DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Supplement 2009

Website: www.aimSciences.org

pp. **171–180**

THE COARSE-GRAIN DESCRIPTION OF INTERACTING SINE-GORDON SOLITONS WITH VARYING WIDTHS

Ivan Christov

Department of Engineering Sciences and Applied Mathematics
Northwestern University, Evanston, IL 60208-3125, USA

C. I. Christov

Department of Mathematics, University of Louisiana at Lafayette
Lafayette, LA 70504-1010, USA

Abstract. We study the dynamics of the sine-Gordon equation's kink soliton solutions under the coarse-grain description via two "collective variables": the position of the "center" of a soliton and its characteristic width ("size"). Integral expressions for the interaction potential and the quasi-particles' cross-masses are derived. However, these cannot be evaluated in closed form when the solitons have varying widths, so we develop a perturbation approach with the velocity of the faster soliton as the small parameter. This enables us to derive a system of four coupled second-order ODEs, one for each collective variable. The resulting initial-value problem is very stiff and numerical instabilities make it difficult to solve accurately, so a semi-empirical iterative approach to its solution is proposed. Then, we demonstrate that, even though it appears the solitons pass through each other, the quasi-particles actually "exchange" their pseudomasses during a collision.

1. **Introduction.** In 1962, Perring and Skyrme [1] proposed (what is now called) the sine-Gordon equation as a model field equation governing the interaction of bosons and mesons. The latter has the dimensionless form ($\hbar = c = m = 1$)

$$u_{tt} - u_{xx} = -\sin u, \tag{1}$$

where $u(x,t)$ is the field variable, and subscripts denote differentiation with respect to an independent variable. Over the years, equation (1) has appeared in many other contexts: e.g., the study of dislocations in materials, fluxons in a Josephson junction and DNA transcription [2], to name a few. Arguably, the most striking feature of the sine-Gordon equation (and a number of other *integrable* nonlinear wave equations) is the existence of *soliton* solutions. It has long been know that soliton dynamics resemble very much those of classical "point" particles [3]. Precisely how this analogy is quantified, and what the mechanism of the soliton interaction is, remains the subject of research [4]. However, thanks to the *coarse-grain description* (CGD for short), a type of variational approximation [5] pioneered in [6], there is now a systematic way available of deriving dynamic equations of motion for point particles corresponding to solitons governed by nonlinear wave equations, even non-integrable ones. Unlike integrable equations, which have an extensive and unifying mathematical underpinning allowing one to study exactly phenomena such

2000 *Mathematics Subject Classification.* