Galerkin Spectral Method for the 2D Solitary Waves of Boussinesq Paradigm Equation

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Abstract. We consider the 2D stationary propagating solitary waves of the so-called Boussinesq Paradigm equation. The fourth- order elliptic boundary value problem on infinite interval is solved by a Galerkin spectral method. An iterative procedure based on artificial time ('false transients') and operator splitting is used. Results are obtained for the shapes of the solitary waves for different values of the dispersion parameters for both subcritical and supercritical phase speeds.

Keywords: Spectral methods, Boussinesq equation, CON system, operator splitting.
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INTRODUCTION

One of the most important feature of generalized wave equations containing nonlinearity and dispersion, is that they possess solutions of type of permanent waves which behave in many instances as particles. When the governing system is fully integrable, such waves are called *solitons*. In 1D, a plethora of deep mathematical results have been obtained for solitons [1, 2]. The success was contingent upon the existence of an analytical solution of the respective nonlinear dispersive equation. Naturally, the predominant part of the theoretical results were confined to the 1D case. At this stage, it is of crucial importance to investigate also the 2D case, because of the different phenomenology and the practical importance. No analytical solutions are available in the literature for the Boussinesq equation. Interesting results are obtained for the so-called Kadomtsev-Petviashvili equation (KP), which has fourth spatial derivatives only in one spatial direction, while in the other direction, the highest-order derivative is the second one. Interesting analytical results are obtained for localized waves which are localized in the direction with the fourth-order derivative, and are periodic in the other direction (see, e.g., [3, 4, 5] and the literature cited therein). For the time being, the full Boussinesq equation still remains less amenable to analytical techniques, which requires the development of numerical techniques.

Numerical treatment of localized solutions imposes special requirements on the technique to be used because no boundary conditions are specified at given points, but rather the square of solution is required to be integrable over the infinite domain. Such solution is said to belong to the $L^2(-\infty,\infty)$ space. A number of difficulties are encountered on the way of application of difference or/and finite-element numerical methods to the problems in $L^2(-\infty,\infty)$. One of the worst setbacks lies in the inevitable reducing of the infinite interval to a finite one. Such a procedure introduces artificial eigenvalue problems. This difficulty can be surmounted if a spectral method is used with basis system

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of localized functions which automatically acknowledge the requirement that the solution belongs to $L^2(-\infty,\infty)$ space. Here we make use of a complete orthonormal (CON) system of functions proposed in [6]. The product property of this system is crucial because it allows one to use a Galerkin type of expansion the latter being much simpler and faster in implementation than the pseudo-spectral algorithm. The application of the CON system [6] to 2D was initiated in [7] for a model problem of quadratic Klein–Gordon equation (KGE) and continued in [8] for the Boussinesq Equation.

Here we investigate numerically the stationary propagation of localized waves for Boussinesq Paradigm equation (see [9, 10]) which involves a fourth-order spatial derivative and second-time-second-space mixed derivative. We add an artificial time ('false transient') and solve the resulting 2D parabolic system by operator splitting.

BOUSSINESQ PARADIGM MODEL

The first soliton-supporting Generalized Wave Equation (GWE) was derived by Boussinesq [11] (see, also the overview in [5]) who found its permanent solution to be of *sech* type. The existence of a localized solution proved that a balance between dispersion and nonlinearity can exist. Later on Korteweg and De Vries [12] derived the evolution equation for the wave amplitude in the moving frame. The same *sech* is a solution also to KdV equation. The original Boussineq equation turned out to be incorrect in the sense of Hadamard, and the numerical works were focused on the *ad-hoc* change of the sign of the fourth-order derivatives. A regularized version was proposed by [13, 14] by changing the fourth-order spatial derivative to a mixed second in time and second in space derivative. The Regularized Long-Wave Equation from [13] (called also Benjamin–Bona-Mahony equation or BBME) opened the way for direct numerical investigation of this model. An energy consistent rigorous derivation of Boussinesq approximation was presented in [9, 10] where it was shown that both dispersion terms connected with the different fourth derivatives must be present in a physically correct derivation and the term 'Boussinesq Paradigm Equation' (BPE) was coined.

