$$u^{(iv)}(x) + 2u''(x) + u(x) = 1 - Bu^2(x), \quad u = u' = 0\,, \quad \text{for} \quad x = \pm 1\,, \tag{12}$$

where the nonhomogeneous term is taken equal to unity for the sake of the illustration and the coefficient B can be chosen to be large enough in order to enhance the effect of the nonlinearity.

Acknowledging the symmetry of the boundary conditions we expand the sought function u as $u(x) = \sum_{k=1}^{\infty} u_k c_k(x)$. We also expand unity into series with respect to c_m. Then following the Galerkin procedure the nonlinear problem is recast as the following nonlinear algebraic system:

$$(1 + \kappa_i^4)u_i + 2\sum_{j=1}^{N} u_j \beta_{ij} = \frac{2\sqrt{2}\tanh\kappa_i}{\kappa_i} - B\sum_{m=1}^{N}\sum_{n=1}^{N} u_m u_n h_i^{mn},$$

for $i = 1, \ldots, N$, where β_{ij} and h_i^{mn} are defined in formulas (6) and (11), respectively. We solve the latter with iterative method in which the linear operators are inverted, while the nonlinear term is taken from the previous iteration. When the method is applied to the multidimensional case, the simple iterative procedure may not be efficient enough, but for the purposes of the present work it is fully sufficient.

First, we show the performance of the method for a linear case, $B = 0$, when the following analytical solution is available

$$u(x) = 1 - \frac{2\cos x(\cos 1 + \sin 1) + 2x\sin 1 \sin x}{2 + \sin 2}. \tag{13}$$

We solve the above system and verify that the convergence is indeed fifth order algebraic. The pointwise error of the obtained spectral solution is presented in Figure 3 for $N =$

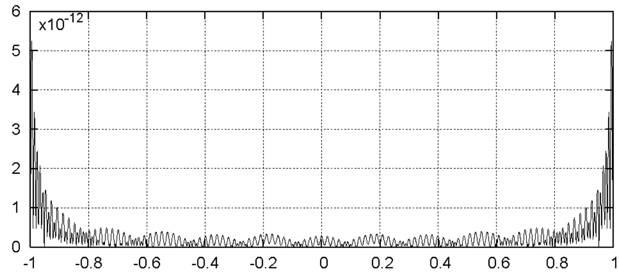

FIGURE 3. The difference between the spectral and analytical solution (Eq. (13)) for $N = 200$.

200. The maximal error is of order of 10^{-12} which is fully consistent with the fifth order of approximation 200^{-5}.

Unfortunately, there is no analytical solution when $B \neq 0$. In order to verify the performance of the method we resort here to another approach. We take a simple function that satisfies the boundary conditions, say $u(x) = (1-x^2)^2$ and expand it into even beam functions. We then apply product formula (11) to compute its square. The results are then compared to the exact function $g(x) = u^2(x) = (1-x^2)^4$ and to the direct spectral expansion of $g(x)$. This demonstrates the performance of formula (11) in representing the square of a known function.

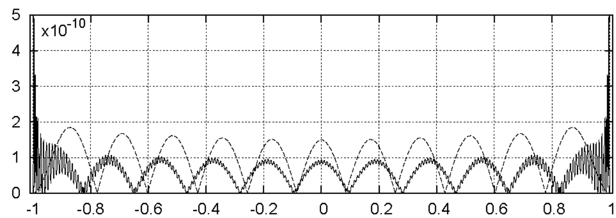

FIGURE 4. Comparisons for the expansion of the square of the function $u = (1-x^2)^2$. Solid line: with the analytical expression for u^2. Dashed line: with the squared spectral expansion of u.

In order for this comparison to take place, formulas for expanding even powers of x into even beam functions had to be derived:

$$\int_{-1}^{1} x^n c_m \mathrm{d}x = \frac{1}{\sqrt{2}}\left[\frac{2}{\kappa_m}(\tanh\kappa_m - \tan\kappa_m) - \frac{4n}{\kappa_m^2} + \sqrt{2}\frac{n(n-1)(n-2)(n-3)}{\kappa_m^4}\int_{-1}^{1} x^{n-4} c_m \mathrm{d}x\right],$$

$$\int_{-1}^{1} x^2 c_m(x)\,\mathrm{d}x = \sqrt{2}\left[\frac{1}{\kappa_m}(\tanh\kappa_m - \tan\kappa_m) - \frac{4}{\kappa_m^2}\right],$$

$$\int_{-1}^{1} x^4 c_m(x)\,\mathrm{d}x = \frac{1}{\sqrt{2}}\left[\left(\frac{2}{\kappa_m} + \frac{48}{\kappa_m^5}\right)(\tanh\kappa_m - \tan\kappa_m) - \frac{16}{\kappa_m^2}\right],$$

$$\int_{-1}^{1} x^6 c_m(x)\,\mathrm{d}x = \frac{1}{\sqrt{2}}\left[\left(\frac{2}{\kappa_m} + \frac{6!}{\kappa_m^5}\right)(\tanh\kappa_m - \tan\kappa_m) - \frac{24}{\kappa_m^2} - \frac{4\cdot 6!}{\kappa_m^6}\right],$$

$$\int_{-1}^{1} x^8 c_m(x)\,\mathrm{d}x = \frac{1}{\sqrt{2}}\left\{ \frac{2}{\kappa_m}(\tanh\kappa_m - \tan\kappa_m) - \frac{32}{\kappa_m^2} + \frac{8\cdot7\cdot6\cdot5}{\kappa_m^4}\left[\left(\frac{2}{\kappa_m}+\frac{48}{\kappa_m^5}\right)(\tanh\kappa_m - \tan\kappa_m) - \frac{16}{\kappa_m^2}\right]\right\}.$$

The most important finding here is that the acknowledgment of the nonlinear term does not degrade the fifth order of the convergence, as testified by Figure 5.

