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On Boussinesq's paradigm in nonlinear wave propagation

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Abstract

Boussinesq's original derivation of his celebrated equation for surface waves on a fluid layer opened up new horizons that were to yield the concept of the soliton. The present contribution concerns the set of Boussinesq-like equations under the general title of 'Boussinesq's paradigm'. These are true bi-directional wave equations occurring in many physical instances and sharing analogous properties. The emphasis is placed: (i) on generalized Boussinesq systems that involve higher-order linear dispersion through either additional space derivatives or additional wave operators (so-called double-dispersion equations); and (ii) on the 'mechanics' of the most representative localized nonlinear wave solutions. Dissipative cases and two-dimensional generalizations are also considered.
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Résumé

Sur la paradigme de Boussinesq pour la propagation d'ondes non linéaires. L'obtention originale de sa célèbre équation gouvernant les ondes de surface sur une couche fluide par Boussinesq a ouvert de nouveaux horizons qui devaient conduire au concept de soliton. La présente contribution concerne l'ensemble des équations du type Boussinesq sous le titre général de « paradigme de Boussinesq ». Celles-ci sont de véritables équations bi-directionnelles qui apparaissent dans de nombreuses situations physiques et partagent des propriétés analogues. L'accent est mis sur : (i) les système généralisés de Boussinesq qui impliquent une dispersion linéaire d'ordre supérieur soit en raison de la présence de dérivées spatiales d'ordre supérieur, soit avec la contribution d'autres opérateurs d'onde (équation à « double dispersion ») ; et (ii) la « mécanique » des solutions les plus représentatives d'ondes non linéaires localisées qui en résulte. Des généralisations dissipatives et à deux dimensions d'espace sont également envisagées.
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Keywords: Computational fluid mechanics; Boussinesq; Nonlinear wave; Elastic solids; Crystal lattice; Solitons; Korteweg–de Vries equation

Mots-clés : Mécanique des fluides numérique ; Boussinesq ; Ondes non linéaires ; Solides élastiques ; Réseau cristallin ; Solitons ; Équation de Korteweg–de Vries

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1. Introduction

The governing equation for *bi-directional* solitary waves propagating on the free surface of a constant depth fluid was first presented by J.V. Boussinesq [1,2] and Rayleigh [3]; this has come to be known as the *Boussinesq equation*, or *BO equation*. Korteweg and de Vries [4] derived the *unidirectional* version of the Boussinesq equation which now bears their name, viz, the Korteweg–de Vries or *KdV equation*. Indeed, Boussinesq provided a theory where higher-order linear dispersion was introduced in the long-wave limit and the crucial role of the balance between nonlinearity and dispersion was shown albeit not clearly seen by later authors (for an illuminating paper on the subject, see Ursell [5]).

Boussinesq derived initially a well-posed equation which contained also a mixed fourth order derivative for the dispersion alongside with the purely spatial fourth order derivative. Then, in an attempt to have a form of the equation that allows finding an analytical solution (the famous *sech*) he went on to replace the time derivatives in the mixed-derivative term by purely spatial ones. Such an approximation is called sometimes Linear Impedance Assumption (LIA) because it makes use of the ansatz $u_t = -cu_x$ which is true for steady linear waves propagating with phase speed c. Without the LIA, the Boussinesq equation contains the two types of dispersion (in the same fashion as the equation for transversal vibration of elastic beam does) and is mathematically correct (see [6], for the details). As shown in [6] if the unnecessary LIA was not applied by Boussinesq, his equation would have had the form

$$u_{tt} - c_0^2 u_{xx} - \left(\frac{\mathrm{d}F(u)}{\mathrm{d}u}\right)_{xx} = (\beta_1 u_{tt} - \beta_2 u_{xx})_{xx} \tag{1}$$

where $\beta_1 = \frac{1}{2}\beta$, $\beta_2 = \frac{1}{3}\beta$, $\beta > 0$ and $F(u) = \frac{1}{3}\alpha u^3$. Generally speaking, $F(u)$ may be sought of as a polynomial in u, starting with second degree.

The original equation had a positive sign in front of the dispersion connected with the fourth-order spatial derivative ($\beta_1 = 0$, $\beta_2 < 0$) and turned out to be mathematically improper, being incorrect in the sense of Hadamard (the initial-value problem is ill-posed). The reason for this is that in the absence of nonlinearity, small perturbations would amplify as frequency may become imaginary. The equation, known nowadays as BO is the incorrect equation (called sometimes 'bad' BO), while if a sufficient strong surface tension is added in the model (making the coefficient β_2 positive), the equation is correct (called 'good' BO).

The LIA was used by Boussinesq [2] in one more place: when treating the nonlinear terms. It was shown in [7] that in doing so, he has also destroyed the Galilean Invariance of the model. This is a minor point, but it can have important effect on the collision of a solitary wave changing the sign of the phase shift [7]. In [7] it was shown how to obtain a conservative formulation of the model within the same order of approximation as the original Boussinesq derivations. In a sense, [7] presents the essential Boussinesq idea without the minor mathematical setbacks, such as incorrectness in the sense of Hadamard, and lack of Galilean invariance. Incredibly enough, the more elaborated Boussinesq system from [7] once again possesses a *sech* solution which shows the generic validity of Boussinesq insight.

In [6], Eq. (1) was referred to as 'Boussinesq's Paradigm Equation' (PBE), and the problem with the 'good' and 'bad' equations is alleviated altogether.

The Boussinesq-like equation deduced from the long-wave-length limit of a chain of atoms interacting nonlinearly [8], is once again incorrect in the sense of Hadamard, i.e., it is a '*bad*' BO equation! According to Rosenau's analysis [9], this is simply due to a 'bad' expansion in the discrete-continuum approximation. A 'good' expansion yields the 'good' BO equation that we rewrite in a generic form as—the so-called *regularized long-wave BO equation* (RLWE)—($\beta_1 > 0$, $\beta_2 = 0$).

$$u_{tt} - u_{xx} - \left[F(u)\right]_{xx} - \beta u_{xxtt} = 0 \tag{2}$$

In fluid dynamics the derivation of RLWE is presented in [10]. In the context of solid mechanics, the mixed fourth-order space–time derivative was introduced by Ostrovskii and Sutin [11], who incorporated the effect of so-called lateral inertia in a one-dimensional dynamical description of elastic rods. From the point of view of (*small*) dispersion, this term is equivalent to the negative of the fourth-order space derivative u_{xxxx}. For $F(u) = u^2$, (2) is the 'good' or so-called 'improved' BO equation; for $F(u) = u^3$, Eq. (2) is called the 'modified improved' BO equation. The case $F(u) = u^p$, $p > 3$, was studied by Clarkson et al. [12]. Clearly, the different 'improved' equations fall in the general class of PBE (Eq. (1)).