Here, we consider the two dimensional BPE, namely

$$u_{tt} = \Delta[u - u^2 + \beta_1 u_{tt} - \beta_2 \Delta u],$$

where $\beta_i > 0,\ i = 1,2$ are the dispersion parameters, and Δ is the Laplace operator. Upon introducing an auxiliary function q one can show that the BPE follows from the following system:

$$u_{tt} = \Delta q, \quad q = u - u^2 + \beta_1 u_{tt} - \beta_2 \Delta u. \tag{1}$$

The solitary wave solution is subject to the asymptotic boundary conditions (a.b.c):

$$u \to 0, \quad q \to 0 \quad \text{as} \quad x \to \pm\infty,\ y \to \pm\infty. \tag{2}$$

Let c_1 and c_2 be the components of the phase speed of the center of the steadily propagating localized wave. In a frame moving with the wave one can introduce new independent variables $\tilde{x} = x - c_1 t$ and $\tilde{y} = y - c_2 t$, and get for the stationary localized solution an elliptic system. Since re-scaling the spatial variables ξ and η does not change the nature of the asymptotic boundary value problem, we introduce the scalings $\xi = \lambda\tilde{x}$, $\eta = \mu\tilde{y}$ and arrive at the following system

$$
\begin{aligned}
0 &= \lambda q_{\xi\xi} + \mu q_{\eta\eta} - [\lambda^2 c_1^2 u_{\xi\xi} + 2\lambda\mu c_1 c_2 u_{\xi\eta} + \mu^2 c_2^2 u_{\eta\eta}], \\
0 &= q - u + u^2 - \beta_1[\lambda^2 c_1^2 u_{\xi\xi} + 2\lambda\mu c_1 c_2 u_{\xi\eta} + \mu^2 c_2^2 u_{\eta\eta}] + \beta_2[\lambda^2 u_{\xi\xi} + \mu^2 u_{\eta\eta}].
\end{aligned} \tag{3}
$$

Using the scaling parameters is crucial because they allow one to adjust the characteristic length of the CON system of functions, to the characteristic length of the sought solution. The importance of the scaling parameters was demonstrated in previous works of the authors [15, 16]. In the present paper we will use the optimal values of the scaling parameters without going into details how they were selected.

FALSE TRANSIENTS AND OPERATOR SPLITTING

We introduce artificial time in each equation of Eqs. (3), and use a time stepping scheme as an iterative procedure to obtain the solution of the elliptic system. In order to achieve second order of approximation in time we use staggered time stages for the two functions. To make the time stepping computationally efficient we use the method of operator splitting [17], namely

$$
\begin{aligned}
\frac{\tilde{q} - q^{n-\frac{1}{2}}}{\tau} &= \lambda^2 \tilde{q}_{\xi\xi} + \mu^2 q_{\eta\eta}^{n-\frac{1}{2}} - [\lambda^2 c_1^2 u_{\xi\xi}^n + 2\lambda\mu c_1 c_2 u_{\xi\eta}^n + \mu^2 c_2^2 u_{\eta\eta}^n], \\
\frac{q^{n+\frac{1}{2}} - \tilde{q}}{\tau} &= \mu^2 q_{\eta\eta}^{n+\frac{1}{2}} - \mu^2 q_{\eta\eta}^{n-\frac{1}{2}}
\end{aligned} \tag{4}
$$

$$
\begin{aligned}
\frac{\tilde{u} - u^n}{\tau} &= \lambda^2(\beta_2 - c_1^2\beta_1)\tilde{u}_{\xi\xi} + \mu^2(\beta_2 - c_2^2\beta_1)u_{\eta\eta}^n + q^{n+\frac{1}{2}} - \frac{1}{2}(\tilde{u} + u^n), \\
&\quad + (u^n)^2 - 2\beta_1\lambda\mu c_1 c_2 u_{\xi\eta}^n \\
\frac{u^{n+1} - \tilde{u}}{\tau} &= \mu^2(\beta_2 - c_2^2\beta_1)(u_{\eta\eta}^{n+1} - u_{\eta\eta}^n) - \frac{1}{2}(u^{n+1} - u^n).
\end{aligned} \tag{5}
$$

