On Beam-like Functions with Radial Symmetry

N. C. Papanicolaou* and C. I. Christov†

*Dept. of Computer Science, University of Nicosia, P.O. Box 24005, 1700 Nicosia, Cyprus
†Dept. of Mathematics, University of Louisiana at Lafayette, LA 70504-1010, USA

Abstract. In this work, we introduce a complete orthonormal (CON) set of functions as the eigenfunctions of a fourth-order boundary problem with radial symmetry. We derive the relation for the spectrum of the problem and solve it numerically. For larger indices n of the eigenvalues we derive accurate asymptotic representations valid within $o(n^{-2})$.

Two model fourth order problems with radial symmetry which admit exact analytic solutions are featured: a simple problem involving only the fourth-order radial operator and a constant and the other also involving the second-order radial operator. We show that for both cases, the rate of convergence is $O(N^{-5})$ which is compatible with theoretical predictions. The spectral and analytic solutions are found to be in excellent agreement. With 20 terms the absolute pointwise difference of the spectral and analytical solutions is of order 10^{-7} which means that the fifth order algebraic rate of convergence is fully adequate.

Keywords: Radial beam functions, Galerkin spectral method, fourth-order boundary value problems with radial symmetry, Bessel functions, asymptotic methods.
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INTRODUCTION

In a previous work (see [4] and the literature cited therein) we showed that the set of beam functions was most suitable for solving fourth order boundary value problems with homogeneous boundary conditions.

We now seek to extend this idea to radial-symmetric fourth-order problems in cylindrical coordinates. Chandrasekhar [2] describes sets of functions suitable for fourth order problems with cylindrical and spherical boundaries. His functions satisfy four boundary conditions (the non-slip conditions) and also involve Bessel functions of the second kind Y_n and modified Bessel functions of the second kind K_n. However, Y_n and K_n are highly singular at the origin and thus they cannot be used in the incide of a circular domain. Here, we develop what we call 'Radial Beam Functions'. Our set of functions does not have singularities, and satisfies two homogeneous boundary conditions at the outer boundary $r = \alpha$. Radial Beam Functions (or functions derived in a similar vein) will be very useful for solving biharmonic problems, such as viscous flows in cylindrical pipes (Hagen-Poiseuille flows), and deformation of elastic plates. In this first work, we focus on radial beam functions inside a cylinder of radius $a = 1$.

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THE STURM-LIOUVILLE PROBLEM

Consider the following eigenvalue problem

$$\mathscr{L}u - \lambda^4 u = L^2 u - \lambda^4 u = 0, \qquad u(1) = \left.\frac{du}{dr}\right|_{r=1} = 0, \qquad L \stackrel{\text{def}}{=} \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}, \tag{1}$$

which can also be rewritten as follows

$$(L-\lambda^2)(L+\lambda^2)u = (L-\lambda^2)v = 0, \quad (L+\lambda^2)u = v. \tag{2}$$

The equation for function v is

$$r^2 v''(r) + rv'(r) - \lambda^2 r^2 v(r) = 0, \tag{3}$$

whose solutions are the the modified Bessel functions $I_0(\lambda r)$ and $K_0(\lambda r)$.

For the inside of the cylinder we only have to keep the function I_0. Then we have to find the solution of the following nonhomogeneous equation

$$r^2 u''(r) + ru'(r) + \lambda^2 r^2 u(r) = 2Cr^2 I_0(\lambda r). \tag{4}$$

The particular solution of this equation is $U(r) = C\lambda^{-2} I_0(\lambda r)$, while the homogeneous equation has as solutions the two Bessel functions $J_0(\lambda r)$ and $Y_0(\lambda r)$. Guided by the same considerations, we keep in this note only the function J_0 which is not singular in the origin. Thus the general solution of equation (1) has the form

$$u(r) = AJ_0(\lambda r) + BI_0(\lambda r), \tag{5}$$

where A and B are two arbitrary constants that has to be identified from the boundary conditions. The latter give us the following system

$$AJ_0(\lambda) + BI_0(\lambda) = 0, \quad AJ_0'(\lambda) + BI_0'(\lambda) = 0.$$

This system can have nontrivial solution only if

$$J_0(\lambda) I_0'(\lambda) - I_0(\lambda) J_0'(\lambda) = 0, \tag{6}$$

which gives the spectrum. An alternative form can be presented using the properties $J_0'(z) = -J_1(z), \quad I_0'(z) = I_1(z)$ (see, e.g., [5, 1]). Then the spectral equation reads

$$J_0(\lambda) I_1(\lambda) + I_0(\lambda) J_1(\lambda) = 0. \tag{7}$$

ASYMPTOTIC APPROXIMATION OF THE EIGENVALUES

The solutions λ_n of the spectral equation (7) can be found numerically. We also derive an asymptotic formula which will provide a good approximation for large n.

From [3] we can find the third order asymptotic series expansions for Bessel $J_\nu(x)$ and modified Bessel functions $I_\nu(x)$ of order ν, for the case of large x. Introducing those into (7) and neglecting the terms of order $O(|\lambda|^{-3})$ we get

$$J_0(\lambda)I_1(\lambda)+I_0(\lambda)J_1(\lambda) \propto \frac{e^{\lambda}}{\pi\lambda}\Big\{(1-\frac{24}{128\lambda^2}-\frac{3}{8\lambda})\cos(\frac{4\lambda-\pi}{4})+(\frac{1}{8\lambda}-\frac{3}{64\lambda^2})$$
$$\times\sin(\frac{4\lambda-\pi}{4})+(1+\frac{24}{128\lambda^2}+\frac{1}{8\lambda})\cos(\frac{4\lambda-3\pi}{4})-(\frac{3}{8\lambda}+\frac{3}{64\lambda^2})\sin(\frac{4\lambda-3\pi}{4})\Big\}$$
$$=\frac{e^{\lambda}}{\sqrt{2\pi\lambda}}\left\{\cos\lambda(-\frac{1}{4\lambda}-\frac{9}{32\lambda^2})+\sin\lambda(2+\frac{1}{4\lambda})+O(|\lambda|^{-3})\right\}.$$

Then, in the adopted third-order approximation, the equation for the spectrum reduces to

$$\cos\lambda(-\frac{1}{4\lambda}-\frac{9}{32\lambda^2})+\sin\lambda(2+\frac{1}{4\lambda})=0. \tag{8}$$

To find an approximate solution of equation (8) for large λ, we will stipulate that $\lambda = n\pi+\varepsilon_n+\alpha\varepsilon_n^2$, where ε_n is a small addition to the n-th root of the zeroth-order equation $\sin\lambda = 0$ and α a parameter to be determined. Then,

$$\cos\lambda = \cos(n\pi+[\varepsilon_n+\alpha\varepsilon_n^2]) = (-1)^n(1-\tfrac{1}{2}\varepsilon_n^2)+O(\varepsilon_n^3) \tag{9}$$
$$\sin\lambda = \sin(n\pi+[\varepsilon_n+\alpha\varepsilon_n^2]) = (-1)^n(\varepsilon_n+\alpha\varepsilon_n^2)+O(\varepsilon_n^3). \tag{10}$$

Introducing into equation (8), keeping only terms up to order $O(\varepsilon_n^2)$, we arrive at

$$(-8\alpha+\tfrac{11}{2}+132n\pi+64n^2\pi^2\alpha+8n\pi\alpha)\varepsilon_n^2+(-8+64n^2\pi^2+8n\pi)\varepsilon_n+(-8n\pi-9)=0.$$

To achieve a third-order approximation we set the coefficient of ε_n^2 in the last equality equal to zero, and solve for α to obtain

$$\alpha = \frac{-6.5-132n\pi}{-8+8n\pi+64n^2\pi^2}\;,\quad \varepsilon_n = \frac{8n\pi+9}{-8+8n\pi+64n^2\pi^2}\quad\text{and thus} \tag{11}$$
$$\lambda(n) = n\pi+\frac{8n\pi+9}{-8+8n\pi+64n^2\pi^2}+\frac{-6.5-132n\pi}{-8+8n\pi+64n^2\pi^2}\Big(\frac{8n\pi+9}{-8+8n\pi+64n^2\pi^2}\Big)^2 \tag{12}$$

which approximates λ_n up to order $O(\frac{1}{n^3\pi^3})$.

