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CONSERVATIVE NUMERICAL SCHEME IN COMPLEX ARITHMETIC FOR COUPLED NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. For the Coupled Nonlinear Schrödinger Equations (CNLSE) we construct a conservative fully implicit numerical scheme using complex arithmetic which allows to reduce the computational time fourfold. We obtain various results numerically and investigate the role of the nonlinear and linear coupling. For nontrivial but moderate nonlinear coupling parameter, we find that the polarization of the system changes, but no other effects are present. For large values of the nonlinear coupling parameter, additional (faster) solitons are created during the collision of the initial ones. The linear coupling is shown to manage the self-focusing/dispersion and to make the additional solitons appear for smaller nonlinear coupling. These seem to be new effects, not reported in the literature.

1. **Introduction.** The investigation of soliton supporting systems is of great importance both for the applications and for the fundamental understanding of the phenomena associated with propagation of solitons. Recently, elaborate models such as Coupled Nonlinear Schrödinger Equations (CNLSE) appeared in the literature. They involve more parameters and possess richer phenomenology but, as a rule, are not fully integrable and require numerical approaches. The non-fully-integrable models possess as a rule three conservation laws: for "(wave) mass", (wave) momentum, and energy and these have to be faithfully represented by the numerical scheme.

A numerical scheme with internal iterations was first proposed for the single NLS in the extensive numerical treatise [7] and named Crank-Nicolson implicit scheme. The concept of the internal iterations was first applied to CNLSE in [3] and extended in [6] in order to ensure the implementation of the conservation laws on difference level within the round-off error of the calculations. The CNLSE is investigated numerically also in [4]. Here, we follow generally the works [3, 6] but focus on a new complex-variable implementation of the conservative scheme. This

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allows us to invert (albeit complex-valued) but five-diagonal matrices while the real-valued scheme requires the inversion of nine-diagonal matrices [6]. To this end, we generalize the computer code for Gaussian elimination with pivoting developed earlier for real-valued algebraic systems in [2]. This gives a significant advantage in the efficiency of the algorithm.

The numerical validation of the new code includes comparisons with [3, 6] which show that the complex-numbers implementation of the scheme gives identical results with the real-numbers codes but is approximately four times as efficient. Several featuring examples of interacting solitons in CNLSE are elaborated

Creating and validating a numerical scheme based on complex-number arithmetic is important for the future application of the conservative-scheme approach in two spatial dimensions. The system for the real and imaginary parts of the wave function has an intricate form that precludes using operator splitting in the real-valued version of the algorithm. Without splitting the needed computational resources in 2D are enormous which makes the approach rather unpractical. In this instance, the present approach can be a good basis development of a 2D numerical scheme based on operator splitting.

2. **Coupled Nonlinear Schrödinger Equations.** In optics, the most popular model is the cubic Schrödinger equation which describes the single-mode wave propagation in a fiber. It has the form

$$\mathrm{i}\psi_t + \beta\psi_{xx} + \alpha|\psi|^2\psi = 0, \tag{1}$$

where $\mathrm{i} = \sqrt{-1}$ and $\psi(x,t)$ is complex valued wave function. Depending on the sign of coefficient α, the localized solutions of equation (1) are either the hyperbolic secants ("bright solitons") or hyperbolic tangents ("dark solitons"). Since the fibers also allow propagation of multiple orthogonally polarized modes, a multi-component version of equation (1) has been actively investigated during the last decade. A general form of the Coupled Nonlinear Schrödinger Equations (CNLSE) reads

$$\begin{aligned} &\mathrm{i}\psi_t + \beta\psi_{xx} + \left[\alpha_1|\psi|^2 + (\alpha_1 + 2\alpha_2)|\phi|^2\right]\psi + \gamma\psi + \Gamma\phi = 0, \\ &\mathrm{i}\,\phi_t + \beta\phi_{xx} + \left[\alpha_1|\phi|^2 + (\alpha_1 + 2\alpha_2)|\psi|^2\right]\phi + \gamma\phi + \Gamma\psi = 0, \end{aligned} \tag{2}$$

where β is the dispersion coefficient and α_1 describes the self-focusing of a signal for pulses in birefringent media. Complex-valued coefficients γ and Γ are responsible for the linear coupling between the two equations. Respectively α_2 governs the nonlinear coupling between the equations. It is interesting to note that when $\alpha_2 = 0$, the nonlinear coupling is not presented despite the fact that some "cross" terms proportional to α_1 appear in the equations. In fact, when $\gamma = \Gamma = \alpha_2 = 0$, the solution of the two equations are identical, $\psi \equiv \phi$, and equal to the solution of single NLSE, equation (1) with nonlinearity coefficient $\alpha = 2\alpha_1$. The sum $(\alpha_1 + 2\alpha_2)$ is called sometimes "cross-phase modulation".

Functions ψ and ϕ have various interpretations in the context of optic pulses including the amplitudes of x and y polarizations in a birefringent nonlinear planar waveguide, pulsed wave amplitudes of left and right circular polarizations, etc. The quantity γ is called normalized birefringence, and Γ is the relative propagation constant. The presence of the two new parameters, γ and Γ, in equations (2) makes the phenomenology of the system (2) much richer. In particular, they allow to study the phenomena such as "self-dispersion" and "cross-dispersion", dissipation, etc. (see [6] and the literature cite therein).

For $\Gamma, \gamma = 0$, equation (2) is alternatively called the Gross-Pitaevskii equation or an equation of Manakov-Type. It was solved analytically for the case $\alpha_2 = 0, \beta = \frac{1}{2}$ by Manakov [5] via inverse scattering transform who generalized an earlier result by Zakharov and Shabat [8, 9] for the scalar cubic NLSE (i.e. equation (2-ψ) with $\phi(x,t) = 0$). In a birefringent medium, $\alpha_1 + 2\alpha_2$ distinguishes the description of elliptic, circular and linear polarizations. In this more interesting case, integrability is lost, and numerical methods have to be used to study the evolution of the system.

Let us define energy, E, and mass, M, of the wave system as

$$E = \int_{-\infty}^{\infty} \Big[-\beta \left(|\psi_x|^2 + |\phi_x|^2 \right) + \frac{\alpha_1}{2} (|\psi|^4 + |\phi|^4) + (\alpha_1 + 2\alpha_2) \left(|\phi|^2 |\psi|^2 \right)$$

$$+ \gamma \left(|\psi|^2 + |\phi|^2 \right) + 2\Gamma \Re(\bar{\phi}\psi) \Big] \mathrm{d}x, \qquad M = \int_{-\infty}^{\infty} \left(|\psi|^2 + |\phi|^2 \right) \mathrm{d}x, \tag{3}$$

respectively. Then it is readily proved (see [1] and the literature cited therein) that these quantities are conserved on the solutions of equation (2) , namely

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0, \qquad \frac{\mathrm{d}E}{\mathrm{d}t} = 0. \tag{4}$$

If one is to construct a numerical algorithm, the above conservation laws have to be embodied in the scheme in order to faithfully represent the physics of the problem.

3. **Conservative Difference Scheme.** Consider a uniform mesh in the interval [-L_1, L_2],

$$x_i = (i-1)h, \quad h = (L_1 + L_2)/(N-1) \quad \text{and} \quad t^n = n\tau,$$

where N is the total number of grid points in the interval and τ is the time increment. Respectively, ψ_i^n and ϕ_i^n denote the value of the ψ and ϕ at the ith spatial point and time stage t^n. Clearly, $n = 0$ refers to the initial conditions.

Our purpose is to create a difference scheme that is not only convergent (consistent and stable), but also reflects the conservation laws equation (4). A fully nonlinear scheme that satisfies the conservation laws reads

$$\begin{aligned} \mathrm{i}\frac{\psi_i^{n+1} - \psi_i^n}{\tau} =& \frac{\beta}{2h^2} \left(\psi_{i-1}^{n+1} - 2\psi_i^{n+1} + \psi_{i+1}^{n+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n \right) \\ &+ \frac{\psi_i^{n+1} + \psi_i^n}{4} \Big[\alpha_1 \left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2 \right) + (\alpha_1 + 2\alpha_2) \left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2 \right) \Big] \\ &+ \frac{\gamma}{2} (\psi_i^n + \psi_i^{n+1}) + \frac{\Gamma}{2} (\phi_i^n + \phi_i^{n+1}) \end{aligned} \tag{5}$$

$$\begin{aligned} \mathrm{i}\frac{\phi_i^{n+1} - \phi_i^n}{\tau} =& \frac{\beta}{2h^2} \left(\phi_{i-1}^{n+1} - 2\phi_i^{n+1} + \phi_{i+1}^{n+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n \right) \\ &+ \frac{\phi_i^{n+1} + \phi_i^n}{4} \Big[\alpha_1 \left(|\phi_i^{n+1}|^2 + |\phi_i^n|^2 \right) + (\alpha_1 + 2\alpha_2) \left(|\psi_i^{n+1}|^2 + |\psi_i^n|^2 \right) \Big] \\ &+ \frac{\gamma}{2} (\phi_i^n + \phi_i^{n+1}) + \frac{\Gamma}{2} (\psi_i^n + \psi_i^{n+1}). \end{aligned} \tag{6}$$

Such a scheme will be stable when time-stepping with respect to the physical time, provided that the iterative procedure to resolve the nonlinear terms is convergent.