Here τ is the time increment with respect to the fictitious time and it plays the role of an iteration parameter. We can show the consistency of our scheme if we rewrite it as

$$
\begin{aligned}
(I - \tau\Lambda_{\xi\xi})\tilde{q} &= q^{n-\frac{1}{2}} + \tau\Lambda_{\eta\eta} q^{n-\frac{1}{2}} - \tau[c_1^2\Lambda_{\xi\xi}u^n + 2\lambda\mu c_1 c_2 u_{\xi\eta}^n + c_2^2\Lambda_{\eta\eta}u^n], \\
\tilde{q} &= q^{n+\frac{1}{2}} - \tau\Lambda_{\eta\eta} q^{n+\frac{1}{2}} + \tau\Lambda_{\eta\eta} q^{n-\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
[I - \tau(\beta_2 - c_1^2\beta_1)\Lambda_{\xi\xi} + \tfrac{1}{2}\tau]\tilde{u} &= [I - \tfrac{1}{2}\tau + \tau(\beta_2 - c_2^2\beta_1)\Lambda_{\eta\eta}]u^n + \tau q^{n+\frac{1}{2}} - 2\beta_1\lambda\mu c_1 c_2 \tau u_{\xi\mu}^n, \\
\tilde{u} &= u^{n+1} - (\beta_2 - c_2^2\beta_1)\tau\Lambda_{\eta\eta}(u^{n+1} - u^n) + \frac{\tau}{2}(u^{n+1} - u^n),
\end{aligned}
$$

where I is the identity matrix operator and $\Lambda_{\xi\xi} \equiv \lambda^2 \frac{d^2}{d\xi^2}$, $\Lambda_{\eta\eta} \equiv \mu^2 \frac{d^2}{d\eta^2}$. We get "the full-time-step scheme" if we eliminate $\tilde{q}$ and $\tilde{u}$ by multiplying the second equation of each system of equations by $(I - \tau\Lambda_{\xi\xi})$ and $[I - \tau(\beta_2 - c_1^2\beta_1)\Lambda_{\xi\xi} + \frac{1}{2}\tau]$ respectively.

GALERKIN SPECTRAL METHOD IN $L^2(-\infty,\infty)$

To investigate the problem we use a Galerkin spectral method. The advantage of the Galerkin method over the collocation method is that it has only one type of error: the truncation error, while in the case of the latter one has to deal also with the discretization error. The disadvantage is that it cannot operate effectively unless a product formula is known for the functions of the basis set. A system with the required property was proposed in [6] consisting of the real and imaginary parts of the Wiener functions [18]

$$C_n(x) = \frac{\rho_n - \rho_{-n-1}}{\sqrt{2}}, \quad S_n = \frac{\rho_n + \rho_{-n-1}}{i\sqrt{2}}, \quad \rho_n = \frac{1}{\sqrt{\pi}}\frac{(ix-1)^n}{(ix+1)^{n+1}}, \quad n = 0,1,2,\ldots \tag{6}$$

which are the Fourier transforms of the Laguerre functions. The significance of the system Eq. (6) for nonlinear problems was demonstrated in [6], where the product formula was derived. More information regarding the system and its applications can been found in [6, 7, 15, 16].

Now, we expand the sought solution into series with respect to C_n and S_n:

$$\begin{aligned}
u^l(\xi,\eta) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left[a^l_{mn}C_m(\xi)C_n(\eta) + b^l_{mn}S_m(\xi)S_n(\eta)\right],\\
\tilde{u}(\xi,\eta) &= \sum_{i=0}^{\infty}\sum_{g=0}^{\infty}\left[\tilde{p}_{ig}C_i(\xi)C_g(\eta) + \tilde{r}_{ig}S_i(\xi)S_g(\eta)\right],\\
q^{l+\frac{1}{2}}(\xi,\eta) &= \sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\left[d^{l+\frac{1}{2}}_{kj}C_k(\xi)C_j(\eta) + e^{l+\frac{1}{2}}_{kj}S_k(\xi)S_j(\eta)\right],\\
\tilde{q}(\xi,\eta) &= \sum_{w=0}^{\infty}\sum_{z=0}^{\infty}\left[\tilde{f}_{wz}C_w(\xi)C_z(\eta) + \tilde{g}_{wz}S_w(\xi)S_z(\eta)\right].
\end{aligned} \tag{7}$$

We insert the spectral expansion (7), into Eqs. (4), (5) and obtain for the even functions:

$$\begin{aligned}
\frac{\tilde{f}_{mn} - d^{l-\frac{1}{2}}_{mn}}{\tau} &= \lambda^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\tilde{f}_{kj}\chi^{\xi}_{kj} + \mu^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}d^{l-\frac{1}{2}}_{kj}\chi^{\eta}_{kj} - \lambda^2c_1^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^l_{kj}\chi^{\xi}_{kj}\\
&\quad - 2\lambda\mu c_1c_2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^l_{kj}\theta^{\xi}_{kj}\theta^{\eta}_{kj} - \mu^2c_2^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^l_{kj}\chi^{\eta}_{kj}\\
\frac{d^{l+\frac{1}{2}}_{mn} - \tilde{f}_{mn}}{\tau} &= \mu^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}d^{l+\frac{1}{2}}_{kj}\chi^{\eta}_{kj} - \mu^2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}d^{l-\frac{1}{2}}_{kj}\chi^{\eta}_{kj}\\
\frac{\tilde{p}_{mn} - a^l_{mn}}{\tau} &= \lambda^2(\beta_2 - c_1^2\beta_1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}d^{l+\frac{1}{2}}_{kj}\chi^{\xi}_{kj} + \mu^2(\beta_2 - c_2^2\beta_1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^l_{kj}\chi^{\eta}_{kj} + d^{l+\frac{1}{2}}_{mn}\\
&\quad - \frac{1}{2}(\tilde{p}_{ig} - a^l_{mn}) - 2\beta_1\lambda\mu c_1c_2\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^l_{kj}\theta^{\xi}_{kj}\theta^{\eta}_{kj}\\
&\quad + \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\left[a^l_{k_1m_1}a^l_{k_2m_2}\beta_{k_1k_2m_1m_2,mn} + b^l_{k_1m_1}b^l_{k_2m_2}\alpha_{k_1k_2m_1m_2,mn}\right]\\
\frac{a^{l+1}_{mn} - \tilde{p}_{mn}}{\tau} &= \mu^2(\beta_2 - c_2^2\beta_1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^{l+1}_{kj}\chi^{\eta}_{kj} - \mu^2(\beta_2 - c_2^2\beta_1)\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}a^l_{kj}\chi^{\eta}_{kj} - \frac{a^{l+1}_{mn} - a^l_{mn}}{2}.
\end{aligned}$$

Similarly we have a coupled system for the odd coefficients. The trivial solution is avoided by means of re-scaling $a_{i,j} = \delta a_{i,j}$ and imposing an additional condition on the first coefficient, namely $a_{0,0} = 1$. Hence, the equation for $a_{0,0}$ is an equation to determine δ, namely:

$$\delta = \frac{-d_{00} + a_{00} - (\beta_2 - c_2^2\beta_1)\sum_{k=0}^{2} a_{0k}\chi_{k0} - (\beta_2 - c_1^2\beta_1)\sum_{k=0}^{2} a_{k0}\chi_{0k}}{\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty} a_{k_1m_1}^{l} a_{k_2m_2}^{l} \beta_{k_1k_2m_1m_2,00}}.$$

NUMERICAL RESULTS

It was shown first in [9] that the BPE possess a *sech* solution of the following type

$$u = -\frac{3}{2}\frac{c^2-1}{\alpha}\mathrm{sech}^2\left(\frac{x-ct}{2}\sqrt{\frac{c^2-1}{\beta_1 c^2 - \beta_2}}\right), \tag{8}$$

for $|c| < \min\{1, \sqrt{\beta_2/\beta_1}\}$ or $|c| > \max\{1, \sqrt{\beta_2/\beta_1}\}$. The former interval relates to subcritical solitons, while the latter interval comprises supercritical ones. Unlike the BE, the subcritical solitons do not exist for the whole interval $c < 1$. This is a specific feature of the BPE equation. In [9] was investigated also the interaction of different combinations of solitons of the different genera and shown that the subcritical are much less stable than the supercritical.

The supercritical solitons are well defined for the 1D case. However, considering a supercritical phase speed in the 2D case leads to a spatial operator that is not elliptic. For instance, if $c_1 = 0$ and $c_2 > \max\{1, \sqrt{\beta_2/\beta_1}\}$, the coefficients of the second spatial derivative in Eq. (5) are $\beta_2 > 0$ and $\beta_2 - c_2^2\beta_1 < 0$, respectively. For this reason, we focus in the paper on the subcritical case only. For definiteness, we set $\beta_2 = 1$.

When $c_1 = c_2 = 0$ the solution is expected to be axisymmetric. Our computations do confirm this with high accuracy. The soliton has perfect bell shape and the contour lines are concentric circles.

