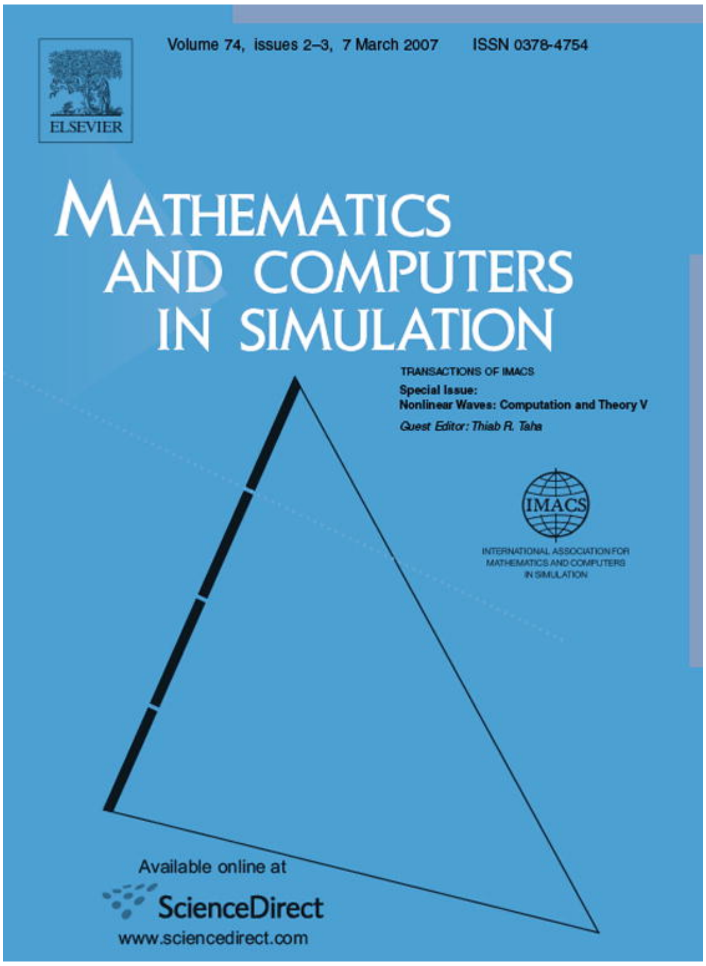
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Maxwell–Lorentz electrodynamics as a manifestation of the dynamics of a viscoelastic metacontinuum

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Abstract

We prove that, when linearized, the governing equations of an incompressible viscoelastic continuum can be rendered into a form identical to that of Maxwell's equations of electrodynamics. The divergence of deviator stress tensor is analogous to the electric field, while the vorticity (the *curl* of velocity field) is interpreted as the magnetic field. The elastic part of constitutive relation explains Maxwell's displacement current, and is responsible for the propagation of gradient (shear) waves. In turn, the viscous part is associated with the Ampere's and Ohm's laws for the current. This analogy is extended further and the nonlinearity of the material time derivative (the advective part of acceleration) is interpreted as the Lorentz force. The classical wave equations of electrodynamics are also derived as corollaries. Thus an interesting and far reaching analogy between the viscoelastic continuum and the electrodynamics is established.

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Keywords: Maxwell–Lorentz electrodynamics; Shear waves; Viscoelastic liquid

1. Introduction

"... according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. According to the general theory of relativity space without ether is unthinkable; ...". A. Einstein ([14], p. 23).

Along with gravitation, electromagnetic phenomena are an example of an action at a distance. The latter needs a transmitter, a kind of continuous media called a *field*. The only feasible explanation of the action at a distance is the propagation of the internal stresses in a continuum. For this reason, it is still a valid avenue of research to attempt to understand the luminiferous (electromagnetic) field as a material continuum in which the *internal stresses* are the transmitter of the long-range interactions. When considering the propagation of electrodynamic oscillations in space devoid of matter, the obvious candidate for the field is the elastic medium because – as shown by Cauchy (see Ref. [28]), much before the time when the light was recognized as an electromagnetic phenomenon – it gives a good quantitative

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prediction for the shear-wave phenomena (light) and explains quantitatively very well the experiments of Young and Fresnel. In our previous works [5,6,7(Part I)], the derivation of Maxwell's equations in vacuo from the equations of Hookean elastic continuum was presented. We showed that the conjecture of Cauchy that the elastic continuum should have vanishing dilation modulus (so-called "volatile" continuum) does not hold and the other extreme – a virtually incompressible continuum – is fully compatible with Maxwell's electrodynamics. While the Hookean elastic medium predicts splendidly the phenomena connected with high-frequency shear oscillations (light), it has one major deficiency: it does not allow currents in the electrostatic limit. In the present work, an attempt is made to generalize the model to account also for currents in the electrostatic limit. The only way to do this, according to our conjecture, is to assume that the electromagnetic field is a material continuum possessing the properties of both elastic bodies and viscous liquids.

Scrutinizing the theory of electrodynamics reveals two different facets: elastic and viscous. The viscous "personality" shows up in slow processes, notably in electrostatics, especially in the combination of Ampere's and Ohm's laws. The electric field creates a current in a very similar fashion as force creates flux in a viscous liquid. Clearly, Ohm's resistance law is a direct analogy of the viscous drag law in Newtonian (Navier–Stokes) liquids. In turn, the elastic nature shows trough the so-called "displacement current" conjectured by Maxwell. Although introduced in somewhat of an *ad hoc* manner by Maxwell, the displacement current was crucial to explain the propagation of light and the phenomenon of capacitance (see the illuminating discussion in Ref. [17], p. 274).