Here is to be mentioned that the local truncation error of the scheme is $O(\tau^4 + h^4)$, but the more important characteristic is the global truncation error. This is especially true for the truncation of the terms containing time derivatives because in time we have an initial value problem and the proliferation of the error is a major issue. We were able to show that the global error of the time approximation is $O(\tau^2)$, but the proof is very technical and will be discussed elsewhere.

Following [3] we have proved that the scheme in question conserves the discrete analogs of mass and energy, from (4). Namely, for all $n \geq 0$, we have

$$\begin{aligned} M^n &= \sum_{i=2}^{N-1} \left(|\psi_i^n|^2 + |\phi_i^n|^2\right) = \text{const}, \\ E^n &= \sum_{i=2}^{N-1} \frac{-\beta}{2h^2} \left(|\psi_{i+1}^n - \psi_i{}^n|^2 + |\phi_{i+1}^n - \phi_i{}^n|^2\right) + \frac{\alpha_1}{4} \left(|\psi_i^n|^4 + |\phi_i^n|^4\right) \\ &\quad + \frac{\alpha_1 + 2\alpha_2}{2} \left(|\psi_i^n|^2 |\phi_i^n|^2\right) + \frac{\gamma_r^\phi}{2} |\phi_i{}^n|^2 + \frac{\gamma_r^\psi}{2} |\psi_i{}^n|^2 + \Gamma \Re[\overline{\phi}_i^n \psi_i^n] = \text{const}. \end{aligned}$$

The scheme equations (5), (6) cannot be implemented directly, because it is nonlinear with respect to the variables ψ_i^{n+1} and ϕ_i^{n+1}. We follow the idea of [3] and introduce internal iterations, namely

$$\begin{aligned} \mathrm{i}\frac{\psi_i^{n+1,k+1} - \psi_i^n}{\tau} =& \frac{\beta}{2h^2} \left(\psi_{i-1}^{n+1,k+1} - 2\psi_i^{n+1,k+1} + \psi_{i+1}^{n+1,k+1} + \psi_{i-1}^n - 2\psi_i^n + \psi_{i+1}^n\right) \\ &+ \frac{\psi_i^{n+1,k} + \psi_i^n}{4} \Big[\alpha_1 \left(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2\right) \\ &+ (\alpha_1 + 2\alpha_2) \left(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2\right)\Big] \\ &+ \frac{\gamma}{2}(\psi_i^n + \psi_i^{n+1,k+1}) + \frac{\Gamma}{2}(\phi_i^n + \phi_i^{n+1,k+1}) \end{aligned} \tag{7}$$

$$\begin{aligned} \mathrm{i}\frac{\phi_i^{n+1,k+1} - \phi_i^n}{\tau} =& \frac{\beta}{2h^2} \left(\phi_{i-1}^{n+1,k+1} - 2\phi_i^{n+1,k+1} + \phi_{i+1}^{n+1,k+1} + \phi_{i-1}^n - 2\phi_i^n + \phi_{i+1}^n\right) \\ &+ \frac{\phi_i^{n+1,k} + \phi_i^n}{4} \Big[\alpha_1 \left(|\phi_i^{n+1,k+1}||\phi_i^{n+1,k}| + |\phi_i^n|^2\right) \\ &+ (\alpha_1 + 2\alpha_2) \left(|\psi_i^{n+1,k+1}||\psi_i^{n+1,k}| + |\psi_i^n|^2\right)\Big] \\ &+ \frac{\gamma}{2}(\phi_i^n + \phi_i^{n+1,k+1}) + \frac{\Gamma}{2}(\psi_i^n + \psi_i^{n+1,k+1}). \end{aligned} \tag{8}$$

Now for the current iteration of the unknown functions (superscript $n+1, k+1$) we have an implicitly coupled system of two tridiagonal systems with complex coefficients. The coupling here is essential. Without it, it is hard to secure absolute stability of the scheme. We begin from an initial condition $\psi^{n+1,0} = \psi^n$ and conduct the internal iterations (repeating the calculations for the same time step $(n+1)$ with increasing value of the superscript k) until convergence, i.e., when both the following criteria are satisfied

$$\begin{aligned} \max_i |\psi_i^{n+1,k+1} - \psi_i^{n+1,k}| &\leq 10^{-12} \max_i |\psi_i^{n+1,k+1}|, \\ \max_i |\phi_i^{n+1,k+1} - \phi_i^{n+1,k}| &\leq 10^{-12} \max_i |\phi_i^{n+1,k+1}|. \end{aligned} \tag{9}$$

Note that the initial condition is a very good guess which is within $O(\tau)$ of the sought solution for ψ. This makes the convergence of the internal iterations very fast. We have performed the due numerical experiments to verify this fact. Even for very large values of the time increment τ we did not encounter instability of the internal iterations. Yet, keeping τ in pace with the physical time scale of the process is important, because if the particular motion is evolving quickly in time, and the time increment is rather large, then the number of internal iterations increase significantly. In each case under consideration we selected τ such that no more than six internal iterations were required to reach the precision of 10^{-12}.

After the internal iterations converge, one gets the solution of the nonlinear scheme by setting $\psi^{n+1} \equiv \psi^{n+1,k+1}$ and $\phi^{n+1} \equiv \phi^{n+1,k+1}$. We mention here that for physically reasonable time increments τ the number of internal iterations needed for convergence is four to six, which is a small price to pay to have fully implicit, nonlinear and conservative scheme.

4. **Scheme Implementation and Validation.** As mentioned above, the linearized scheme equations (7), (8) is inextricably coupled. In order to be solved, the respective two tridiagonal linear algebraic systems are to be recast as a single system. We use a new vector of unknowns

$$\mu_{2i} = \psi_i^{n+1,k+1}, \quad \mu_{2i+1} = \phi_i^{n+1,k+1}, \qquad i = 1, \dots N,$$

which is twice longer than the original vectors ψ_i and ϕ_i. Respectively the band of the matrix increases to five.

$$\begin{pmatrix}
1 & \cdots & & & & & & & \cdots & 0\\
0 & 1 & & & & & & & \cdots & 0\\
0 & \cdots & & & & & & & \cdots & 0\\
0 & \cdots & \frac{\beta}{2h^2} & 0 & -\frac{\beta}{h^2}+\frac{\mathrm{i}}{\tau}+\frac{\gamma_\psi}{2}+f_l & \frac{\Gamma_\phi}{2} & \frac{\beta}{2h^2} & 0 & \cdots & 0\\
0 & \cdots & 0 & \frac{\beta}{2h^2} & \frac{\Gamma_\psi}{2} & -\frac{\beta}{h^2}+\frac{\mathrm{i}}{\tau}+\frac{\gamma_\phi}{2}+f_{l+1} & 0 & \frac{\beta}{2h^2} & \cdots & 0\\
0 & \cdots & & & & & & & \cdots & 0\\
0 & \cdots & & & & & & & 1 & 0\\
0 & \cdots & & & & & & & \cdots & 1
\end{pmatrix}$$

$$\times \begin{pmatrix} \mu_1\\ \mu_2\\ \cdots\\ \mu_l\\ \mu_{l+1}\\ \cdots\\ \mu_{2N-1}\\ \mu_{2N}\end{pmatrix} = \begin{pmatrix} 0\\ 0\\ \cdots\\ (\frac{\mathrm{i}}{\tau}-f_l-\frac{\gamma_\psi}{2}+\frac{\beta}{2h^2}\Lambda_{xx})\psi_l^n-\frac{\Gamma_\phi}{2}\phi_l^n\\ (\frac{\mathrm{i}}{\tau}-f_{l+1}-\frac{\gamma_\phi}{2}+\frac{\beta}{2h^2}\Lambda_{xx})\phi_l^n-\frac{\Gamma_\psi}{2}\psi_l^n\\ \cdots\\ 0\\ 0\end{pmatrix} \tag{10}$$

where $l = 1, 2, \ldots 2N$ and the first two and the last two lines acknowledge the trivial boundary conditions imposed on the unknown functions.

For the inversion of the five-diagonal $2N \times 2N$ matrix (10) we created an algorithm based on Gaussian elimination with pivoting which is a generalization of a similar algorithm for real system developed in [2]. The details of the new algorithm and the FORTRAN code will be published elsewhere.

As it should have been expected, the computational time needed for the scheme with complex arithmetic to complete the calculations for a given set of parameters is four times shorter than the scheme with real arithmetic. Indeed, for the cases presented here the computational time on PC P4 3.66 GHz required by the scheme in complex arithmetic ranged from 5 min to 5 min and 20 sec, while the scheme in real arithmetic required 20 min to 21 min.