The qualitative dependence of the shape of the solitary wave on the phase speed is similar for the different values of β_1, albeit the quantitative aspects can differ. The main feature is similar to the case of Boussinesq equation [19, 8, 20] in the sense that the support of the profile is becoming larger with the increase of the phase speed, but there is a relative contraction in the direction of translation. This behavior is well seen in Figure 1 for $c_2 = 0.7$ and different β_1. The case β_1 corresponds exactly to the results of [19, 8, 20] for the respective phase speed, save the change of sign in the cited work which was used to make the solitons positive. The figure shows that the relative contraction is more pronounced for larger β_1. In this sense, it has a similar effect as a larger phase speed. The effect of the increase of β_1 on the height (depth) of the soliton is once again comparable to a higher phase speed: larger β_1 makes the depth of the depression smaller (depression becomes more shallow). It is interesting to note that if one period in y-direction of the KP solution is cut out and considered as a single localized structure (see [4]), there is a qualitative agreement with our results. There are depressions in the direction of motion (in the front and the back of the soliton) in [4] similar to the reported in Figure 1.

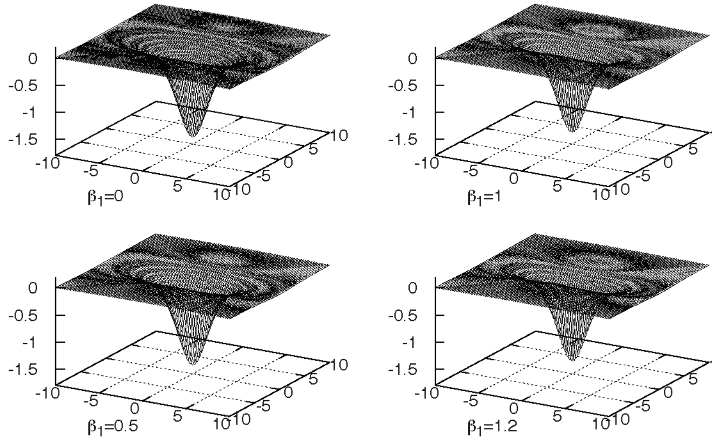

FIGURE 1. Subcritical Solitary waves for $c_2 = 0.7$ and different β_1

In order to substantiate the above delineated analogy between the effects of β_1 and c_2, we present in Figure 2 the case of $\beta_1 = 4$ for four different values of the phase speed. We have chosen such large value for β_1 because then the effect of c is much

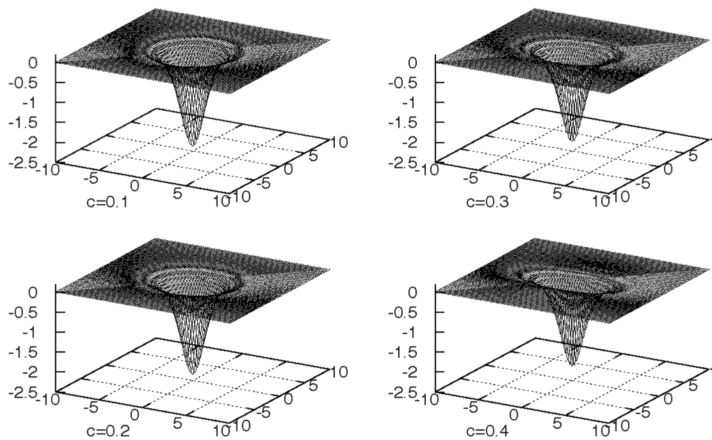

FIGURE 2. Subcritical Solitary waves for $\beta_1 = 4$ and different c_2

more pronounced, as it is the case for 1D given in Eq. (8). One sees that for larger c the depressions are less profound and shorter in the direction of motion. The main hump is relatively contracted in the direction of motion while the overall support of the soliton is enlarged proportionally to the pseudo-Lorentzian factor $\sqrt{1-c_2^2}$. However for larger β_1, the fore- and rear- runners become less pronounced, which is a quantitative difference.