A note is due on the usage here of the concept of continuous medium. We do not consider the propagation of electromagnetic waves in classical continua. We refer the reader to the exhaustive treatise [15] for the interaction between electromagnetic field and elastic continuum. Here we attempt to understand the electromagnetic phenomenon through its analogy to viscoelastodynamics. The notion of continuum in the present work is about the underlying nature of the field. It should be more properly addressed as a *metacontinuum* to distinguish it from the ubiquitous continuous model that is an approximation for systems of very large number of interacting particles. Later in this paper, we will discuss the problem of interaction between the metacontinuum and the moving material charges by considering the latter as phase patterns propagating *over* the field, rather than point objects moving *through* the field.

The absolutivity of the speed of light means that the electromagnetic waves are propagating in a continuum that is not affected by the motion of material bodies. Unfortunately, the original concept of an absolute continuum, called "the ether", relied squarely on the belief that it permeates the space between atoms and that the particles and charges (the "matter") are moving through it. In the present work, we will abandon this assumption. Rather, we put forward the concept that a particle is our "coarse-grain" perception for a localized wave in the metacontinuum. According to soliton theory, the localized waves can interact as particle-like objects which are often called "quasi-particles".

2. The model of viscoelastic liquid

In the Eulerian framework, the Cauchy balance equations read (see e.g., [27])

$$\mu\left[\frac{\partial \boldsymbol{v}}{\partial t}+\boldsymbol{v}\cdot\nabla\boldsymbol{v}\right]=-\nabla p+\nabla\cdot\mathcal{T},\qquad \boldsymbol{v}\equiv\frac{\partial \boldsymbol{u}}{\partial t}, \tag{1}$$

where μ is the density of the metacontinuum, p the pressure, $\boldsymbol{u}$ the displacement vector, $\boldsymbol{v}$ the velocity vector, and $\mathcal{T}$ stands for the so-called "deviator" stress tensor. The Cauchy balance, Eq. (1), is coupled by the continuity equation

$$\frac{\partial\mu}{\partial t}+\nabla\cdot(\mu\boldsymbol{v})=0. \tag{2}$$

The constitutive relation for viscoelastic continuum admits two different physical models. One of them considers a medium which is basically an elastic body endowed with dissipation. This is the so-called Voigt model (or in more general setting—Kelvin–Voigt) [3]. The essential mechanical trait of Kelvin–Voigt model is that an individual volume can retain its content during its motion while experiencing deformation. A different viscoelastic model is due to Maxwell from his work on the theory of gases [23] where he assumes that the continuum is behaving basically as a liquid (the individual parcels do not preserve their actual content of material points). However, when faster changes in the deformation are in place, there is some elastic relaxation of the stress. As it has been already above mentioned, we choose the Maxwell viscoelastic liquid in order to explain the flow-like nature of the currents and the magnetic phenomenon.

Maxwell liquids are extensively studied in engineering fluid mechanics. The constitutive relation for the Maxwell model of viscoelastic liquid involves a time derivative of the stress tensor. The simplest approach is to use merely the partial time derivative, but then the constitutive relation is not material invariant (or material frame indifferent). The natural generalization is to use the convective derivative of the stress tensor. For homogeneous and isotropic deformations, the Cauchy balance Eq. (1) is coupled by the following relation (see e.g., [2,12] for a detailed coordinate form)

$$\tau\left[\frac{\partial \mathcal{T}}{\partial t} + \boldsymbol{v}\cdot\nabla\mathcal{T}\right] + \sigma\mathcal{T} = \mathcal{E} \stackrel{\text{def}}{=} \frac{\eta}{2}[\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^{\mathrm{T}}] + \lambda\mathcal{I}\nabla\cdot\boldsymbol{v}, \tag{3}$$

where $\mathcal{E}$ is the tensor of rate of deformations and $\mathcal{I}$ is the unit tensor.

Along with the convective derivative, the so-called upper- and/or lower-convected derivatives are considered (see Ref. [2,25]). At this stage, it is not clear what is the best choice for the invariant derivative to be used in the Maxwell constitutive law for viscoelastic liquids. This is more true for the viscoelastic metacontinuum considered here. The problem is addressed in Refs. [8,9], where the pertinent formulation for the rheological law involving relaxation is discussed in connection with the displacement current. It turned out that Oldroyd's upper convected derivative is the one that suits best the formulation of the electrodynamics as a viscoelastic medium. It goes beyond the framework of the present paper to present the details. Moreover, for the purposes of illustration of the transition form the viscoelastic model to electrodynamics, we consider here the linearized version.

In Eq. (3) τ is the Maxwell relaxation time and σ is a coefficient of proportionality which can be taken $\sigma = 1$ when viscosity is always present. For $\sigma \neq 0$ the rheological relation can be rescaled, but it is more convenient to keep σ it in the constitutive relation for the sake of generality. Note that for $\sigma = 0$, we recover the constitutive relation for a purely elastic medium.

The interpretation of the two coefficients η and λ is different in the pure viscous and pure elastic cases. When $\tau = 0$, $\sigma \neq 0$ one has a purely viscous model for which $\eta\sigma^{-1}$ and $\lambda\sigma^{-1}$ have, respectively, the meaning of shear and bulk coefficients of viscosity. When $\sigma = 0$, $\tau \neq 0$, Eq. (3) can be considered as the time derivative of the Hookean elastic constitutive law and then $\eta\tau^{-1}$ and $\lambda\tau^{-1}$ have the meaning of shear and dilational Lamé coefficients for an elastic medium.

The above introduced viscoelastic liquid has a rich phenomenology and we have to select the scales for the different parameters in order to get into the ranges where it pertains to the actually observed phenomenology in electrodynamics. As already mentioned, the elastic nature dominates the picture for fast oscillations, and we will assume that

$$\sigma \ll \tau, \tag{4}$$

which gives a reasonable model for a viscous liquid that behaves almost as an elastic body when the time scales of the motion are very short.