THE SYSTEM OF EIGENFUNCTIONS

The nth eigenfunction of (1) is given by

$$v_n(r) = A_nJ_0(\lambda_n r)+B_nI_0(\lambda_n r) \tag{13}$$

where A_n is arbitrary and $B_n = -\frac{A_nJ_0'(\lambda_n)}{I_0'(\lambda_n)} = \frac{A_nJ_1(\lambda_n)}{I_1(\lambda_n)}$. We will prescribe the value of $A_n = 1$ and thus $B_n = \frac{J_1(\lambda_n)}{I_1(\lambda_n)}$. We will now show the following theorem:

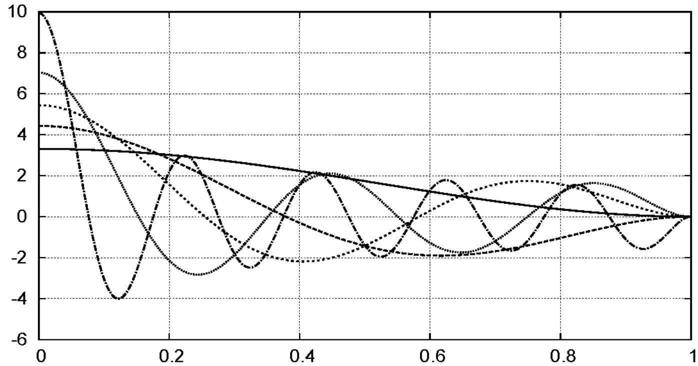

FIGURE 1. The profiles of the radial beam functions u_1,u_2,u_3,u_5,u_{10}

Theorem 1. *The set of eigenfunctions* $\{v_n(r)\}_{n=1}^{\infty}$ *as defined by (13) is orthogonal on* $[0,1]$ *with respect to the weight* $w(r) = r$*, i.e.,*

$$< v_n, v_m >_w = \int_0^1 r v_n v_m \mathrm{d}r = \begin{cases} 0 & \text{if } n \neq m, \\ c \neq 0 & \text{if } n = m. \end{cases} \tag{14}$$

Proof. Suppose λ_n, λ_m two different eigenvalues of (1). Then

$$\lambda_n^4 < v_n, v_m >_w = \int_0^1 r\lambda_n^4 v_n v_m \mathrm{d}r = \int_0^1 [r\mathscr{L} v_n(r)]\, v_m \mathrm{d}r = \int_0^1 \frac{d}{dr} r \frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr}\right) v_n v_m \mathrm{d}r.$$

Integrating by parts twice and using the boundary conditions we obtain

$$\lambda_n^4 \langle v_n, v_m \rangle_w = \int_0^1 \frac{1}{r}\frac{d}{dr}\left(rv_m'(r)\right)\frac{d}{dr}\left(rv_n'(r)\right)\mathrm{d}r. \tag{15}$$

$$\lambda_m^4 \langle u_v, v_m \rangle_w = \int_0^1 \frac{1}{r}\frac{d}{dr}\left(rv_m'(r)\right)\frac{d}{dr}\left(rv_n'(r)\right)\mathrm{d}r. \tag{16}$$

Subtracting (15)-(16) we obtain

$$(\lambda_n^4 - \lambda_m^4) < v_n, v_m >_w = 0.$$

But $\lambda_n \neq \lambda_m$ therefore $< v_n, v_m >_w = 0$ when $n \neq m$. □

Clearly when $n=m$ the number $< v_n, v_n >_w = \int_0^1 r v_n^2(r) \mathrm{d}r = ||v_n||^2$, the square of the L_2 norm of v_n. To normalize $\{v_n(r)\}_{n=1}^{\infty}$ we simply calculate the norm and divide. Thus,

$$||v_n||^2 \equiv \langle v_n, v_n \rangle_w = \int_0^1 r\left[J_0^2(\lambda_n r) + B_n^2 I_0^2(\lambda_n r) + 2B_n J_0(\lambda_n r) I_0(\lambda_n r)\right] \mathrm{d}r$$

$$= \frac{1}{2}[J_0^2(\lambda_n) + J_1^2(\lambda_n)] + \frac{1}{2}B_n^2[I_0^2(\lambda_n) - I_1^2(\lambda_n)].$$

In the above calculations we have used

$$r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}(J_0(ar))\right) = a^3 r J_1(ar), \quad r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}(I_0(ar))\right) = a^3 r I_1(ar),$$
$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}(J_0(ar)) = -a^2 J_0(ar), \quad \frac{1}{r}\frac{d}{dr}r\frac{d}{dr}(I_0(ar)) = a^2 I_0(ar),$$
$$J_1(0) = 0,\ I_1(0) = 0,\ J_0(0) = 1,\ I_0(0) = 1,\ J_0(1) \approx 0.765198,\ I_0(1) \approx 1.26607.$$

Thus, our normalized eigenfunctions now read

$$u_n(r) = \frac{\sqrt{2}}{\sqrt{[J_0^2(\lambda_n) + J_1^2(\lambda_n)] + \frac{J_1^2(\lambda_n)}{I_1^2(\lambda_n)}[I_0^2(\lambda_n) - I_1^2(\lambda_n)]}}\left(J_0(\lambda_n r) + \frac{J_1(\lambda_n)}{I_1(\lambda_n)} I_0(\lambda_n r)\right).$$

The first couple of the eigenfunctions are presented in Figure 1.

EXPANSIONS INTO SERIES OF RADIAL BEAM FUNCTIONS

In order for us to apply our spectral technique we will need formulas for expanding various expressions into series of radial beam functions.

Expansion of Unity. Unity can be expressed into a series of radial beam functions as follows

$$1 = \sum_{n=1}^{\infty} o_n u_n(r), \quad o_n = \frac{\sqrt{2}}{\lambda_n}\left([J_0^2(\lambda_n) + J_1^2(\lambda_n)] + \frac{J_1^2(\lambda_n)}{I_1^2(\lambda_n)}[I_0^2(\lambda_n) - I_1^2(\lambda_n)]\right)^{-1/2} J_1(\lambda_n), \tag{17}$$

In calculating of the above coefficients the following Bessel integrals are used

$$\int x J_0(ax) \mathrm{d}x = \frac{1}{a} x J_1(ax)\ , \quad \int x I_0(ax) \mathrm{d}x = \frac{1}{a} x I_1(ax)\ . \tag{18}$$

Expansion of the Second-Order Cylindrical Differential Operator. In problems with cylindrical symmetry, the second-order cylindrical operator L from (1) is often involved.

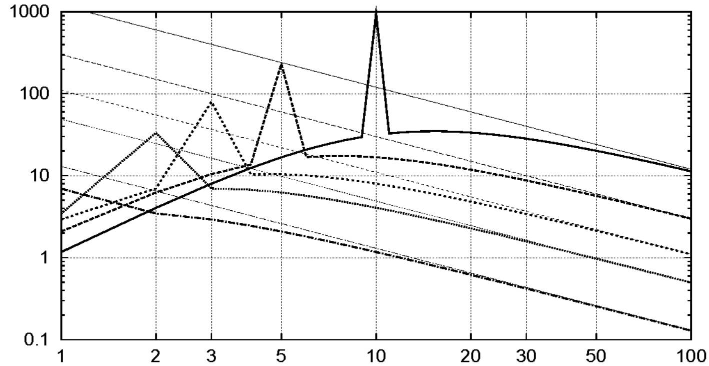

FIGURE 2. The convergence rate of the expansion of second-order operator; From bottom to top: curved lines: the coefficients g_{1n}, g_{2n}, g_{3n}, g_{5n}, g_{10n}, accompanying straight lines: the best fits $13n^{-1}$, $49n^{-1}$, $110n^{-1}$, $300n^{-1}$, $1200n^{-1}$