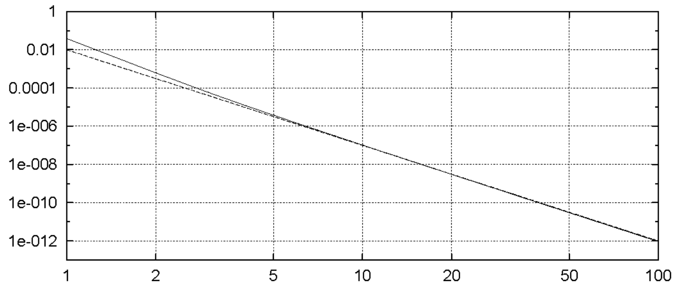

FIGURE 5. Fifth order rate of convergence for the solution of the nonlinear equation. Solid line: u_i; dashed line: $u_i = 0.01\,i^{-5}$.

It must be noted that we did compare the spectral solution of Eq. (12) (the one whose coefficients are presented in Figure 5) to two different numerical solutions based of finite differences and on a Chebyshev tau method. The results differ slightly from the results of the two alternative techniques.

The finite difference scheme is of order $O(h^2)$. When tested for the linear problem, the agreement with the exact analytic solution was limited to two or three significant digits. Thus some small disagreement with the *beam*-Galerkin technique is expected because even for 10000 points the error of the finite difference method is of order 10^{-8}.

TABLE 2. Condition Number Estimates using IMSL routine DLFCRG.

N	(Beam)	(Chebyshev)
10	0.37216988E+03	0.22352482E+09
20	0.17644881E+04	0.35951175E+10
50	0.20897013E+05	0.47854602E+13
100*	0.22308957E+06	0.70174753E+15
150	0.10256813E+07	0.12419855E+17
200	0.31280912E+07	0.45545491E+18
250	0.75100639E+07	0.45545491E+18
300	0.15431671E+08	0.16423978E+19
350	0.28433312E+08	0.48536036E+19
400	0.48335713E+08	0.12401339E+20

* From this value of N and onward, routine DLFCRG declares the Chebyshev matrix algorithmically singular.

The problems with the Chebyshev-tau method are different. This method, just like the beam-Galerkin method reproduces the analytical solution of the linear problem ($B = 0$)

very well. However, the condition number of the Chebyshev-tau matrix increases dramatically with the number of terms N (see Table 2). So, the error increases significantly when we multiply with the inverse many times. Note that when investigating actual non-stationary physical problems, thousands or millions of such multiplications are required, because for some parameter values we must run the program for thousands of periods.

Our assertion is that the *beam*-Galerkin method is more reliable than the other two.

CONCLUSIONS

The earlier developed *beam*-Galerkin method is extended to nonlinear problems. To this end, the formulas for expressing the product of two beam functions are examined and consequently implemented in a fourth order nonlinear b.v.p. with square nonlinearity. It is proven that the overall convergence rate for a fourth-order linear b.v.p is algebraic fifth order, provided that the derivatives of the sought function up to fifth order exist.

It is observed that the coefficients $h_k^{n,m}$ of the product of two even beam functions converge with algebraic rates ranging from order $O(1/k^5)$ for small indices $n,m \leq 10$ to $O(1/k^{4.5})$ when $n \geq 80$ or $m \geq 80$. This deterioration is attributed to resonant peaks at $k = n+m$ and $k = n-m$ but it does not affect the overall fifth rate of convergence.

Our findings have been validated in two ways. Firstly, by expressing the square of a known function $u^2(x)$ using the product formula and comparing it to its direct expansion, and the exact analytic square of the function. It was shown that the error introduced by the expansion of the nonlinear term was less than 10^{-9}. Secondly, by solving a fourth order nonlinear b.v.p. with square nonlinearity. The results are compared to finite-differences and a Chebyshev tau method and shown to disagree insignificantly. The finite difference scheme is of order $O(h^2)$ (which gives error of order 10^{-8} even for 10000 points), whilst the condition number of the Chebyshev-tau matrix increases drastically with the number of terms N, leading to compounding of the error during iterations. Thus, the *beam*-Galerkin method is the most reliable for the problem under consideration.

REFERENCES

1. L. Rayleigh, *Theory of Sound*, Dover, New York, 1945.
2. C. I. Christov, *Ann. Univ. Sof., Fac. Math. Mech.* **76**, 87–113 (1982/87).
3. G. Poots, *Quart. J. Mech. Appl. Math.* **11**, 357–273 (1958).
4. N. C. Papanicolaou, and C. I. Christov, *Journal of Neural Parallel and Scientific Computations* **10**, 339–354 (2002).
5. N. C. Papanicolaou, *A Galerkin Spectral Method for Fourth-Order Boundary Value Problems*, Ph.D. thesis, University of Louisiana at Lafayette (2003).
6. N. Papanicolaou, and C. I. Christov, *Annuaire de l'Universit de Sofia* **94**, 71–83 (2000).
7. J. P. Boyd, *Fourier and Chebyshev Spectral Methods*, Dover, New York, 2000.