This means that when fast oscillatory motions are considered, the term proportional to σ plays the role of a very small dissipation (attenuation in this case) of the respective waves. This complies with the observations that the plane light waves can propagate at long distances with no appreciable attenuation. We will return in more detail to this point later in the text. Note that Eq. (4) does not imply that the metacontinuum will behave as a slightly viscous gas. The careful examination of the statics shows that when stationary motion is concerned, the material behaves as a viscous liquid with shear coefficient of viscosity $\eta\sigma^{-1} \gg 1$ for small σ. This means that in creeping motions, the medium will behave as an extremely viscous liquid. These two properties: small attenuation of fast oscillations and very viscous behavior of the steady flows are the properties of a jello-like material. In this sense, the metacontinuum proposed here is not an ethereal substance. G.G. Stokes came closest to this idea in 1845 (see Ref. [28], p. 128) suggesting that substances like pine pitch and shoemaker's wax are so rigid as to support elastic vibrations, yet can allow other bodies passing through them.

Note that we have kept the convective part of the material time derivative in Maxwell's constitutive relation, not merely the partial time derivative. In general, the material time derivative is the relevant one, when a material medium is considered. In an analogous situation for the Maxwell–Cattaneo law for finite-speed heat conduction, we have shown that the omission of the convective part of the derivative leads to paradoxical results (see Ref. [10]). The issue of material invariance of a constitutive law is much deeper, however. Unlike Newton's second law, which is the basis for the Cauchy balance law, the material invariance of a constitutive relation is more complicated. Oldroyd [25] pointed out that there are different ways to make a constitutive law material invariant. He introduced the so-called upper- and lower-convected derivatives (see also Ref. [2]). These, are in fact, connected to Lie derivatives (see e.g., [21]). It goes beyond the scope of present work to discuss different approaches to material invariance of the constitutive laws and will be done in another work.

Yet, when considering the small-amplitude oscillations, we will use, as usual, the linearized version, i.e. the partial time derivative. Taking the operation *div* of the linearized version of Eq. (3) yields

$$\tau \frac{\partial \nabla \cdot \mathcal{T}}{\partial t} + \sigma \nabla \cdot \mathcal{T} = \eta \Delta \boldsymbol{v} + \lambda \nabla(\nabla \cdot \boldsymbol{v}) = -\eta \nabla \times \nabla \times \boldsymbol{v} + (\lambda + \eta)\nabla(\nabla \cdot \boldsymbol{v}). \tag{5}$$

Then $\mathcal{T}$ is eliminated between Eqs. (1) and (5) to obtain

$$\frac{\tau\mu}{\eta}\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \frac{\mu\sigma}{\eta}\frac{\partial \boldsymbol{v}}{\partial t} = \Delta \boldsymbol{v} + \frac{\lambda}{\eta}\nabla(\nabla \cdot \boldsymbol{v}) - \frac{1}{\eta}\nabla p, \tag{6}$$

which can be rewritten as

$$\frac{\tau\mu}{\eta}\left(\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \frac{\sigma}{\tau}\frac{\partial \boldsymbol{v}}{\partial t}\right) = -\nabla \times \nabla \times \boldsymbol{v} + \frac{\eta + \lambda}{\eta}(\nabla \cdot \boldsymbol{v}) - \frac{1}{\eta}\nabla p. \tag{7}$$

The speeds of propagation of shear (c) and compressional (c_s) disturbances, are given by

$$c = \left(\frac{\eta}{\tau\mu}\right)^{1/2}, \quad c_s = \left(\frac{\eta + \lambda}{\tau\mu}\right)^{1/2}, \quad \delta = \frac{\eta}{\eta + \lambda}, \tag{8}$$

where the ratio δ is introduced for convenience.

To consider the two rheological coefficients (moduli η, λ) means to assume that both the shear and the dilational/compressional waves are observable for the material under consideration. Since the ground breaking works of Young and Fresnel, it is well established that the electromagnetic waves (e.g., light) are a purely transverse (shear) phenomenon. This observation requires us to reduce the complexity of the model and to find a way to neglect the term proportional to the dilational modulus λ. Cauchy assumed that $\lambda = 0$ and ended up with the theory of "volatile aether" (see Ref. [28]). Upon a close examination we found that such an approach cannot explain Maxwell's equations. We assume the other extreme, namely that $\lambda \gg \eta$. In the proper place in the text to follow we will outline the place where this assumption is essential for the complete derivation of Maxwell's model. Under the above assumption we get for the ratio $\delta \ll 1 \to \delta^{-1} \gg 1$. If the rest of the coefficients entering Eq. (7) are of lower order of magnitude than $\delta \ll 1$, one can expand the density μ, displacements $\boldsymbol{u}$ and velocities $\boldsymbol{v}$ into asymptotic power series with respect to δ, namely

$$\mu = \mu_0 + \delta\mu_1 + \cdots, \quad \boldsymbol{u} = \boldsymbol{u}_0 + \delta\boldsymbol{u}_1 + \cdots, \quad \boldsymbol{v} = \boldsymbol{v}_0 + \delta\boldsymbol{v}_1 + \cdots. \tag{9}$$

Introducing Eq. (9) into Eq. (7) and combining the terms with like powers we obtain for the first two terms

$$\nabla \cdot \boldsymbol{v}_0 = 0, \tag{10}$$

$$c^{-2}\left(\frac{\partial^2 \boldsymbol{v}_0}{\partial t^2} + \gamma\frac{\partial \boldsymbol{v}_0}{\partial t}\right) + \nabla \times \nabla \times \boldsymbol{v}_0 + \frac{1}{\eta}\nabla p = \nabla(\nabla \cdot \boldsymbol{v}_1). \tag{11}$$