Since there is no analytical solution to compare with, the scheme was thoroughly validated through the standard numerical tests involving halving the spacing and time increment. The global truncation error of time approximation was verified as follows. We fixed the spacing h and the time interval (say 1800 time units). Then we ran the calculations with three different time increments, $\tau = 0.04$, $\tau/2$ and $\tau/4$. The solution for $\tau/4$ is considered as an "analytical solution" and compared the other two solutions point-wise (in the common grid points) at the last moment of time, $T = 1800$. Let us denote the respective solutions $\psi^{(1)}, \psi^{(2)}, \psi^{(3)}$ (and the same for the other component ϕ). Now we can define the norms of the differences between the solutions with the three different time increments, namely

Using the three solutions we define the norms of the following two differences

$$e_1 = \max_x \{|\psi^{(1)}(x,T) - \psi^{(2)}(x,T)|, |\phi^{(1)}(x,T) - \phi^{(2)}(x,T)|\} = 2.261 \times 10^{-3},$$
$$e_2 = \max_x \{|\psi^{(2)}(x,T) - \psi^{(3)}(x,T)|, |\phi^{(2)}(x,T) - \phi^{(3)}(x,T)|\} = 5.6185 \times 10^{-4}.$$

The ratio $e_1/e_2 = 4,024206 \approx 2^2$ which confirms the second order of accuracy in time. In a similar fashion we found that the global spatial truncation error is also second-order, $O(h^2)$.

Another crucial validation is possible by a comparison with the results [3, 6]. We did repeat a couple of the more involved cases using both the scheme with real arithmetic and the presented here scheme with complex arithmetic. The results with the same scheme parameters turn out to indistinguishable within the order of round-off error.

5. **Discussion on Some Numerical Experiments.** We assume the initial condition to be of the form of a single propagating soliton for each of the functions

$$\psi(x,t), \phi(x,t) = A \operatorname{sech}[b(x - X - ct)] \exp\left[\mathrm{i}\left(\frac{c}{2\beta}x - X - nt\right)\right],$$
$$b^2 = \frac{1}{\beta}\left(n + \gamma - \frac{c^2}{4\beta}\right), \qquad A = b\sqrt{\frac{2\beta}{\alpha_1}}, \qquad u_c = \frac{2n\beta}{c}, \tag{11}$$

and n is the carrier frequency. Here X is the spatial position where the soliton is situated in the initial moment. It is different for the different functions ψ, ϕ. Note also that when n and γ are selected, then one has to check if $c < 2\sqrt{\beta(n+\gamma)}$.

First we begin with investigating the role of the nonlinear coupling parameter α_2. As already above mentioned, $\alpha_2 = 0$ means that there is no interaction between the two orthogonal modes ψ and ϕ, despite of the fact that $\alpha_1 \neq 0$ means that terms proportional to $|\psi|^2$ are present in the equation for ϕ, and vice versa. It is important to verify this behavior in the actual numerical solution. Indeed, as shown in Fig. 1, no cross-signals are excited during the interaction for this case. It

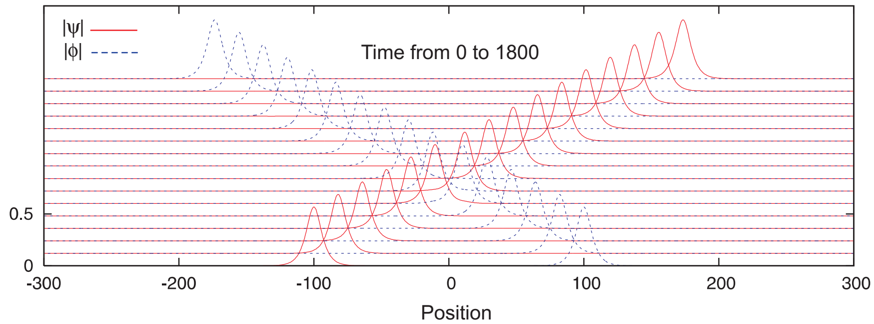

FIGURE 1. Head-on collision for equal carrier frequencies and phase speeds: $n_{\text{left}} = n_{\text{right}} = 0.05$; $c_{\text{left}} = 0.2$, $c_{\text{right}} = -0.2$; $\alpha_1 = 0.25$, $\alpha_2 = 0$, $\beta = 1$, $\Gamma = \gamma = 0$ and number of grid points 3600.

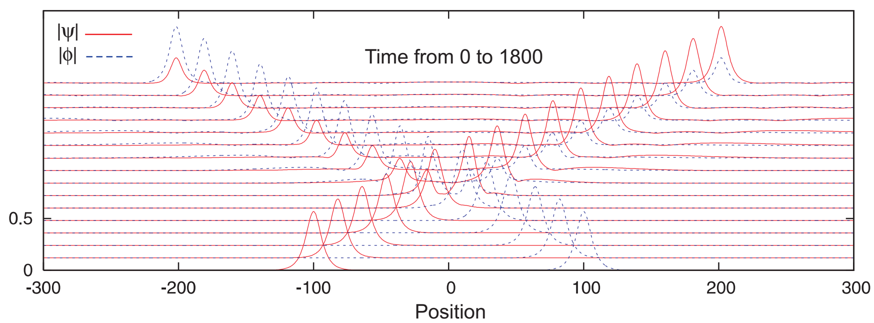

FIGURE 2. The same as in Fig. 1 with $\alpha_2 = 0.1$.

is interesting to note that the effect of α_2 is not monotone. In Fig. 2 we see that the amplitude of the excited secondary cross-signals for $\alpha_2 = 0.1$ is slightly larger than the respective amplitude in Fig. 3 where $\alpha_2 = 0.25$. Yet, the energy of the excited additional oscillation is larger for larger α_2.

The trend of increasing of the additional oscillations is clearly seen in Fig. 4 for $\alpha_2 = 0.5$. The genesis of a standing soliton in the place of the collision is observed. The further increase of $\alpha_2 = 1$ brings into view a new phenomenon: the excitation of faster solitons of smaller amplitude. In addition, the standing soliton in the point of collision becomes more intensive and its support becomes shorter, commensurate with the support of the original two solitons The excitation of additional, faster solitons for strong nonlinear interaction $\alpha_2/\alpha_1 = 4$ is a new effect which is observed for the first time. It was possible because of the strict conservative properties and the high efficiency of the developed here numerical scheme.

Another important feature of the collision for a case when $\alpha_2 > \alpha_1$, is that alongside with the excited new pair of coupled solitons, the main solitons become narrower after the collision. The comparison of Fig. 5 and Fig. 6 shows that when

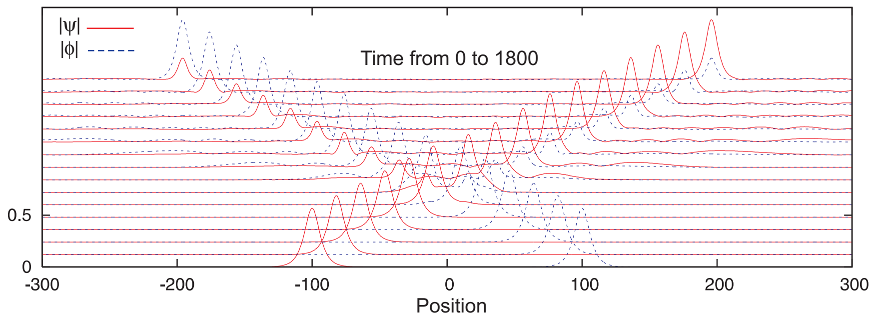

FIGURE 3. The same as in Fig. 1 with $\alpha_2 = 0.25$.

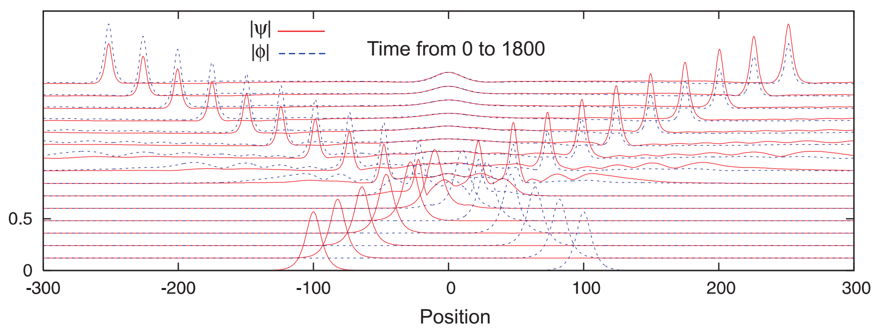

FIGURE 4. The same as in Fig. 1 with $\alpha_2 = 0.5$.

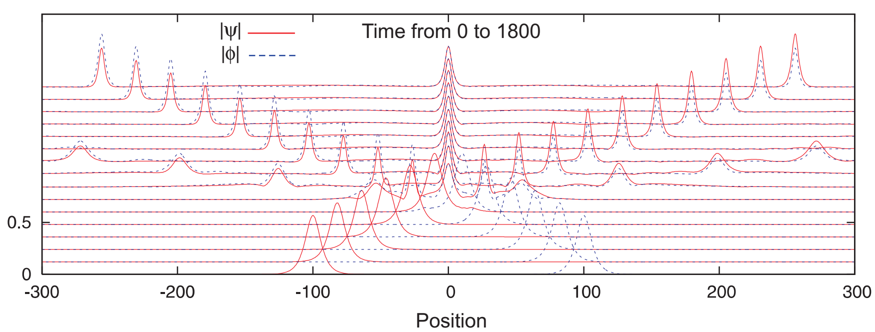

FIGURE 5. The same as in Fig. 1 with $\alpha_2 = 1$.