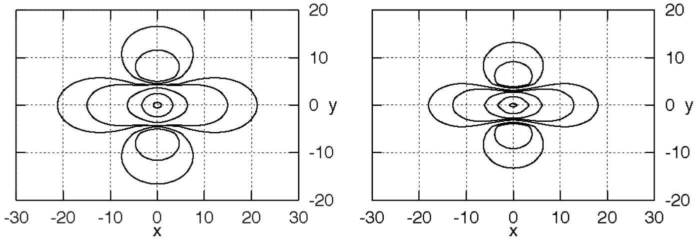

FIGURE 3. Contour lines $-0.02, -0.01, 0.01, 0.02, 0.1, 0.4, 1.2$ for the case $c_2 = 0.7$. Left panel: $\beta_1 = 0$. Right panel: $\beta_1 = 1$

The fact that the support of the localized wave decreases with the increase of β_1 (with all the other parameters being the same) is illustrated in Figure 3 where some contours of the 2D profiles are plotted. Once again, one sees in the left panel that the localized wave for $\beta_1 = 0$ spans a larger support than in the right panel where is presented the result for $\beta_1 = 1$ with the same phase speed. The contour plots shows that the sign of the shape function changes: two positive regions (elevations) can be see in the front and the back of the wave. For the Boussinesq equation, this has been first observed in our previous works [19, 8].

A very important issue for soliton problems, especially in 2D, is the asymptotic behavior of the profile at infinity. To study the behavior we consider vertical cross-sections of the solution and compare different cases. Such cross sections are shown in Figure 4 for fixed $c = 0.7$ and $\beta_2 = 1$ and different values of β_1. In the left panel we present the

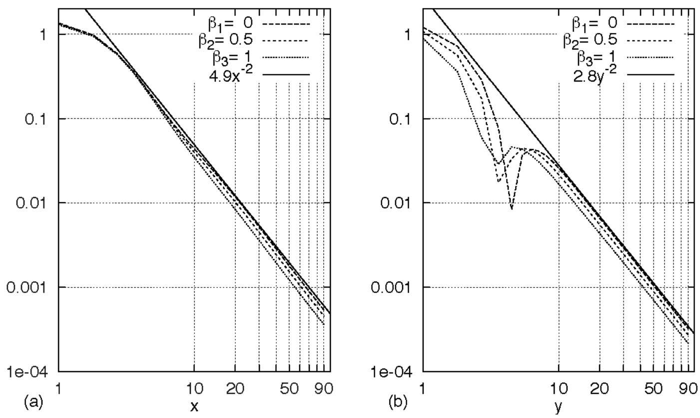

FIGURE 4. Asymptotic behavior of the profiles for $c_2 = 0.7$, $\beta_2 = 1$ and different β_1. Left: cross-section $y = 0$; right: cross-section $x = 0$

cross-section that is perpendicular to the propagation direction of the localized wave (the 'transverse' cross-section), while the right panel gives the longitudinal cross-section. One can see that increasing the parameter β_1 makes the profile decay faster, i.e., the support of the soliton is smaller. We have added to the figure asymptotic lines that correspond the power-functions if the type of inverse square. The comparison tells us that the profiles decay at infinity as the inverse square of the respective variable. This is in perfect agreement with the numerical and spectral results for Boussinesq Equation [19, 8], and with the perturbation theory from [20] where the inverse square law for the decay of the profile is proved for small c_2.

What is shown in Figure 4 is the absolute value of the wave amplitude as a function of the respective coordinate (always positive). In the right panel of the figure is shown the profile in the longitudinal cross-section. The fact that the shape changes its sign appears as a dip in the profile.

CONCLUSIONS

In the present paper we use a Galerkin spectral method to compute the shapes of the stationary propagating 2D localized solutions of the so-called Boussinesq Paradigm Equation (PBE). In many instances, the results presented here are similar to the results for the Boussinesq Equation (BE) which is a particular case of the BPE considered here. The new phenomenology is connected with the additional dispersion introduced by the second-time-second-space mixed derivative. The important physical finding about BE from our previous works —that while the overall length of the support increases, the dimension in the direction of motion is contracted relative to the transverse direction— is confirmed in this work to hold also for BPE. The effect of the second dispersion is quantified and it is shown the latter acts to contract the profile in the direction of translation of the wave. In a sense, the additional dispersion has the same effect on the shape of the solitary wave as an increase of the phase speed. Yet, one has to be aware that the additional dispersion significantly reduces the interval of values of the phase speed for which stationary propagating shapes exist.

The results obtained here are encouraging and open the possibility for the time dependent problem which will allow investigation of the interaction of solitons in the cases when no analytical solutions are available.

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