The velocity field must be solenoidal within the first order of approximation of the small parameter δ. However, a slight deviation from solenoidality in the first order cannot be discarded from the equation because it is multiplied by the large parameter. For this reason the term $\nabla \cdot \boldsymbol{v}_1$ persists in Eq. (11). The good news is that the said term does not really change the type of the system of equations because it can be included in the pressure term, by the substitution

$$\hat{p} = p - (\lambda + \eta)\nabla \cdot \boldsymbol{v} = p - \delta(\lambda + \eta)\nabla \cdot \boldsymbol{v}_1 = p - \eta\nabla \cdot \boldsymbol{v}_1. \tag{12}$$

Now, Eq. (11) adopts the form

$$c^{-2}\left(\frac{\partial^2 \boldsymbol{v}_0}{\partial t^2} + \gamma\frac{\partial \boldsymbol{v}_0}{\partial t}\right) + \nabla \times \nabla \times \boldsymbol{v}_0 + \frac{1}{\eta}\nabla \hat{p} = 0, \tag{13}$$

which has the form of the momentum equation for incompressible Maxwell liquids. Indeed, from Eq. (10) one can deduce that the Eq. (2) takes the form

$$\frac{\partial \mu_0}{\partial t} + \boldsymbol{v}_0 \cdot \nabla\mu_0 = 0 \quad \Rightarrow \quad \mu_0 = \text{const}, \tag{14}$$

which means that the metacontinuum must behave as an incompressible continuum when its bulk modulus λ is much larger than the shear modulus η.

There are no conceptual difficulties to extend the model to include a slight compressibility of the metacontinuum. Such a generalization opens the way to look for the speed of sound of the metacontinuum, which will be much greater (proportionally to $1/\delta$) than the speed of light. However, this investigation goes beyond the frame of the present paper whose sole purpose is to show that Maxwell's equations are a strict corollary from the model of incompressible viscoelastic liquid.

In conclusion of this section, we mention that Eq. (13) governs the propagation of transverse (shear) waves and accomplishes the goal of Cauchy who attempted to explain the light waves are shear waves of a material medium.

3. From metacontinuum to Maxwell–Lorentz electrodynamics

3.1. Definition of electric and magnetic fields

It is important to understand the effect of the internal stresses acting in a continuum. The effect is that the divergence of the deviator stress tensor will be manifested as a body force in the particular point. Let us denote formally

$$\boldsymbol{E} \stackrel{\text{def}}{=} -\nabla\cdot\mathcal{T}, \tag{15}$$

and recast the Cauchy balance law into the form

$$-\mu\frac{\partial \boldsymbol{v}}{\partial t} - \nabla\hat{p} = \boldsymbol{E}. \tag{16}$$

Hereafter, we omit the index "0" without fear of confusion. Note that the "−" sign in Eq. (15) is taken only for the sake of compliance with the traditional notations. Now $\boldsymbol{E}$ is a body force experienced at each point of the metacontinuum. We can give a name to it, say "electric force".

Proceeding further, we can define a "magnetic field", $\boldsymbol{H}$, as the vorticity of the continuum

$$\boldsymbol{H} \stackrel{\text{def}}{=} \nabla\times\boldsymbol{v}, \tag{17}$$

or in terms of the magnetic induction

$$\boldsymbol{B} = \mu\nabla\times\boldsymbol{v} = \mu\boldsymbol{H}. \tag{18}$$

Here $\boldsymbol{H}$ is called "magnetic field" instead of "magnetic force"; it is not a force in the strict sense, it is the vorticity of the velocity field.

The system (16) and (18) presents the equations of electrodynamics in terms similar to the form employing vector and scalar potential. There are different ways to introduce the potentials: Coulomb gauge, Lorenz gauge (see Ref. [18], p. 220; or [16], Ch. 15). The above derivations tells us that the vector and scalar potentials are not simply useful devises to render the electrodynamics in an equivalent form. One can see that if an absolute continuum exists, then the velocity vector, $\boldsymbol{v}$ and the pressure p of the metacontinuum would appear as a particular choice of vector and scalar potentials. In some sense, a proper way to call the governing equations of the material metacontinuum is "physical gauge"

3.2. Maxwell equations as corollaries

Taking the the operation *curl* of Eq. (16) and acknowledging Eq. (18) we obtain Faraday's law of magnetic induction

$$\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{B}}{\partial t}, \tag{19}$$

which is a mere corollary from the conservation of the linear momentum for the metacontinuum.

Now we are poised to derive the second of Maxwell's equations. Introducing the definition of electric field Eq. (15) into Eq. (5), one obtains

$$\frac{\tau}{\eta}\frac{\partial\boldsymbol{E}}{\partial t} + \frac{\sigma}{\eta}\boldsymbol{E} = \nabla\times(\nabla\times\boldsymbol{v}) \equiv \nabla\times\boldsymbol{H}, \tag{20}$$

or in terms of magnetic induction

$$\frac{\tau\mu}{\eta}\frac{\partial \boldsymbol{E}}{\partial t}+\frac{\mu\sigma}{\eta}\boldsymbol{E}=\nabla\times\boldsymbol{B} \quad \Rightarrow \quad \frac{\partial \boldsymbol{E}}{\partial t}+\frac{\sigma}{\tau}\boldsymbol{E}=c^2\nabla\times\boldsymbol{B}. \tag{21}$$

When $\sigma = 0$, the above equation is simply the second of Maxwell's equations, as valid for empty space ([16], p. 15). The first term, $\partial\boldsymbol{E}/\partial t$, is the displacement current introduced by Maxwell using heuristic arguments. From this, we can now give a rigorous explanation for its incorporation. It is the result of the elasticity of the medium. Hence, it is of no surprise, then, that this was the term which allowed Maxwell to postulate the existence of electromagnetic waves (believed to be elastic waves by Fresnel and Cauchy).

Now we face the conceptual question of whether one can have a current in the metacontinuum. The notion of current is involved in Ampere's and Ohm's law. They were derived for what sometimes is called "Macroscopic Electromagnetism" (see Ref. [18], p. 226). This means that the current is considered as a flow of charged particles. As it will become clear in what follows, the charges are patterns in the metacontinuum and it is not *a priori* clear whether the stresses (electric field) can produce a current in the material points of the metacontinuum. Very often, this question is implicitly answered in the literature in the affirmative by simply calling Eq. (21) Ampere's law (see Ref. [19], p. 301). We believe that a more precise terminology is needed. In fact, it is not clear whether the Ohm's resistance law should be valid for the motion of material points of the metacontinuum. One has to postulate that the electric field produces a flow (a current) in the metacontinuum. This will amount to the assumption that there is a current in the "microscopic electromagnetism". Hence, we add an additional assumption that *in vacuo* one can still define a current $\boldsymbol{j}$ as being related to the field according to Ohm's law

$$\boldsymbol{j}=\frac{\sigma}{\eta}\boldsymbol{E} \quad \text{or} \quad \sigma\boldsymbol{E}=\eta\boldsymbol{j}, \tag{22}$$

where $\sigma\eta^{-1}$ plays the role of the conductivity coefficient for empty space.

From this conjecture, it follows that Ampere's law is also valid in vacuo, i.e. it is a property of the rheology of metacontinuum. Then the manifestation of Ohm's law in the "macroscopic electromagnetism" is a trivial corollary. Thus, from Eq. (21) follows the second Maxwell equation, which is a generalization of Ampere's law, namely,

$$\frac{\partial \boldsymbol{E}}{\partial t}+\frac{\boldsymbol{j}}{\varepsilon}=c^2\nabla\times\boldsymbol{B}, \tag{23}$$

where $\varepsilon = \tau\eta^{-1}$ is called the electric permittivity. It can be also expressed as

$$\varepsilon=\frac{1}{c^2\mu}, \tag{24}$$

when the definition for c, Eq. (8), is acknowledged. The permittivity coefficient has a clear dynamical meaning. It is the inverse of the shear Lamé coefficient if the metacontinuum is considered in the elastic limit (viscosity is neglected).

Under the assumption of the validity of Ohm's law in vacuo, we will call either Eqs. (21) or (23) the "Second Maxwell Equation".

To complete this picture, we mention also that the third Maxwell equation $\nabla\cdot\boldsymbol{H}=0$ follows from the definition of the magnetic field. Also, taking *div* of (23) yields $\tau\partial_t(\nabla\cdot\boldsymbol{E})+\sigma\nabla\cdot\boldsymbol{E}=0$, provided that there are no singularities of the magnetic fields (no charges). In the electrostatic case with no charges, one has $\nabla\cdot\boldsymbol{E}=0$. If a point charge of magnitude e is present at point $\boldsymbol{x}_\alpha$, then $\nabla\cdot\boldsymbol{E}=e\delta(\boldsymbol{x}-\boldsymbol{x}_\alpha)$, where $\delta(\cdot)$ is the Dirac delta function. Consecutively, the average divergence will be equal to the density of the point charges.

Thus, we have shown that the linearized equations of viscoelastic liquids admit what can be called *Maxwell's form*. In the framework of such a paradigm, the absolute viscoelastic medium will be called "the metacontinuum" to be distinguished from the ubiquitous continuous media, such as liquids, gases, elastic bodies. The notion of continuum for these mechanical media is a device to reduce the complexity of molecular description. The metacontinuum is the media for propagation of the electromagnetic interactions. To recap the results of this subsection, we repeat that each point of the metacontinuum experiences a body force $\boldsymbol{E}$ ("electric force"), to which the action of the internal elastic stresses is reduced. The vorticity of the velocity field is called "magnetic field". An *action at a distance*, such as the electric field, is the result of the propagation of *internal stresses* throughout the metacontinuum.

3.3. Convective nonlinearity and Lorentz force

The preceding has revealed that there exists an analogy between Maxwell's electrodynamics and the linearized dynamics of a viscoelastic liquid. Here we examine more closely the nonlinear part of the material time derivative to search for its counterpart in electrodynamics. The pertinent question here is of what kind of effects are to be expected due to the convective nonlinearity. To this end, consider the Cauchy balance equations in the so-called Lamb's form [27]

$$\mu\left(\frac{\partial \boldsymbol{v}}{\partial t} - \boldsymbol{v}\times\nabla\times\boldsymbol{v}\right) + \nabla\varphi = -\boldsymbol{E}, \quad \varphi \equiv \hat{p} + \frac{1}{2}|\boldsymbol{v}|^2. \tag{25}$$

This form allows one to assess the forces acting at a given material point of the metacontinuum due to the convective acceleration. The gradient part of the convective acceleration cannot be observed independently from the pressure gradient. In fact, one can measure only the quantity $\varphi_1 \equiv \varphi + \frac{1}{2}\boldsymbol{v}^2$. Thus, the only observable quantity is the term of the acceleration that contains the vorticity. By virtue of our definition of magnetic induction (18), the term under consideration adopts the form

$$F_l = \mu_0 \boldsymbol{v}\times\nabla\times\boldsymbol{v} = \mu_0\boldsymbol{v}\times\boldsymbol{H} = \boldsymbol{v}\times\boldsymbol{B}, \tag{26}$$

which gives the inertial force acting at each point of the medium because of the relative motion.

