On the Inertial Force Experienced by a Solid Body Undergoing Rotation about Two Axes

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Abstract. The theory of rigid body motion is used to derive the governing equations, in terms of the Eulerian angles, of a top rotating about two axes. Then, a formula for the 'lifting' component of the net inertial force (as function of the angle of inclination, the top's two angular velocities and its moments of inertia) is derived for a particular motion termed constrained nutation. In a distinguished limit, the critical value of the angle of inclination, i.e., the value for which the vertical component of the net inertial force acting on the top overcomes the weight of the rotating system, is calculated.

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INTRODUCTION

There are many anecdotal tales of the ability of gyroscopes to lift themselves, including emotional recounts. However, the phenomenon is not new, and a wonderful exposition on how a spinning top may lift itself was given 90 years ago by Prof. Andrew Gray, F.R.S., in response to a letter to the Royal Society [1]. Yet, to the best of our knowledge, the only *quantitative* measurements of the inertial forces (or any resulting weight reduction, when lift is not achieved) of such a gyroscopic device, which involves the rotation of a solid body about *two* axes, are the recent study of Wayte [2] and the informal results of Turner [3]. Quantitative theoretical predictions regarding the lifting force are also absent in the literature.

Motivated by the positive results reported in [1, 2, 3], we present an in-depth study of the dynamics of a solid body rotating about *two* axes. Typically, for such a mechanical system, two types of motion are identified—precession and nutation,—which are explained by appealing to the principles of conservation of energy and angular momentum. On the other hand, we give an equivalent theoretical discussion of the motion within the conceptual framework of *inertial forces*.

POSITION OF THE PROBLEM AND GOVERNING EQUATIONS

Consider an idealized gyroscope consisting of a massive disk (the 'top') rotating about two distinct axes (see Figure 1). Then, by $\vec{\Omega}_p$ we denote the angular velocity of rotation of the top's axis about the vertical axis, while by $\vec{\Omega}_t$ we denote the angular velocity of the

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top's rotation about its axis of symmetry. Then, $\vec{\omega} \stackrel{\text{def}}{=} \vec{\Omega}_{\text{p}} + \vec{\Omega}_{\text{t}}$ is the total angular velocity vector. By *Pxyz* and *OXYZ* we denote the local Cartesian coordinate systems associated with the top and the 'laboratory,' respectively. That is, *OXYZ* is a *fixed* (external) coordinate system with origin *O* located at the intersection between the vertical axis and the line perpendicular to it through the top's center, i.e., *P*.

Furthermore, $\phi(t)$ is the angle that describes the rotation of the top about *Pz*, so that $\dot{\phi}(t)$, where a superimposed dot denotes differentiation with respect to time, is the angular velocity of the top. Similarly, $\psi(t)$ is the angle that describes the rotation of the top's axis of rotation *Pz* about the vertical axis *OZ*, and $\dot{\psi}(t)$ is the corresponding angular speed. In addition, *Pz* meets *OZ* at a certain point, concluding a dynamical angle $\theta(t)$, termed the *angle of inclination*, with the latter. This angle can range between between 0° (the vertical position) and 90° (the horizontal position). Finally, *L* is the distance from the center of the top *P* to the center point *O* of the fixed coordinate system, and *l* is the distance from *P* to the point where the axes *Pz* and *OZ* intersect.

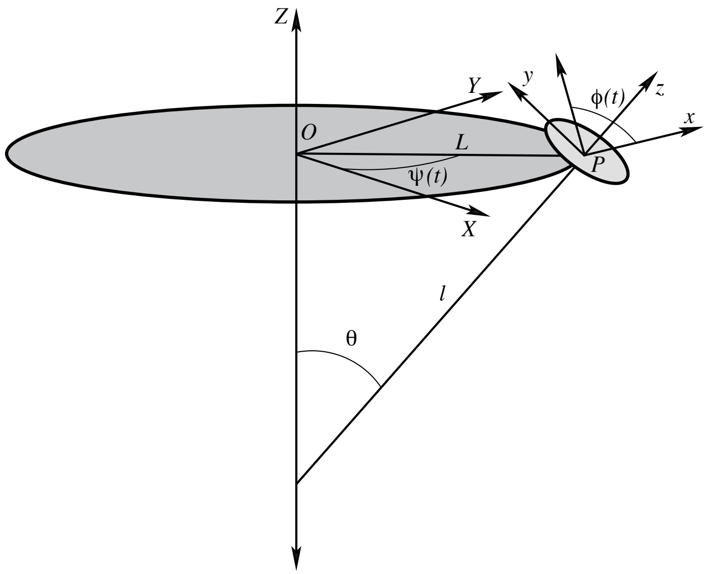

FIGURE 1. Schematic of the idealized gyroscope

Adapting the analysis in [4, 5] to our problem, it is easy to show, via elementary geometry, that the two angular velocity vectors we identified above have the following decompositions in the *Pxyz* coordinate system:

$$\vec{\Omega}_{\text{t}} = \dot{\psi} \sin\theta \sin\phi \; \hat{i} + \dot{\psi} \sin\theta \cos\phi \; \hat{j} + \dot{\phi} \; \hat{k}, \tag{1}$$

$$\vec{\Omega}_{\text{p}} = \dot{\theta} \cos\phi \; \hat{i} - \dot{\theta} \sin\phi \; \hat{j} + \dot{\psi} \cos\theta \; \hat{k}, \tag{2}$$

where $\hat{i}$, $\hat{j}$ and $\hat{k}$ are the unit vectors in the directions of the axes *Px*, *Py* and *Pz*, respectively.

Now, we employ the *Eulerian angles* ϕ, ψ and θ of the system as generalized coordinates in the usual least-action (variational) formulation of mechanics to obtain the governing equations. This idea applied to solid body rotation about two axes first appears in the modern literature in [6, Chap. VI, §69–75].

Using the angular velocities given by (1) and (2) together with the fact that the top is symmetric, we find that the kinetic energy of the system is

$$T = \tfrac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\psi}^2\sin^2\theta\right) + \tfrac{1}{2}I_3\left(\dot{\phi} + \dot{\psi}\cos\theta\right)^2, \tag{3}$$

where I_1 is the moment of inertia of the top about any axis through P perpendicular to the axis Pz, I_3 is the moment of inertia of the top about the axis Pz.

Similarly, it can be shown (see, e.g., [4, 5]) that the potential energy of the system is

$$V = Wl\cos\theta, \tag{4}$$

where W is magnitude of the gravitational force acting on the device (for now, we shall assume all components are massless except for the top, so W is just its weight), and $l = L/\sin\theta$ (recall Figure 1).

Noting that the system is conservative, we appeal to Hamilton's principle to obtain the equations of motion in terms of ϕ, ψ and θ. Using the expressions for the kinetic energy T and the potential energy V given by (3) and (4), respectively, it is an elementary exercise to obtain the governing equations (see, e.g., [4, 5] for more details). These, upon integration of the first two and rearrangement of the third one, are

$$\dot{\psi}\cos\theta + \dot{\phi} = \text{const.} \stackrel{\text{def}}{=} \omega_z, \tag{5a}$$

$$I_1\dot{\psi}\sin^2\theta + I_3\omega_z\cos\theta = \text{const.} \stackrel{\text{def}}{=} p_\psi, \tag{5b}$$

$$I_1\ddot{\theta} - \dot{\psi}\sin\theta(I_1\dot{\psi}\cos\theta - I_3\omega_z) - Wl\sin\theta = 0. \tag{5c}$$

As an aside, we note that (5a) and (5b) represent *integrals of motion* with an immediate physical interpretation: ω_z is the component of the total angular velocity $\vec{\omega}$ along the axis Pz, while p_ψ is the component of the total angular momentum along the axis OZ (see, e.g., [4] for more details).

EQUILIBRIUM VALUE OF θ

Now, suppose that the top is undergoing *stationary* precession, so that the first and second time derivatives of θ can be neglected, and the governing equation of motion for θ, i.e., (5c), adopts the following *quasi-stationary* form:

$$\dot{\psi}\sin\theta(I_1\dot{\psi}\cos\theta - I_3\omega_z) + Wl\sin\theta = 0. \tag{6}$$

It is possible to interpret (6) as a torque balance, and hence a statement of the fact that the change in angular momentum must be balanced by the applied torques, because the generalized forces corresponding to generalized coordinates of rotation (such as the Eulerian angles we have used) are components of torques [5, §5–5]. This balance gives us an equilibrium value $\theta^\bullet$ for the angle of inclination θ.

Now, from the definition that torque is the moment of force and the fact that (6) represents one particular component of the (vector) torque balance, it is clear that (6) can also represent a balance of forces. Thus, we have our inertial-forces interpretation: the first and second terms on the left-hand side of the equality in (6) are the component of the net torque in the direction of the generalized coordinate θ due the inertial forces in the vertical direction and the pull of gravity, respectively. Taking things a step further, since these two forces have the same moment arm, we may define the component of the net inertial *force* in the direction opposite that of gravity as

$$F_I \equiv -l^{-1}\dot{\psi}(I_1\dot{\psi}\cos\theta - I_3\omega_z), \tag{7}$$

and, then, (6) is a direct corollary of d'Alembert's principle, namely $F_I - W = 0$. Therefore, it now follows that, when $\theta = \theta^\bullet$, the vertical component of the inertial forces balances the pull of gravity (see [2, Figure 5] for experimental evidence), so there is no acceleration in the vertical direction. But what if the value of θ is kept *fixed yet unequal* to the equilibrium value $\theta^\bullet$, i.e., the top is in the nutation regime but is constrained?