Therefore it is useful to have Galerkin expansions of the form

$$Lu_n(r) = \sum_{m=1}^{\infty} g_{nm} u_m(r). \tag{19}$$

The formula is obtained by evaluating the inner product $< Lu_n, u_m >_w$. Note that all integrals involved can be evaluated symbolically. Thus,

$$g_{nn} = \frac{\lambda_n^2}{2\,||v_n||^2} \left\{ -J_0^2(\lambda_n) - J_1^2(\lambda_n) + B_n^2 \left(I_0^2(\lambda_n) - I_1^2(\lambda_n) \right) \right\}, \tag{20}$$

$$\begin{aligned} g_{nm} = \frac{\lambda_n^2}{||v_n||\,||v_m||} \Bigg\{ &- \left(\frac{\lambda_n J_0(\lambda_m) J_1(\lambda_n) - \lambda_m J_0(\lambda_n) J_1(\lambda_m)}{\lambda_n^2 - \lambda_m^2} \right) \\ - B_m \left(\frac{\lambda_m I_1(\lambda_m) J_0(\lambda_n) + \lambda_n I_0(\lambda_m) J_1(\lambda_n)}{\lambda_n^2 + \lambda_m^2} \right) &+ B_n \left(\frac{\lambda_n I_1(\lambda_n) J_0(\lambda_m) + \lambda_m I_0(\lambda_n) J_1(\lambda_m)}{\lambda_m^2 + \lambda_n^2} \right) \\ &+ B_n B_m \left(\frac{\lambda_n I_0(\lambda_m) I_1(\lambda_n) - \lambda_m I_0(\lambda_n) I_1(\lambda_m)}{\lambda_n^2 - \lambda_m^2} \right) \Bigg\} \qquad n \neq m. \end{aligned} \tag{21}$$

TEST CASES

Simple Fourth-Order BVP. In order for us to test our method, we at first consider the problem

$$\mathscr{L} u = \frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \left(\frac{1}{r}\frac{d}{dr} r \frac{d}{dr} \right) u = 64, \qquad u(1) = \left. \frac{du}{dr} \right|_{r=1} = 0, \tag{22}$$

which admits the exact analytic solution

$$u_{\text{an}}(r) = 1 - 2r^2 + r^4. \tag{23}$$

To apply our technique we expand the sought function into series

$$u_{\text{sp}}(r) = \sum_{m=1}^{N} a_m u_m(r). \tag{24}$$

Substituting (17) and (24) into (22), using the basic property of the eigenfunctions (1) and taking successive inner products with $u_n(r)$, $n = 1,2,3,\ldots,N$, yields

$$a_n = 64\,\frac{o_n}{\lambda_n^4} = \frac{1}{||v_n||}\frac{128}{\lambda_n^5}J_1(\lambda_n) \ \ , \ \ n = 1,2,3,\ldots,N. \tag{25}$$

From this we may deduce that the convergence rate of the solution coefficients a_n is fifth order algebraic. Indeed, Figure 3 below confirms our assertion.

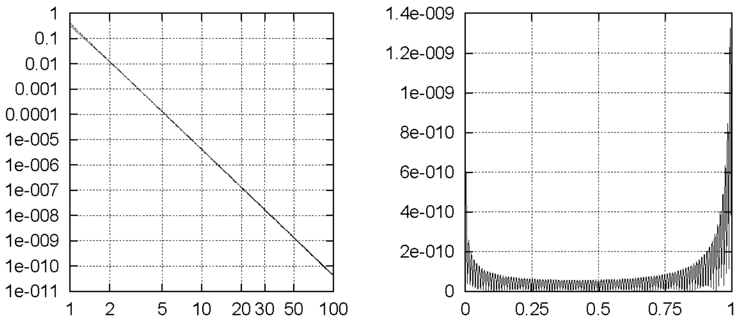

FIGURE 3. Left panel: convergence rate of solution coefficients a_n along with the best fit line $a_n = 0.4\ n^{-5}$. Right panel: absolute difference between the exact analytic solution and the spectral solution with 100 terms

Furthermore, we observe that the maximum absolute error is of order 10^{-9}.