The detection of this force can be done on the basis of its cumulative effect over a distributed localized pattern of deformation. If we consider the pattern as a charged particle, then, integrating (26) over the span of a charge gives the Lorentz force in its well known form. Thus the existence of the Lorentz force is a further corollary from the hypothesis of existence of an absolute viscoelastic metacontinuum. This force is now an integral part of the model, which is an improvement over the classical approach where the Lorentz force has to be considered as an additional empirically observed phenomenon [16–18].

Identifying the Lorentz force as the inertial force created by the convective part of the acceleration calls for careful examination of some of the basic tenets of modern electrodynamics. The problem is connected with the fact that the motion of the metacontinuum is Galilean invariant as testified by the Lorentz force. For an unbiased observer, it is not too hard to spot the Galilean invariance. In fact, it is acknowledged in the monograph ([18], p. 213). In order not to contradict the fundamental postulate that everything in electrodynamics is Lorentz invariant, the author goes on to introduce the caveat that Galilean invariance is an approximation for low speeds of the Lorentz invariance. In our opinion, the cited argument from a most authoritative book [18] is not convincing. To any student of continuum mechanics, the opposite argument is more appealing: the Lorentz invariance is connected with the linearized accelerations and as such is an approximation of the full inertial terms for relatively slow velocities. We do not see the need for excuses when mentioning the fact that the incorporation of the Lorentz force makes the field equations Galilean invariant. Moreover, one does not have to abolish the Galilean invariance of the absolute medium in order to understand the processes without Galilean invariance related to motions of particles and charges. In Section 4, we show that the Galilean invariance of the model of the metacontinuum does preclude the motion of the phase patterns to obey the Lorentz contraction rule.

It has to be mentioned here that the full model of viscoelastic liquid is Galilean invariant if one takes the material derivative in Eq. (5) or one of the derivatives defined in Ref. [25]. This, however, goes beyond the scope of the present work which is focused on the meaning of the viscoelastic metacontinuum for the Maxwell–Lorentz electrodynamics. The presence or absence of Galilean invariance is not essential for this work. The full details are due elsewhere.

3.4. The wave equations of electrodynamics

Upon neglecting the convective part we have

$$\mu\frac{\partial \boldsymbol{v}}{\partial t} + \nabla\varphi_1 = -\boldsymbol{E}, \quad -\left(\tau\frac{\partial \boldsymbol{E}}{\partial t} + \sigma\boldsymbol{E}\right) = \eta\Delta\boldsymbol{v} = -\eta\nabla\times\nabla\times\boldsymbol{v}, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{27}$$

Eliminating $\boldsymbol{E}$ between the above equations on gets a single equation for velocity

$$c^{-2}\left(\frac{\partial^2 \boldsymbol{v}}{\partial t^2} + \gamma\frac{\partial \boldsymbol{v}}{\partial t}\right) + \frac{1}{\eta}\left(\tau\frac{\partial\nabla\varphi}{\partial t} + \nabla\varphi\right) = \Delta\boldsymbol{v}. \tag{28}$$

Taking the *curl* of the latter one gets the following generalized dissipative wave equation

$$c^{-2}\left(\frac{\partial^2 \boldsymbol{H}}{\partial t^2}+\gamma\frac{\partial \boldsymbol{H}}{\partial t}\right)=\Delta \boldsymbol{H}, \tag{29}$$

in terms of the magnetic induction. The coefficient $\gamma = \sigma\tau^{-1}$ is introduced for brevity. Alternatively, one can eliminate $\boldsymbol{v}$ between Eq. (27) to obtain

$$c^{-2}\left(\frac{\partial^2 \boldsymbol{E}}{\partial t^2}+\gamma\frac{\partial \boldsymbol{E}}{\partial t}\right)=-\nabla\times\nabla\times \boldsymbol{E}. \tag{30}$$

The wave equations of electrodynamics contain attenuation terms $\gamma\partial_t \boldsymbol{E}$ or $\gamma\partial_t \boldsymbol{H}$. For small γ these are negligible and the equations are non-dissipative wave equations. However, for γ small but finite, the attenuation term can have a profound impact on the propagation of the electromagnetic waves at long times or large distances. As already mentioned, γ cannot be estimated from ubiquitous Ohm's law for currents in matter because the notion of charge has to be introduced.

One effect of attenuation will be that the distant stars appear dimmer than they would have appeared if $\gamma = 0$. Consequently, the distances estimated on the basis of luminosity would appear larger than they are in reality. As a result of the overestimated distances, the density (and the mass, for that matter) of the Universe would appear much smaller than the projected mass based on the orbital speeds of stars and the galaxies. A way out of this situation is to assume the presence of dark matter [26]. From the point of view of the present work, the discrepancy between the estimates for the mass of the Universe with and without attenuation can be used to evaluate the attenuation factor γ. The details are very elaborate and go beyond the scope of the present work, but there are no principal difficulties in doing this. The relationship between the actual distance, d, and the perceived distance, d^*, when $\gamma = 0$ is

$$d^* = d\mathrm{e}^{\gamma d/c}. \tag{31}$$

The number $\bar{d} = c/\gamma$ has the dimension of length. The value of $\bar{d}$ must be very large because $\gamma \ll 1$ and $c \gg 1$. When the actual distance $d \approx \bar{d}$, the perceived distance d^* will be approximately three times larger. This will make the apparent density of matter in a cube of length $\bar{d}$ to appear $\bar{d}^3$ times smaller than its actual value. Using this relation one can estimate the order of magnitude of $\bar{d}$ (or which is the same, γ.)

4. Localized shear waves in metacontinuum: the wave-particles

"Indeed, one of the most important of our fundamental assumptions must be that the ether not only occupies all space between molecules, atoms or electrons, but that it pervades all these particles. We shall add the hypothesis that, though the particles may move, the ether always remains at rest. We can reconcile ourselves with this, at first sight, somewhat startling idea, by thinking of the particles of matter as of some local modification in the state of the ether. These modifications may of course very well travel onward while the volume-elements of the medium in which they exist remain at rest", H.A. Lorentz [20].