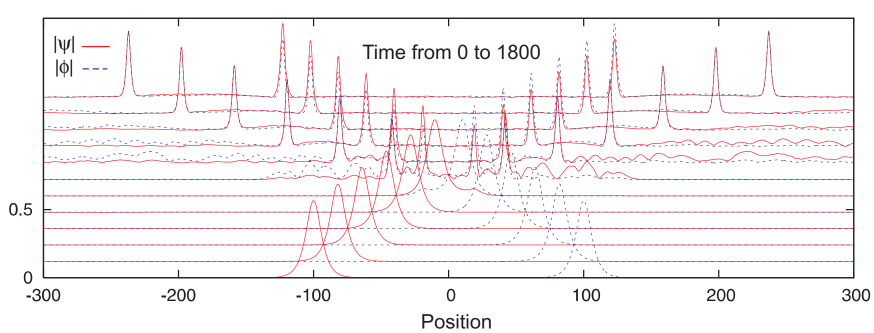

FIGURE 6. The same as in Fig. 1 with $\alpha_2 = 2$.

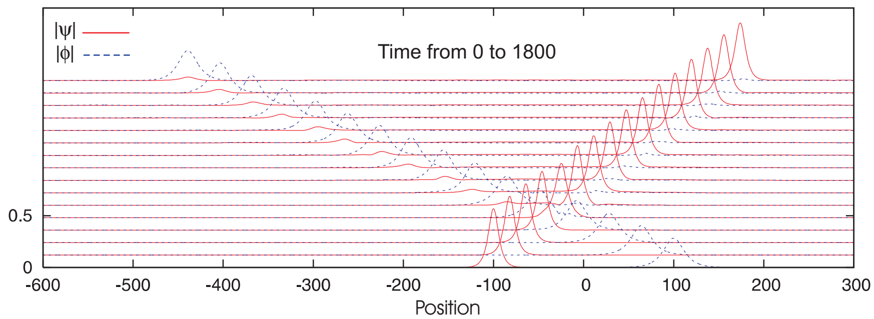

FIGURE 7. Head-on collision for equal carrier frequencies: $n_{\rm left} = n_{\rm right} = 0.05$; $c_{\rm left} = 0.2$, $c_{\rm right} = -0.4$; $\alpha_1 = \alpha_2 = 0.25$, $\beta = 1$, $\Gamma = \gamma = 0$ and number of grid points 3600.

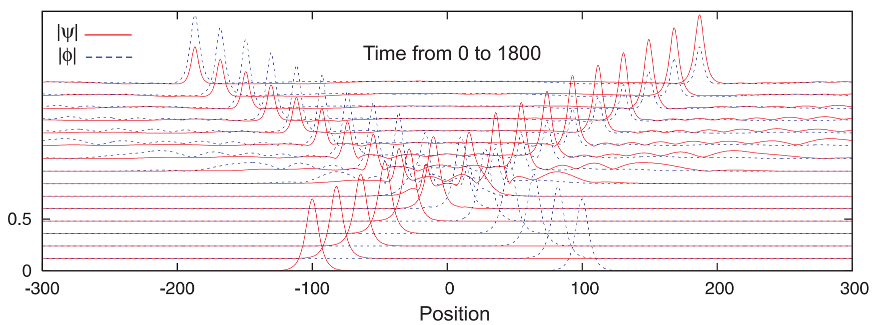

FIGURE 8. The same as in Fig. 3 ($\alpha_2 = 0.25$) but for $\gamma = 0.02$.

$\alpha_2/\alpha_1 = 8$ even the secondary solitons become narrower. This effect needs a special investigation and will be given the proper consideration in a separate work.

The non-monotone effect of the nonlinearity is seen in Fig. 7 where the interaction of two non-equal solitons is presented. The smaller (and faster) soliton acquires a larger associated signal, than the larger (and slower) soliton. In addition, the amplitudes of these cross-signals are much smaller than in the previous case. Clearly, the amplitudes of the excited cross-signals depend strongly on the total energy of the system.

In the end we like to elucidate the role of γ when it is strictly real. As mentioned in [6], the imaginary part of γ plays the role of dissipation/energy input according to its sign. The real part of γ does not destroy the conservative nature of the system. The visual comparison of Fig. 8 and Fig. 3 shows that introducing $\gamma = 0.02$ does not change qualitatively the interaction of the solitons, save the fact that the increased self-focusing makes the soliton steeper. For $\gamma < n$, the general behavior is the same as testified by Fig. 9 where the profile is plotted at particular moment of time for several different γ. There is some small differences in the phase shift but they do not warrant a special investigation.