Fourth-Order BVP involving the Second-Order Radial Differential Operator. We now consider the following problem which also involves the second-order radial differential operator,

$$L^2u - Lu = -2I_1(1)(\approx -1.13031820798497), \qquad u(1) = u'(1) = 0, \tag{26}$$

and admits the exact analytic solution $u_{\text{an}}(r) = -I_0(r) + \frac{1}{2}I_0'(1)r^2 + I_0(1) - \frac{1}{2}I_0'(1)$.

Once again we expand the sought function into a series of radial beam functions $u_{\text{sp}}(r) = \sum_{n=1}^{N} a_n u_n(r)$ and employ formulas (1),(17),(20) to obtain the Galerkin system

$$\lambda_n^4 a_n - \sum_{m=1}^{N} g_{mn} a_m = -1.13031820798497 o_n \ \ , \ \ n = 1,2,3,\ldots,N. \tag{27}$$

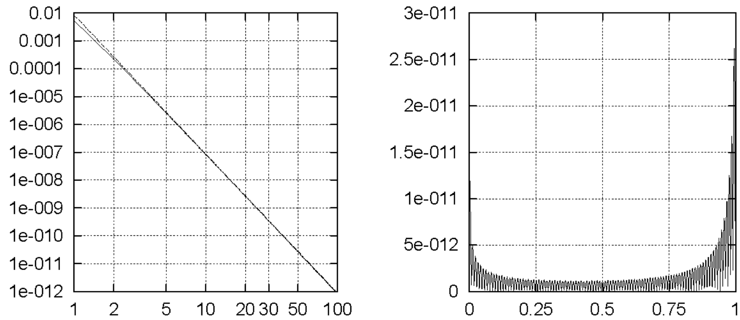

FIGURE 4. Left panel: convergence rate of solution coefficients a_n along with the best fit line $a_n = 0.0086\, n^{-5}$. Right panel: absolute difference between the exact analytic solution and the spectral solution with 100 terms

The matrix multiplying the spectral coefficients $\{a_n\}_{n=1}^{N}$ in (27) is symmetric. We solve (27) using IMSL routine DLSASF for linear systems with a real symmetric coefficient matrix.

We can see in Figure 4 that the convergence rate of the spectral coefficients is once again fifth-order algebraic, and that the overall absolute error when using $N = 100$ terms is $O(10^{-11})$.

CONCLUSIONS

A complete orthonormal set of functions —the so-called Radial Beam functions— is introduced. This is the set of eigenfunctions of a Sturm-Liouville fourth-order boundary problem with radial symmetry. The eigenvalues are found numerically whereas for larger indices n of the eigenvalues, asymptotic representations valid within $o(n^{-2})$ are derived.

The appropriate formulas for expressing unity and the various derivatives of our functions into members of the CON are derived and their convergence rate is verified.

The technique is assessed by applying it to two model fourth order problems with radial symmetry which admit to exact analytic solutions: a simple problem involving only the fourth-order radial operator and a constant and the other also involving the second-order radial operator. In both cases, the convergence rate of the spectral coefficients is $O(N^{-5})$.

The spectral and analytic solutions are found to be in excellent agreement. For the first test problem 200 terms secure a maximum absolute error of order $O(10^{-11})$, whereas for the second test problem this difference is achieved with only 100 terms -a result fully compatible with the theoretical rate of convergence.

The fifth-order algebraic rate of convergence is fully adequate for the problems under consideration. Thus, a new technique suitable for biharmonic fourth-order boundary problems with radial symmetry is introduced. The approach can be easily extended to other fourth-order problems that arise in fluid dynamics and elasticity which involve the bi-Stokesian or even more complex operators.

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