It is hard to put the idea of moving patterns in better words than the above quoted passage from Lorentz. In our opinion, the only way to reconcile the absolute medium (as testified by the absolute speed of propagation of shear waves) and the relative motion of so-called particles is to understand the latter as phase patterns. Since the very notion of a particle presumes a localization, we consider as particles the localized patterns of the metacontinuum. It is now well known that in many continuous systems the localized waves behave as particles. These wave-particles are called *solitons*. Soliton research has been a most rapidly growing scientific field in the last couple of decades. We refer the reader to the excellent review [4] and the extensive monographs on the subject, e.g. [24]. The soliton presents an example of a moving "modification of the state" of the absolute continuum. Solitons do travel (propagate) while the particles of the continuum remain in the vicinity of their original positions. Solitons can interact with each other upon their collisions and regain the original form after they separate enough after the collision is over. The quasi-particle behavior of solitons is now very well studied and documented. For a non-exhaustive list of mechanical models in which the soliton solutions behave as particles, one is also referred to [22], and the literature cited therein.

In the present section, we show the consequence of the idea of considering particles as wave patterns. We only add to the Lorentz definition that the particles, charges, etc can be considered as localized waves propagating over the 3D hypersurface called the *metacontinuum*.

The notion of quasi-particle (sometimes called pseudo-particle) arises from the idea that in an ubiquitous material continuum, the localized waves propagate and interact with each other in the same way that particles do. If we now call the localized waves "particles", then the medium they are propagating in will appear as a concept beyond the mechanics of the particle. For this reason we use the coinage *metacontinuum* to designate the absolute medium which is the carrier of all kind of waves and wave-particles alike.

4.1. Localized vortex patterns in metacontinuum

It is well known that the model of inviscid liquids admits potential vortex solutions. The vortex flow is irrotational except for the central point where a singularity is observed. In fact when no interaction with the boundaries is presumed, the potential solutions exist even for the Navier–Stokes equations of viscous liquids. The singularity cannot be removed in the model of Newtonian liquids. Yet, the point-vortex flow [1] is one of the theoretically best studied solutions of classical hydrodynamics. Lord Kelvin extended the idea of vortex structures to the alleged ether in an attempt to explain atoms. Our point of view is that the vortices of the metacontinuum cannot account for all observable phenomena associated with material particles. As shown in [5,7(Part II)], a fourth dimension is needed in order to explain the Schrödinger wave mechanics. At the same time, the vortex is a perfect model of what is known as "electric charge" because it possesses a topological charge (circulation).

To elucidate this concept we begin with the stationary case in 2D when Eq. (29) reduces to a single scalar Laplace equation for the third component of the magnetic field, say $\psi = \boldsymbol{H}_z$, namely

$$\Delta\psi = 0.$$

In the case of polar symmetry $\psi = F(r)$, where $r = \sqrt{x^2 + y^2}$. Then

$$F(r) = \ln(r). \tag{32}$$

It is well known that in 2D, the fundamental solution to Laplace equation diverges at infinity logarithmically. There are different ways to improve the mechanical model as so to remove this singularity. The most sound way from mechanical point of view, is to consider the higher-gradient elasticity of the medium and to generalize Eq. (29) to one containing a biharmonic operator along with the Laplace operator. This is rather elaborate and goes beyond the scope of the present work. For the velocity components, we have the following expressions

$$v_x = \frac{y}{r^2}, \quad v_y = -\frac{x}{r^2}. \tag{33}$$

The vector field generated by Eq. (33) is shown in Fig. 1(a). This is the well known potential vortex in 2D fluid dynamics (see e.g., [19], Ch. IX). The topological charge of the vortex solution is defined as the integral over a closed curve C

$$\Gamma = \oint_C \boldsymbol{v}\mathrm{d}\boldsymbol{s} = \oint_C [u_x \mathrm{d}x + u_y \mathrm{d}y],$$

where $\mathrm{d}\boldsymbol{s}$ is the elementary arc length along the curve C and the quantity Γ is the circulation. If we consider now the vortex of metacontinuum as the electric charge, then the quantitative value of its *charge* is given by the circulation Γ (the topological charge). Thompson's theorem (see Refs. [1,19]) asserts that in an inviscid liquid the circulation is conserved, i.e., $\Gamma = \text{const}$, which gives in our model the conservation of charge.

Note that even if the metacontinuum is an inviscid liquid, some deviation from Thompson's theorem can be expected for vortices whose centers are accelerating. Such a deviation is associated with the additional acceleration in the Maxwell rheological law. However, no difference is expected for stationary propagating vortices.

Now we have come to the most important distinction between the concept of metacontinuum proposed here and the classical notion of absolute medium "filling" the void space between the "point" particles. According to the concept of present work, point particles *per se* do not exist. It is our course-grain appreciation of a a localized pattern with very small scale of the support. This concept enjoys quite a popularity in modern physics due to the interpretation of solitons as quasi-particles (see Ref. [22] for an overview).

Yet, it is natural to ask the question about the constitution of the metacontinuum. Clearly, it is too early for any kind of practical hypothesis, but it is quite natural to expect that just as the ubiquitous continua do, the metacontinuum is also composed by some particles that are subject to non-equilibrium "thermodynamics" (see Ref. [11]). These "atoms" or "molecules" of the metacontinuum should be of much smaller scale than the elementary particles since the latter are phase patterns on the metacontinuum which appears to them as a continuous medium. Most probably the size of the "molecules" of the metacontinuum should be of order of 10^{-30} m or even smaller.