As mentioned above, concerning its physical origin or the fundamental mechanical modeling at its basis, Boussinesq's celebrated equation was introduced first in the framework of fluid mechanics. However, Boussinesq himself [13] derived a linear version with mixed fourth-order space derivative in an attempt to explain the *anomalous* light dispersion discovered by F.P. Leroux in his 1860–1862 measurements. For this he essentially considered an elastic continuum interacting with 'matter'. The resulting equation is none other than an *improved* BO equation or RLWE, Eq. (2) without the nonlinear term.

Nowadays, it has become normal usage to call the 'BO equation' not only that equation that originates from the theory of nonlinear waves on shallow water, but also all the equations of this type that are encountered in various physical situations including waves in deformable crystals [14], electric waves in dielectrics with nonlinear electronic polarization, waves along certain electromagnetic transmission lines, longitudinal waves in elastic rods [15], [16] (Chapter 9), [17]. Thus we have agreed to call *Boussinesq paradigm* [18] this set of equations which contains simultaneously the following ingredients: (i) bi-directionality of the wave solutions (propagation to the left *and* to the right; presence of a d'Alembertian operator); (ii) nonlinearity of any order; and (iii) dispersion of any order, the latter resulting in the presence of combined space and time derivatives of the fourth order at least. Thus equations such as

$$u_{tt} - c_0^2 u_{xx} - \left[F'(u) + \beta_1 u_{tt} - \beta_2 u_{xx} + \delta u_{xxxx}\right]_{xx} = 0 \tag{3}$$

belong in this paradigm with a different combination of nontrivial coefficients, and they may also be referred to as *generalized* BO equations. Many works have been devoted to a mathematical justification of the derivation of Boussinesq's equation and its generalizations (together with the related generalizations of the KdV equation) in fluids. Rather than dealing with this problem that remains of critical interest in the theory of asymptotics, the present *essay* is more concerned with the 'mechanics of waves' exhibited in the BO paradigm than with the mechanics yielding the BO equation. This follows the spirit of a previous contribution [18].

2. Some properties of the standard Boussinesq equation

2.1. Bell-shaped and hump solitons

Particularly relevant from the historical viewpoint is that Boussinesq found an analytical expression of the *sech* type for the permanent long-wave-length waves which are solutions of the equation he derived. For the PBE, the solution that is being translated at the constant speed c, has the form:

$$u = A\, sech^2\left[b(x - ct)\right], \quad A = 3\frac{c_0^2 - c^2}{2\alpha}, \; b = \frac{1}{2}\sqrt{\frac{c_0^2 - c^2}{\beta_2 - \beta_1 c^2}} \tag{4}$$

For $\beta_2 < \beta_2/c_0^2$ both super-critical and sub-critical solitons exist. The sub-critical one takes place for $c < c_0\sqrt{\beta_2/\beta}_1$ while the supercritical one appears for $|c| > c_0$. Thu sub-critical solutions have negative amplitudes (depressions), while the super-critical ones have positive amplitudes (elevations or humps).

2.2. Conservation laws

From soliton's theory [21] one expects an infinite hierarchy of conservation laws and that is the case with the original BO. However, the presence of the mixed fourth derivative destroys the full integrability. Yet, from the meaningful viewpoint of the 'mechanics of waves', only the first few such laws are most relevant. Since Eq. (1) is derivable from a Lagrangian, one can apply Noether's theory to set forth the first three laws. Indeed, we $\xi(x,t)$ a potential for u in x-space and an auxiliary function w through the relations

$$u = \xi_x, \qquad u_t = w_x = \xi_{xt} = (\xi_t)_x \tag{5}$$

and apply the canonical formulation in Appendix A1 of [14]. The boundary conditions require that velocity u vanish at the boundary ∂R of the region alongside with its spatial derivative, i.e. $u, \partial_x u \to 0$ for $x \to \pm\infty$. Then from Eqs. (3) and (5) can be derived conservation laws for the three main properties as (R is the real line).

2.2.1. 'Mass' of the solution

$$M(R)=\int_R u\,\mathrm{d}x=\int_R \xi_x\,\mathrm{d}x=\xi(+\infty)-\xi(-\infty) \tag{6}$$

Here is to be mentioned that the 'mass' is the traditional way to refer to the above integral. The exact meaning of the integral is in fact the one of a 'volume'. When speaking about quasi-particles, the term 'mass' is properly defined as the coefficient of the second time derivative of the trajectory of the center of the particle (i.e. the inertia of the quasi-particle) which does not coincide with the 'volume'. In this work we will adhere to the standard terminology and refer the integral from Eq. (6) as 'wave mass' or simply 'mass'.

2.2.2. Canonical momentum (also called 'pseudomomentum' [14])

$$P(R):=-\int_R \xi_x\xi_t\,\mathrm{d}x=-\int_R uw\,\mathrm{d}x \tag{7}$$

2.2.3. Canonical energy (or Hamiltonian)

$$E(R)=\int_R \mathcal{H}\,\mathrm{d}x,\quad \mathcal{H}(u,w)=\frac{1}{2}\big[c_0^2u^2+w^2+2F(u)+\beta_1(w_x)^2+\beta_2(u_x)^2+\delta(u_{xx})^2\big]$$
$$=\frac{1}{2}\big[c_0^2\xi_x^2+\xi_t^2+2F(\xi_x)+\beta_1(\xi_{tx})^2+\beta_2(\xi_{xx})^2+\delta(\xi_{xxx})^2\big] \tag{8}$$