NET VERTICAL INERTIAL FORCE

Now, we have to solve the converse problem: knowing the equilibrium angle $\theta^\bullet$ and given fixed angular speeds $\alpha \equiv \dot{\psi}$ and $\beta \equiv \dot{\phi}$, can we derive conditions under which, if θ were fixed at some value unequal to $\theta^\bullet$, there is an excess net inertial force in the vertical direction that is not balanced by gravity?

Under these assumptions, α, β, I_1 and I_3 are given constants. Now, we substitute ω_z from (5a) into (7) and acknowledge that L and θ are fixed so that $l = L/\sin\theta = const.$, to obtain

$$F_I = L^{-1}\alpha\sin\theta\left[(I_3 - I_1)\alpha\cos\theta + I_3\beta\right]. \tag{8}$$

At this point, to make the analysis tractable, we must specify some of the implementation details of the device. We take the top to be a very thin disk of mass M, radius R and thickness $\delta \ll R$ rotating around a massless rod that coincides with the axis Pz. Consequently, $I_3 = \frac{1}{2}MR^2$ and $I_1 = \frac{1}{4}MR^2$ [7, 5]. From (8), it follows that F_I does not depend on the signs of α and β individually, so we take $\alpha > 0$ and $\beta < 0$ without loss of generality. Furthermore, we take $|\beta| \gg |\alpha|$, that is we assume that the top's angular speed is much greater than the platform's (equivalently, that the top rotates much faster about Pz than about OZ).

Upon acknowledging the last few assumptions, (8) becomes

$$F_I \simeq L^{-1}\alpha\beta I_3\sin\theta. \tag{9}$$

This simplified expression allows us to easily find values of θ, say θ^*, that lead to an imbalance between the net inertial force in the vertical direction and the pull of gravity. In particular, we can extend the analysis of the previous section, where we showed how to derive the equilibrium value $\theta^\bullet$ at which $F_I = W$ for a free spinning top. Here, it is important to note that, since we are now considering the *whole* gyroscopic device (the top, the platform, the connecting rods, the motors that drive the rotations, etc), $\hat{W} = \hat{M}g$, where $\hat{M}$ is the *combined* mass of the whole device. So, the value of θ for which $F_I = \hat{W}$ is *different* from the one which (6) yields; i.e., $\theta^* \neq \theta^\bullet$.

Fortunately, using (9), we can identify this *critical* inclination angle θ^* (note the difference between 'critical' and 'equilibrium') at which the lifting force balances the weight $\hat{W}$ of the whole device, namely the solution of

$$\sin\theta^* = \frac{\hat{W}L}{\alpha\beta I_3} = \frac{2\hat{W}L}{\alpha\beta MR^2} \tag{10}$$

for our case, since we have $I_3 = \frac{1}{2}MR^2$.

Finally, for (10) to have a solution, we must have that the right-hand side is ≤ 1. Thus, if we hold all parameters except α fixed, then for a platform spinning fast enough, i.e.,

$$|\alpha| \geq \frac{2\hat{W}L}{|\beta|MR^2}, \tag{11}$$

there exists an angle at which the net vertical inertial force overcomes the pull of gravity.

CONCLUSIONS

In the present work, using the Eulerian angles of the system as generalized coordinates in the variational formulation of classical mechanics, we derived and analyzed the equations of motion of a top rotating about two axes. In the quasi-stationary regime (i.e., when the angle of inclination $\theta(t) \approx \text{const}$), we obtained an expression for the vertical component of the net inertial force due to the inertia from simultaneous rotation of the top about two axes. Moreover, we showed how to find the *equilibrium* value $\theta^\bullet$ of the angle of inclination such that the component of the net inertial force in the vertical direction balances the weight of a free spinning top (this is called steady precession). Then, we continued this line of reasoning to deduce an expression for the *critical* angle of inclination θ^* for a thin disk that is spinning about its axis of symmetry much faster than the rotation rate of the platform it is part of. Since the top's axis of rotation is constrained and it cannot nutate, if $\theta(t) = \theta^*\ \forall t$ then the weight of the *entire* system is balanced by the vertical component of the inertial forces. This result is the first quantitative prediction of gyroscopic weight-reduction/lifting phenomena.

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