4.2. Effect of rectilinear motion on localized phase patterns: Lorentz contraction

The soliton-like vortex solution from the previous subsection is a kind of a *torsional dislocation*. The material points of the metacontinuum may move continuously like in the above considered vortex, while the phase pattern is completely at rest. If we consider now the patterns to be the particles, then we must realize that the laws of motion for the center of a localized pattern will appear on macro scale as the laws of motion for a point-particle.

Should the above described "dislocation" be allowed to move, it would not "plow" through the material points of the continuum. Instead, it would propagate as a *phase pattern*, in much the same way a wave propagates over the water surface. The concept that the charge is a phase pattern removes the most substantial objection against the elastic model of the electromagnetic field, namely, that it is too dense for the particles and charges to move through. Also, no "ether wind" is supposed to trail a propagating dislocation. Such a dislocation does not introduce a further disturbance which the pattern itself is in the metacontinuum apart from the disturbance itself.

Consider now a solution for $\boldsymbol{v}$ which is almost stationary (slowly evolving) in a moving frame. Consider for definiteness the frame moving in the y-direction with phase speed c_y. Introducing a local spatial variable $\eta = y - c_y t$ and neglecting the local time derivatives in the moving frame, one gets

$$\frac{\partial \psi}{\partial t} = -c_y \frac{\partial \psi}{\partial \eta}, \quad \frac{\partial^2 \psi}{\partial t^2} = c_y^2 \frac{\partial^2 \psi}{\partial \eta^2}. \tag{34}$$

and the Laplace equation $\Delta \psi = 0$ recasts to

$$\frac{\partial^2 \psi}{\partial x^2} + (1 - c_y^2) \frac{\partial^2 \psi}{\partial \eta^2} = 0, \quad \Rightarrow \quad \frac{1}{z} \frac{\partial}{\partial z} \left(z \frac{\partial \psi}{\partial z} \right) = 0, \tag{35}$$

where $z \equiv \sqrt{x^2 + \eta^2 (1 - c_y^2)^{-1}}$. The solution of Eq. (35) is once again $\psi = \ln z$.

The isolines of ψ are now ellipses that appear contracted in the direction of motion by the Lorentz factor as shown in Fig. 1(b). Comparing the two panels of Fig. 1, hints at an analogy between the contraction of the localized wave to the Doppler shortening of harmonic waves ahead of a moving source. Considering the localized wave as a quasi-particle (wave-particle) we conclude that *Lorentz Contraction is the Doppler Effect for Wave-Particles*.

In closing this section, we stress the point that the Galilean invariance of the motion inside the metacontinuum reflects its absolute nature, but it does not preclude the wave-particles from behaving more in a Lorentz-invariant fashion because they are *phase patterns*, or "quasi-particles". This could be the resolution of long standing paradox

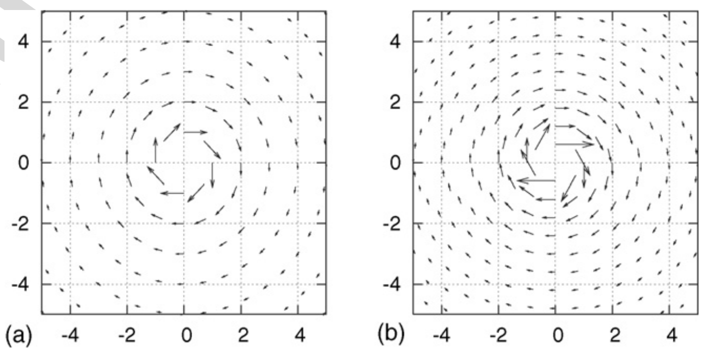

Fig. 1. The localized torsional dislocation in two dimensions for two different phase velocities of propagation. (a) $c_x = 0$, $c_y = 0$. (b) $c_x = 0$, $c_y = 0.8$.

pointed out by Einstein ([13], p. 21) that one cannot rationally reconcile the absolute speed of light with the apparent relativity of rectilinear motion.

Clearly, the wave-particles are subject to Lorentz contraction, and so are the interparticle forces as a result of the Doppler effect of the waves which are transmitting the long-range interactions. This means that a body of charged wave-particles held together by the internal stresses of the metacontinuum (electromagnetic forces) would become shorter in the direction of motion (better said: “direction of propagation”). This means that the Doppler effect and the Lorentz contraction will cancel each other in any interferometry experiment using a split beam and closed light path. If one assumes the presence of an absolute continuum, then the only strict result from the Michelson and Morley experiment *must* be the nil effect. In other words, the nil effect of the Michelson and Morley experiment can be considered as a strong indication of the existence of an absolute continuum. As shown above, the wave-particles, and hence the bodies, are contracting in the direction of motion exactly by same factor by which the planar waves experience Doppler contraction.

5. Concluding remarks

We considered a material medium which is the carrier of the action at a distance in electrodynamics. We call this medium the *metacontinuum* because of its role as the underlying fabric of physical phenomena. We stipulated that the rheology of the *metacontinuum* is that of a viscoelastic liquid. We have shown that the linearized model can be recast into what one can call “Maxwell form” provided that the net result of the internal stresses is interpreted as the electric field and the vorticity is considered as the magnetic field. In other words, with the appropriate dependent variables, we showed that the governing equations of incompressible viscoelastic liquids have, as a corollary, the *full* set of Maxwell–Lorentz electrodynamic equations.

Finally, the elastic part of the constitutive relation explains in full the propagation of electromagnetic waves *in vacuo*. The viscous part is related to Ampere’s laws for current combined with Ohm’s law of resistance. An important conclusion from the analogy proposed here is that, due to the advective nonlinearity of the material time derivative, a force arises, which is analogous to the Lorentz force acting on moving charges. In this sense the hypothesis of a viscoelastic metacontinuum explains both Maxwell’s equations and the Lorentz force acting on moving charges.

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