It is immediately checked that for asymptotic boundary conditions imposed on the entire real line R, the three main mechanical properties of the wave system are conserved, namely

$$\frac{\mathrm{d}M}{\mathrm{d}t}(R)=0,\qquad \frac{\mathrm{d}P}{\mathrm{d}t}(R)=0,\qquad \frac{\mathrm{d}E}{\mathrm{d}t}(R)=0 \tag{9}$$

so that 'mass' and energy are conserved for a BO mathematical hump (verifying vanishing conditions at infinity) that travels *inertially* according to the second of (9). Remarkably enough, Boussinesq [2] deduced two of these conservation laws (the first and third) without the mathematical apparatus now at our disposal. We recognize in the first expression of (8) the *true* total energy for the dynamics of a one-dimensional elastic body in the second-displacement gradient or first strain-gradient theory [14], while the second expression is referred to as a 'pseudo-energy' [19]. In a numerical simulation where space is perforce limited, a finite interval $I=[-L_1,L_2]$ is introduced and the boundary conditions $u,\partial_x u=0$ are imposed at $x=-L_1$ and $x=L_2$. Note that for Eq. (2), only function u is specified at the boundary. In such a case $(9)_1$ and $(9)_3$ are still satisfied over this finite interval, but $(9)_2$ takes the form of *a forced equation of motion* such as

$$\frac{\mathrm{d}P}{\mathrm{d}t}(I)=F_d:=-\big[\mathcal{H}(u,w)\big]_{-L_1}^{L_2} \tag{10}$$

where F_d is the *driving force* created by nonhomogeneous boundary conditions for w. The sixth-order equations (3) requires three conditions on function u: $u=\partial_x u=\partial_{xx}u=0$ and the only way to have different values of the Hamiltonian density at the different ends of the interval is if w (and hence) are subject to nonhomogeneous conditions. This is shown by noting that after one space integration there holds the equation

$$w_t-\beta_1 w_{txx}=\big(c_0^2u+2F'(u)-\beta_2u_{xx}+\delta u_{xxxx}\big)_x \tag{11}$$

Multiplying then $(5)_2$ by $w-\beta_1 w_{xx}$ and (11) by u, adding the two results, and finally integrating over the space interval yields (10). That equation tells us what boundary condition to apply in order to replicate a true inertial motion of the localized wave. Note that another representation of the same problem can be obtained by introducing a potential ς for w, $w=\varsigma_x$, so that we have the following *evolution* problem for the pair (u,ς):

$$u_t=w_x=\varsigma_{xx},\quad \varsigma_t-\beta_1\varsigma_{xx}=c_0^2u+F'(u)-\beta_2u_{xx}+\delta u_{xxxx} \tag{12}$$

A strictly conservative difference scheme can be constructed for this system [19,24].

2.3. Hamiltonian structure

Upon observing that for the Hamiltonian density H

$$\frac{\delta H}{\delta w} = w - \beta_1 w_{xx} \quad \text{and} \quad \frac{\delta H}{\delta u} = c_0^2 u + F'(u) - \beta_2 u_{xx} \tag{13}$$

We can rewrite $(5)_2$ and (11) as follows

$$u_t = w_x = \frac{\partial}{\partial x}\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right]^{-1}\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right] w = \frac{\partial}{\partial x}\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right]^{-1} \frac{\delta H}{\delta w} \tag{14}$$

$$\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right] w_t = \frac{\partial}{\partial x}\left[c_0^2 u + F'(u) - \beta_2 u_{xx}\right] \tag{15}$$

and then the last equation can also be put in the form

$$w_t = \frac{\partial}{\partial x}\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right]^{-1} \frac{\delta H}{\delta u} \tag{16}$$

Eqs. (14), (16) give us the Hamiltonian structure for the problem.

$$u_t = \frac{\partial}{\partial x}\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right]^{-1} \frac{\delta H}{\delta w}, \qquad w_t = \frac{\partial}{\partial x}\left[1 - \beta_1 \frac{\partial^2}{\partial x^2}\right]^{-1} \frac{\delta H}{\delta u} \tag{17}$$

2.4. Interaction of quasi-particles in the Boussinesq paradigm

As already above mentioned, the most important impact of the work of Boussinesq is that it led to the mathematical discovery of the permanent solitary waves which possess some particle-like properties, e.g. preservation of identity (approximate or strict preservation of the shape) upon collisions, and balance and conservation laws for the main characteristics. The collision property of solitary waves was studied to great extent in fully integrable systems for which the so-called two-soliton solutions exist (KdV, Klein–Gordon, BO). In the more practical, and non-integrable, case of the Boussinesq Paradigm Equation (1) with $\beta_1 = 0$, the only approach to studying the dynamics of interaction of solitary waves is to construct a numerical scheme that faithfully represents the main conservation laws, the latter being discussed in the previous subsection. The conservative-scheme approach has been pursued by the authors, among others. The schemes are well documented and it goes beyond the framework of the present paper to present the algorithms. In order to give the flavor of the situation we consider the case $\beta_1 = 1/2$, $\beta_2 = 1/6$ for Eq. (1) which case corresponds to the fluid Boussinesq model and set for definiteness $c_0 = 1$. As already mentioned, the solitons can be either sub-critical or super-critical. The former are depressions, while the latter are elevations. The depressions are pertinent only for sufficiently small phase velocities, because for phase velocities close to the threshold of their existence $c = 1/\sqrt{3} \approx 0.57735$ they are no longer long waves. In turn, the super-critical quasi-particles acquire very large amplitudes for large phase speeds, and then the assumption of weak nonlinearity does not hold. Keeping these limitation in mind, we present in Fig. 1 some typical interactions (see [6,7]) between the solitary waves of BPE (1).

For small supercritical phase speeds the interaction is virtually elastic, save the phase speed. Increasing the phase speed makes the system more nonlinear and some radiation is created (see the upper panel of Fig. 1) which propagates with the characteristic speed and trails the solitons. It is seen that even for such large supercritical phase speeds, the solitons regain their shapes and retain their individuality. In the same way, the sub-critical depressions for low phase speeds collide elastically too (see the lower panel of Fig. 1) provided that the phase speeds are close to the threshold value of $c^* = 1/\sqrt{3} \approx 0.57735$.

All the above shows the rich phenomenology of the quasi-particle interactions and the important new avenue of research opened by Boussinesq's idea of the balance between dispersion and nonlinearity.

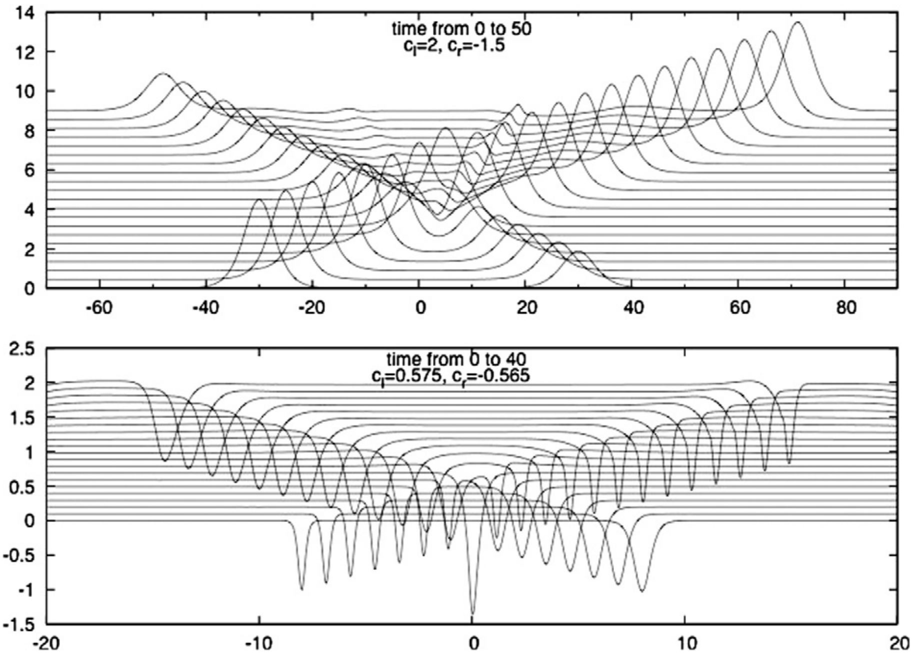

Fig. 1. Interaction of solitons in PBE (1) for $\beta_1 = 3$, $\beta_2 = 1$ and $c_0 = 1$. Upper panel: two super-critical solitons, $c_l = 2$, $c_r = -1.5$. Lower panel: two sub-critical solitons $c_l = 0.575$, $c_r = -0.565$.

Fig. 1. Interaction de solitons pour PBE (1) pour $\beta_1 = 3$, $\beta_2 = 1$ et $c_0 = 1$. Panneau du haut : deux solitons super-critiques, $c_l = 2$, $c_r = -1, 5$. Panneau du bas : deux solitons sous-critiques, $c_l = 0{,}575$, $c_r = -0{,}565$.

3. Generalized Boussinesq systems

3.1. First type

Generalizations of the original Boussinesq equation (1) come in essentially two types illustrated by Eqs. (2) and (3). We consider first the second type with $\beta_1 = 0$, $\beta_2 = \beta$ that includes higher order purely spatial gradients. This corresponds to a class of models extensively studied by the authors [22–25]. The main idea there is that additional gradients obtained in a better approximation of a lattice model provide a better and more realistic linear *dispersion*. Considering these higher order derivatives as perturbations yields singular expansions that are only valid for short distances of travel. These additional terms cannot be treated as perturbations for long transients, whence the need to develop appropriate numerical schemes for these 'stiff' problems while exact integrability is obviously lost. But "who cares about integrability"? [26], insofar as the model seems to be physically faithful. A typical system illustrating all these properties is given by [27]. With $s = v_x$ a shear strain, from a lattice dynamics approach and a long-wavelength limit while neglecting couplings with other strain components, one obtains an equation which is of type of Eq. (3):

$$s_{tt} - c_0^2 s_{xx} - \left[F(s) - \beta s_{xx} + s_{xxxx}\right]_{xx} = 0 \tag{18}$$

where F is a polynomial in s starting with second degree (e.g., a nonconvex function admitting three minima) and β is positive. The boundary conditions for this equation on the infinite interval are $s = s_x = s_{xx} \to 0$, $x \to \infty$. We may say that *both* the nonlinearity and dispersion have been increased compared to the classical BO equation [28]. The Boussinesq idea of balance between the nonlinearity and dispersion holds for the last equation too. There are some differences which manifest themselves in the fact that system (18) admits analytical expression of the type of the ubiquitous *sech* (but to the fourth power) for the solitary-wave solutions only for a single value of c, while numerically the existence of different solitary-wave solutions with a continuous spectrum for c was also shown [27,29]. However, the preservation of the shapes upon collisions is not perfect and pulses are being created during the clash of the two main solitons and these pulses are shown numerically to behave almost as solitons, save some 'aging' (self-similar behavior as the amplitude of a pulse decreases with time while its support increases resulting in 'redshifting' of the frequency of the pulse). This effect was first observed in [23], and confirmed in [20]. Remarkably, such pulses practically pass through each other without changing qualitatively their shapes with perfect conservation of 'mass' and 'energy', so that these pulses may qualitatively be claimed as solutions. For c close to c_0, the amplitude of such a soliton approaches zero transforming the latter into a linear harmonic wave.

An equivalent evolution system can be used in lieu of Eq. (18).

$$s_t = q_{xx}, \qquad w = s_{xx}, \qquad q_t = c_0^2 s + F(s) - \beta w + w_{xx} \tag{19}$$

In fact, remembering that $s = v_x$, employing the canonical formalism (Appendix A1, [14]), and introducing w by $w = s_{xx}$, the mass, canonical momentum and energy are given by

$$M(R) = \int_R s\,\mathrm{d}x = [v]_{-\infty}^{+\infty} \tag{20}$$

$$P(R) := -\int_R v_x v_t\,\mathrm{d}x = \int_R s q_x\,\mathrm{d}x \tag{21}$$

$$E = H(R) = \int_R \frac{1}{2}\left\{v_t^2 + c_0^2 v_x^2 - 2F(v_x) + \beta v_{xx}^2 + v_{xxx}^2\right\}\mathrm{d}x$$
$$= \int_R \frac{1}{2}\left\{q_x^2 + c_0^2 s^2 - 2F(s) + \beta s_x^2 + w^2\right\}\mathrm{d}x \tag{22}$$

where we assumed that $v_t(-\infty) = 0$. It is checked that we have

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}P}{\mathrm{d}t} = F_d = -[s_{xx}^2]_{-\infty}^{+\infty} \tag{23}$$

the last of which indicates what conditions at infinity must be imposed on the solution so that it is a localized nonlinear wave in *inertial* motion (vanishing of the driving force F_d). A strongly implicit conservative scheme [25] was used in order to always preserve both M and E, while the driving force F_d for a finite interval of integration is felt only when the soliton hits the boundaries and rebounds from them. More on this in Refs. [25] and [18]. All what needs to be noted is that solutions of the *subsonic* solitary-wave type of Eq. (20) can transform to damped oscillatory ones by changing the coefficient β of fourth-order dispersion. Increasing β one reaches a threshold above which the localized waves acquire oscillatory tails, corresponding to solutions usually called *Kawahara solitons* [30] (see, also [31] for a precise numerical algorithm for identifying the shapes of stationary Kawahara solitons). As to *supersonic* solutions, regardless of the sign of β, solutions of a new type arise, being made of a central hump and lateral oscillating 'wings' of long or infinite support, so-called 'nanopterons' [32] (if the wings have small amplitude). These are 'nonlocal' solutions as obtained in Ref. [25].

The shallow-layer BO problem is revisited in Ref. [25] and shown that it can be recast into a generalized Boussinesq equation with purely spatial dispersion operators similarly to Eq. (20) as it does involve sixth-order space derivatives. However, it is shown to possess a wave mechanical interpretation of a totally different type, including a so-called *gyroscopic* or *d'Alembertian inertia* term [18], pp. 148–149.

3.2. Second type

We put in this second type generalized BO systems that involve mixed space-time derivatives or multiple wavelike operators. Of course the distinction between space and time derivatives is not so much relevant from the solution viewpoint since all depends on the sign of coefficients, but it does make an essential difference in the wave mechanical interpretation that can be reached. Eqs. (1) and (3) are in the first class, as is the following *nonlinear Maxwell–Rayleigh equation* (our denomination [33,18]), also a generalization of Boussinesq's proposal:

$$u_{tt} - \left[u + F(u) + \beta u_{tt}\right]_{xx} + \beta u_{tttt} = 0, \quad \beta > 0 \tag{24}$$

Of course this can be rewritten as

$$u_{tt} - u_{xx} - \left[F(u)\right]_{xx} - \beta(u_{xx} - u_{tt})_{tt} = 0 \tag{25}$$

which is an example of a *double-dispersion equation* with two wave operators present. In the theory of rods [11,12, 17], the two wave operators are indeed different, viz, typically

$$u_{tt} - c_0^2 u_{xx} - \big[F(u)\big]_{xx} + \beta(u_{tt} - c_1^2 u_{xx})_{xx} = 0 \tag{26}$$

As in the original Boussinesq fluid-mechanics derivation, it is the elimination of the behavior transverse to the direction of propagation that introduces linear dispersion. The solution of systems of the type (26) has been extensively studied by use of the direct technique of Weierstrass elliptic functions by Samsonov et al. [17] to whom we refer the reader. We rather contemplate the strange wave mechanics associated with Eq. (24) and its system of conservation laws after introducing the potential $\bar{u}$ by $u = \bar{u}_x$. We have thus total mass, canonical momentum and energy defined by

$$M(R) = \int_R \bar{u}_x \,\mathrm{d}x = \int_R u \,\mathrm{d}x = [\bar{u}]_{-\infty}^{+\infty} \tag{27}$$

$$P(R) = -\int_R \bar{u}_x \frac{\delta l}{\delta \bar{u}_t} \,\mathrm{d}x = -\int_R u\big\{q - \beta(q_{xxt} - q_{xxx})\big\} \,\mathrm{d}x \tag{28}$$

$$E(R) = \int_R \left(k + \left(\frac{1}{2}\bar{u}_x^2 - U(\bar{u}_x)\right)\right) \mathrm{d}x = \int_R \frac{1}{2}\big(u^2 + q_x^2 - \beta q_{xt}^2 - 2U(u) + \beta u_t^2\big) \,\mathrm{d}x \tag{29}$$

The second of each of (28) and (29) hold true because (26) can be rewritten as the *Hamiltonian system*

$$u_t = q_{xx}, \qquad q_t = \beta(q_{txx} - q_{xxx}) + u + F(u) \tag{30}$$

and we can define a local kinetic energy and Lagrangian density by

$$k = \frac{1}{2}\big(\bar{u}_t^2 + \beta(\bar{u}_{xt}^2 - \bar{u}_{tt}^2)\big), \qquad l = k - \left(\frac{1}{2}\bar{u}_x^2 - U(\bar{u}_x)\right) \tag{31}$$

The mechanical interpretation of this kinetic energy poses a problem, although we identify a 'lateral inertia' part while the potential energy is standard. Total canonical momentum will not be conserved, hump-like solutions being driven by a driving force unless specific conditions at infinity are applied. A similar difficulty arises with the regularized long-wave Boussinesq equation, cf. [18].

3.3. Quasi-particles in Generalized Boussinesq systems

As already mentioned, the solitary waves for the Generalized Boussinesq equations with higher-order dispersion come in different varieties: the ubiquitous bell-shaped or hump (*sech*) solitons and Kawahara solitons. While the dynamics of humps can be expected to be qualitatively similar to the PBE case (save some increased radiation due to the stronger dispersion), the question of what is the dynamics of Kawahara solitons is rather intriguing and can be answered only numerically. In [25] we have investigated the last case and found that Kawahara solitons behave as quasi-particles too. In Fig. 2 we show the collision of two Kawahara solitons as governed by Eq. (18).

It is seen that apart from somewhat increased production of oscillations, the predominant part of the interaction is solitonic, and the intricate Kawahara shapes recover completely their identity after the interaction.

4. Dissipative Boussinesq systems

Dissipation is naturally introduced in fluid dynamics through viscosity processes. This is also the case in solids but more rarely. Presence of dissipation in the BO equation generally destroys the balance between nonlinearity and dispersion. We can expect an attenuation of the bell-shaped solitary waves. More mathematically, added dissipation spoils immediately integrability of the model and little can be done analytically. But dissipation is a physically relevant phenomenon that justifies the numerous studies of its influence on the BO equations of different types and their solutions [34–39]. Typically, dissipation introduces terms with odd-order derivatives in the BO equation (accordingly, terms with even-order derivatives in the corresponding generalized KdV equation). Yet, if an energy production (input) is allowed in the model, another version of Boussinesq Paradigm can be encountered: balance between the energy input and dissipation modulated by the presence of nonlinearity and dispersion.

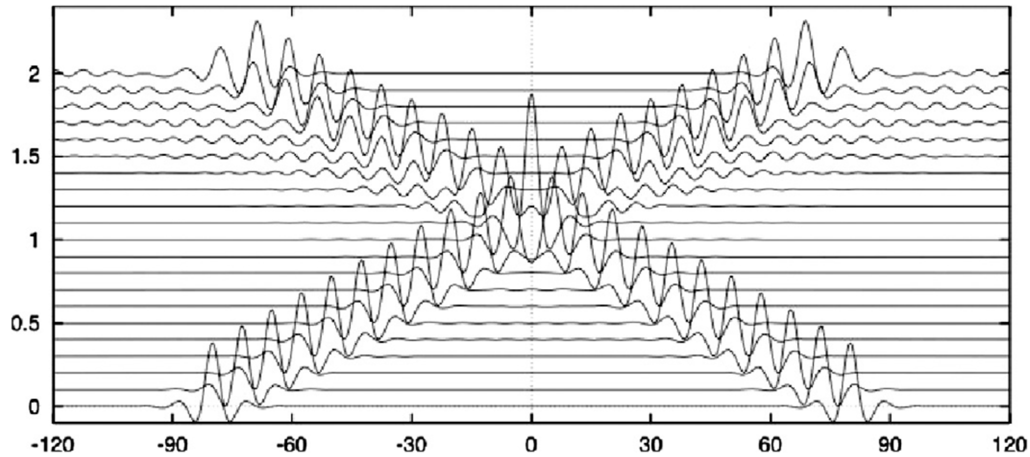

Fig. 2. Interaction of Kawahara solitons in Eq. (18) for unit characteristic speeds and equal amplitude.

Fig. 2. Interaction de solitons de Kawahara d'après l'Éq. (18) pour des vitesses caractéristiques unitaires et des amplitudes égales.

An example of dissipative modification of KdV is provided by the equation known as the *KdV–Kuramoto–Sivashinsky* equation (in fact a generalization of this equation is also available when Marangoni effect in the surface tension on the surface of a fluid layer is accounted for and known as *Kuramoto–Sivashinsky–Velarde* equation). For more details on the derivation of KdV–KSV one is referred to [29,30,35] where the shapes of the permanent waves are computed and the uni-directional propagation is thoroughly investigated leading in [34] to the introduction of the notion of a 'dissipative soliton'.

The generalization to the case of bi-directional waves requires that an equation of Boussinesq type be considered, i.e., an equation with second time derivative. Such a generalization was outlined in [41], and further studied in [42]. We limit ourselves to the case without Marangoni effect, in order to elucidate the interplay between the nonlinearity on one hand, and dispersion, energy input and dissipation—on the other. The prototypical Boussinesq-type of equation with dissipation and energy input then reads

$$v_{tt} = \left[c_0^2 v - \frac{1}{2}\alpha v^2 - \beta v_{xx} - \alpha_2 v_t - \alpha_4 v_{xxt}\right]_{xx} \tag{32}$$

Or rewritten as a system (showing its similarity to the above treated cases)

$$v_t = q_{xx}, \qquad q_t = c_0^2 v - \frac{1}{2}\alpha v^2 - \beta v_{xx} - \alpha_2 v_t - \alpha_4 v_{xxt} \equiv c_0^2 v - \frac{1}{2}\alpha v^2 - \beta v_{xx} - \alpha_2 q_{xx} - \alpha_4 q_{xxxx} \tag{33}$$

This conserves many traits of a wave equation for a short time scale. The long-time scale essentially reduces to the evolution equation of KdV–KS type and was thoroughly examined in [40] and [34]. Although kink solutions also exist but for different limit conditions on v, with $v = u_t$, mass and energy for *hump*-type solutions of Eq. (32) are given by (with limit conditions $v(\pm\infty) = 0$)

$$M(R) = \int_R v\,\mathrm{d}x = -\frac{1}{c}\int_R u_x\,\mathrm{d}x = \frac{1}{c}[u]_{+\infty}^{-\infty} \quad \text{and} \quad E(R) = \int_R \frac{1}{2}\left[v^2 + q_x^2 - \frac{1}{3}\alpha v^3 + \beta v_x^2\right]\mathrm{d}x \tag{34}$$

so that energy is not conserved in general but the following balance law holds

$$\frac{\mathrm{d}E}{\mathrm{d}t}(R) = \alpha_2 \int_R v_t^2\,\mathrm{d}x - \alpha_4 \int_R v_{xt}^2\,\mathrm{d}x \equiv \alpha_2 \int_R q_{xx}^2\,\mathrm{d}x - \alpha_4 \int_R q_{xxx}^2\,\mathrm{d}x \tag{35}$$

containing both an attenuation due to higher-order dissipation and an energy supply.

This issue of dissipation vs. energy supply or *attenuation vs. amplification* is an important one in generalized BO systems with dissipation. In particular, balance may occur between nonlinearity, dispersion, dissipation and activation,

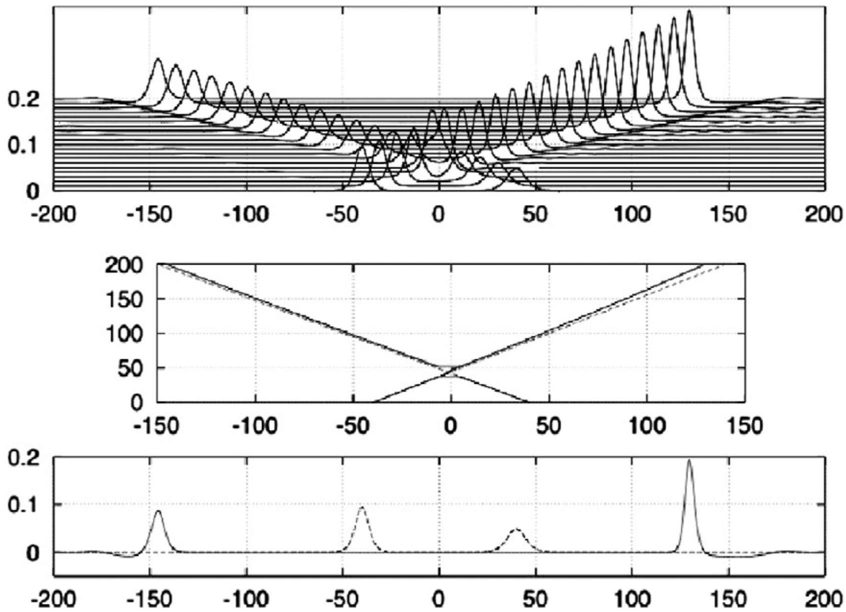

Fig. 3. Interaction of two humps as governed by Eqs. (32) for $\alpha_2 = 1$, $\alpha_4 = 0.4$, $c_0 = 1$, $\beta = 1$. Upper panel shows the collision. Middle panel shows the trajectories of the centers of quasi-particles, Lower panel compares the initial and final wave profiles.

Fig. 3. Interaction de deux solitons « bosse » gouvernés par les Éqs. (32) pour $\alpha_2 = 1$, $\alpha_4 = 0{,}4$, $c_0 = 1$, $\beta = 1$. Le panneau du haut montre la collision ; celui du centre montre les trajectoires des centres des quasi-particules. Celui du bas compare les profiles initiaux et finaux des ondes.

favoring the propagation of localized waves of permanent shape. Amplification may take place in certain favorable circumstances (e.g., in a narrowing rod, during the occurrence of rogue waves in fluids, in systems with external energy supply [38,43]).