What is more important is that increasing γ to 0.05 (which is in fact the value of n), we get strong amplification of the nonlinear interaction. As shown in Fig. 10, the interaction for $\gamma = 0.05$ and $\alpha_2 = 0.25$ exhibits the birth of the standing soliton which happens for $\gamma = 0$ only for $\alpha_2 \approx 0.6 - 0.7$.

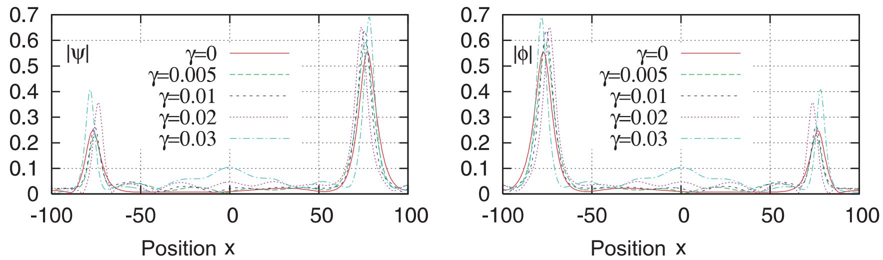

FIGURE 9. Profiles of functions ψ (left) and ϕ (right) at $t = 840$ and different values of γ: $n_{\text{left}} = n_{\text{right}} = 0.05$; $c_{\text{left}} = 0.2$, $c_{\text{right}} = -0.2$, $\alpha_1 = \alpha_2 = 0.25$, $\beta = 1$.

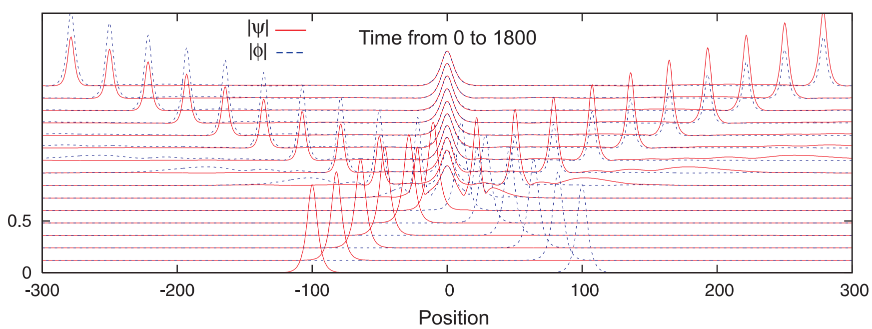

FIGURE 10. The same as in Fig. 3 ($\alpha_2 = 0.25$) but for $\gamma = 0.05$.

6. **Conclusion.** In this paper we develop a complex-arithmetic implementation of a conservative difference scheme for Coupled Nonlinear Schrödinger Equations (CNLSE). To this end a special solver for Gaussian elimination with pivoting is developed for inverting five-diagonal complex-valued matrices, which is a generalization of the solver created earlier by one of the authors (see [2]).

The algorithm is validated by the mandatory numerical tests involving doubling the mesh size and halving the time increment, as well as by the direct comparison in couple of cases with the real-arithmetic schemes [3, 6]. The advantages of the new scheme are that the band of the matrix is twice smaller and that the overall number of unknowns is also twice smaller. Our numerical tests show that, as expected, it is significantly faster than the scheme from [3, 6].

The new tool developed here allowed us to investigate physically important sets of parameters of the CNLSE under consideration. The nonlinear coupling results in changing the original polarization of the two signals from vertical/horizontal to a generally slanted one. This means that although the initial conditions are nontrivial for only one of the functions in any of the locations, after the interaction both functions acquire nontrivial amplitudes in both locations. First we unearth the influence of the nonlinear coupling parameter α_2 and show that when it is twice and more times larger than the main nonlinear parameter, α_1 the collision of the two main solitons produces two more solitons that are smaller and faster. For even larger α_2, a standing soliton is born in the exact place of the interaction and it persists without changing shape or dispersing while the moving solitons continue to the boundaries of the region. The creation of additional quasi-particles appears to be a new effect whose investigation was possible because of the strict conservative properties of the scheme.

Second we investigate the role of self-focusing (parameter γ). We dwell only on the case when this parameter has a real value, because, the imaginary part brings dissipation/energy input and the system ceases to be conservative. For $\gamma < n$ (n is the carrier frequency), the role of the self-focusing is to make the solitons steeper, but the overall qualitative picture of the interaction remains the same. For $\gamma \geq n$, the self-focusing acts also to amplify the effect of the nonlinear coupling and additional structures, such as standing solitons, can appear after the interaction for much smaller α_2 than for the case of $\gamma = 0$.

It is important to stress the point that even new solitons (quasi-particles) are born after the interactions, the total energy is conserved within the round-off error.

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