The fact that the new kind of balance can also result in a quasi-particle behavior of the localized solution was demonstrated in [41,42]. The crucial difference here is that a stationary propagating localized shape does not exist for a continuous spectrum of phase velocities. This means that a hump of general form is not a stationary 'creature'. As shown in [41,42] a *sech* evolves during its translation even before colliding with another *sech*, and yet the actual interaction is quite close to solitonic. This means that on time scales shorter than the scale of 'aging' of the solitary shape, the latter behaves similarly to the quasi-particles in conservative systems. Fig. 3 depicts the situation.

It is clearly seen that the coherent structures retain some individuality upon collision, but the energy-input mechanism speeds up their evolution, and the terminal shapes are actually larger and with higher energy than the initial ones. This reflects the fact that during the collision, the total shape of the wave profile has larger energy production than energy dissipation. When the shapes separate enough, they continue their slower evolution as governed by the balance of the above two effect. The quasi-particles grow/shrink until a shape is reached which makes the r.h.s. of Eq. (35) equal to zero.

It is known that strictly stationary solitary wave solutions of Eq. (34) (and for that matter of KdV–KSV) exist only for a discrete spectrum of phase speeds. The mathematical issues connected with the stationary shape are elucidated in [40] where a precise numerical approach is developed to identify the value of the phase velocity for which the shape exists and to compute the shape itself. As shown in [42], the shape is the same for the sub-critical and super-critical structures, just the phase speeds are different. We present here in Fig. 4 the interactions of two stationary dissipative solitons as governed by Eq. (34).

It is seen in the figure that despite of the non-conservative (and non-integrable) nature of the problem, the interactions are indeed solitonic, i.e. the shapes of the two quasi-particles are strictly restored after the interaction. This means that the new kind of balance can be the key to a phenomenon which can be called 'dissipative solitons' or 'dissipative quasi-particles'. This shows the fertility of the original Boussinesq idea to look for solutions that are sustained by some kind of balance, but otherwise propagate as the solution of simple hyperbolic equations.

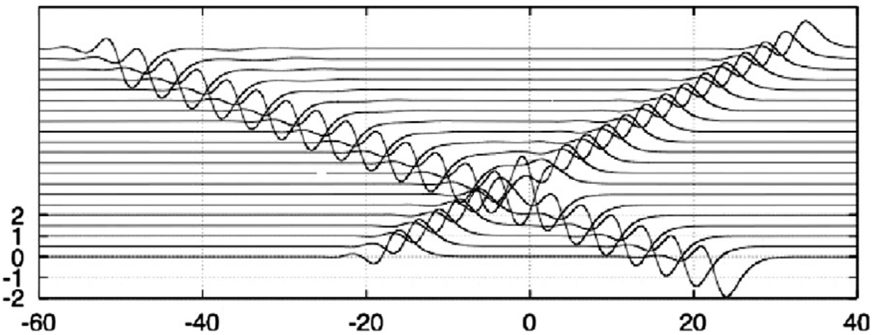

Fig. 4. Interaction of two initially stationary dissipative structures for characteristic speed $c_0 = 3$, $\beta = 1$, $\alpha_2 = \alpha_4 = 1$ and phase speeds (subcritical and supercritical, respectively): $c_l = 2.453$, $c_r = -3.667$.

Fig. 4. Interaction de deux structures dissipatives initialement en mouvement stationnaire pour une vitesse caractéristique $c_0 = 5$, $\beta = 1$, $\alpha_2 = \alpha_4 = 1$ et des vitesses de phase (respectivement sous-critique et super-critique) $c_l = 2{,}453$, $c_r = -3{,}667$.

5. Two-dimensional systems

Two-dimensional generalizations of the BO equations naturally occur in studying two-dimensional lattices and their continuum limit, and the dynamics of elastic plates. This is exemplified by the work of Porubov et al. [44] where the following type of equation is obtained for one-displacement component $u(x, y)$ (in the direction of propagation x):

$$\begin{aligned} &u_{tt} - a_1 u_{xx} - a_2 u_{yy} - (3a_1/2 + a_3)(u_x^2)_x \\ &\quad - (a_1/2 + a_4)\big[(u_y^2)_x + 2(u_x u_y)_y\big] - a_4 h^2 \Delta u_{xx} + a_6 h^2 u_{xxtt} = 0 \end{aligned} \tag{36}$$

where Δ is the two-dimensional Laplacian. This reduces to a *double*-dispersion equation in the one-dimensional case $u(x)$ only.

For a case involving a macro-displacement u and a micro-displacement v in a microstructured material, but with a 'weak' transversality (quasi-uni-dimensional system), a typical example is provided by the system of two equations where dissipative contributions (odd order derivatives) have been incorporated [45]

$$\begin{aligned} &u_{tt} - u_{xx} - \kappa b_1 v_{xx} - \varepsilon\alpha_1 (u^2)_{xx} - \gamma\alpha_2 u_{xxt} \\ &\quad + \delta\alpha_3 u_{xxxx} - \delta\alpha_4 u_{xxtt} + \gamma\delta(\alpha_5 u_{xxxxt} + \alpha_6 u_{xxttt}) + h.o.t. = 0 \\ &(b_1 + \mu)v_{tt} - \mu v_{xx} - b_1 u_{yy} + h.o.t. = 0 \end{aligned} \tag{37}$$

With scaling $\delta = 0(\varepsilon)$, $\varepsilon \ll \gamma \ll 1$, α_2 of order ε and all dissipative terms of the same order, introducing slow and fast variables we would obtain a generalization of the *Kadomtsev–Petviashvili equation* that admits two-dimensional travelling localized wave solutions provided $\alpha_3 > \alpha_4$. Solutions in the shape of a 'Mexican hat' are obtained along the propagation direction while a monotonous decay is observed in the transverse direction (see Fig. 5). However, a transverse *amplification* is obtained for definite values of α_5 and α_6 and negative α_2. Increase in amplitude is accompanied by a decrease in the width of the wave.

In the case of Eq. (38), numerical simulations of the evolution of a two-dimensional Gaussian profile benefit from the transverse localization feature and exhibit dramatically first the formation of a two-dimensional hump in a short time and then the increase in amplitude accompanied by the formation of depressions; see figures in Ref. [44].

The full-fledged 2D Boussinesq equation has been investigated much less than the Kadomtsev–Petviashvili model in which in one of the direction the fourth derivatives are missing. The case with full bi-harmonic spatial operator turns out to a hard problem for numerical treatment, especially of one is concerned with finding the 2D shape that is propagating strictly stationary. This is a bifurcation problem over infinite domain and requires special approaches. We have presented in [47] some preliminary results in this case using a difference scheme, and confirmed them in [48] using a specialized spectral method based on a complete orthonormal system of functions proposed in [60]. We consider the 'good' BO in 2D, namely

$$u_{tt} = \Delta u - \Delta(u^2) - \Delta^2 u \tag{38}$$

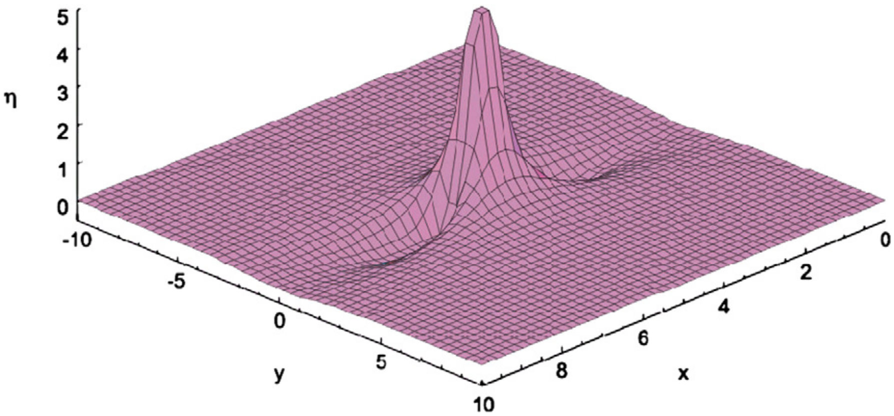

Fig. 5. A typical 2D shape: 'Mexican-hat' section with side depletions along the propagation direction x, and monotonous decrease on both sides orthogonally to that direction. After [45], p. 516.

Fig. 5. Une forme 2d typique : un « chapeau mexicain » avec des creux latéraux dans la direction de propagation x et une décroissance monotone de chaque côté, perpendiculairement à cette direction. D'après [45], p. 516.

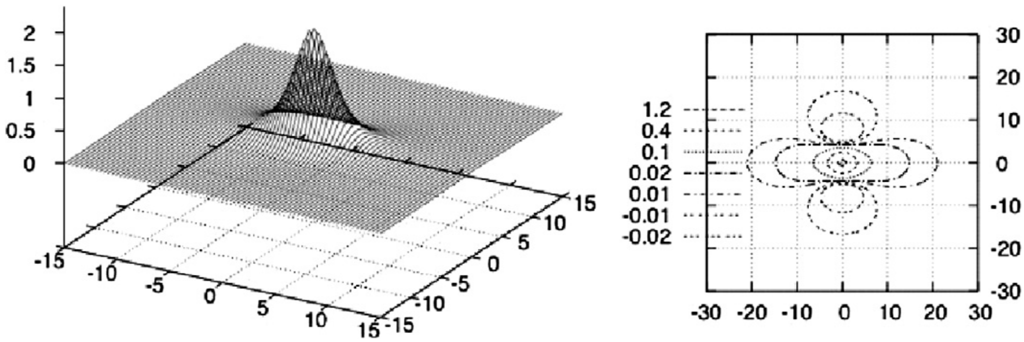

Fig. 6. The shape of the two-dimensional stationary soliton in the 'good' 2D Boussinesq equation propagating in y-direction with phase speed $c = 0.7c_0$, i.e. with phase velocity $(c_x, c_y) = (0, 0.7c_0)$. Note that in this case $c_0 = 1$.

Fig. 6. La forme d'un soliton 2d « stationnaire » pour la « bonne » équation de Boussinesq à deux dimensions, se propageant dans la direction y avec une vitesse de phase $c = 0{,}7c_0$, soit $(c_x, c_y) = (0, 0{,}7c_0)$. A noter que $c_0 = 1$ dans ce cas.

for which the solitons are sub-critical. It is well known in 1D that the supercritical solitons undergo a Lorentz-like elongation in the direction of motion, eventually becoming infinitely long waves upon reaching the characteristic speed. It is of crucial importance to understand the nature of the elongation/contraction property in 2D. Our results show that, indeed, the total support of the soliton does increase with the increase of the characteristic speeds, but there is relative contraction in the direction of the motion, and fore- and back-runners are formed that are depressions, when the main hump is an elevation. This behavior is shown in Fig. 6 for the case of the rather large (but still sub-critical) propagation speed. Note that some mathematical properties of systems such as (38), with added damping, are reported by Varlamov for cylindrical symmetry in the present issue [61].

6. Conclusions

Here above we have intentionally remained with the bi-directional BO equation, although the *reductive-perturbation method* [46] associates more or less automatically generalized KdV equation with generalized BO equations, in particular KdV higher-order systems (e.g., [49,50]), so that we could speak as well of a *KdV paradigm*. We preferred to stay most of the time with the bi-directional systems which are closer to mechanical interpretation in solid mechanics. In particular, we know that the models issued from lattice-elastodynamics allow an exhibition of a complete zoo of soliton-like behaviors (cf. [51]). Sometimes, however, different techniques are needed to exhibit some specific types of solutions such as the mechanical equivalent of bright solitons. Thus the kinematic wave-mechanics of Lighthill, Whitham and Hayes (cf. [52]) and the Whitham–Benney–Newell averaged Lagrangian method may be called for obtaining such solutions directly form the BO equation (cf. [53]). There may also be abuses in the consideration of the BO–KdV equation in modeling some physical problems. This is the case with nonlinear surface wave propagation for seismic purposes [54], which in fact requires a much more careful and complicated modeling (in particular to account for the typical in-depth (transverse) behavior of a surface wave over a solid complex structure [55].

From the point of view of the mechanics of waves, we have emphasized the role played by canonical conservation laws, in particular that of *canonical momentum*. This was first noticed in [56] in relation with the so-called canonical formulation of continuum mechanics. In particular, the driving forces that may appear inconspicuously in the equation of canonical momentum are related to the notion of the *Eshelby stress tensor* often advertised by one of the authors (GAM). This also affords a means to treat perturbations and the noninertial motion of some known solitonic solutions in accord with the study of the dynamical response of nearly integrable systems [57]. On establishing relations existing between total canonical mass, momentum and energy for some specific solutions, new types of 'point mechanics' emerge. This is the case for generalized systems such as (18)—for these see Refs. [58,59].

In conclusion, we have gone far away from the initial penetrating work of Boussinesq, uncovering a whole menagerie of systems which are but slight generalizations of his own invention. However, to treat these mathematically and numerically, it was necessary to exploit methods and techniques that are much more recent. This is but an additional tribute to the inventiveness and foresight of J.V. Boussinesq.

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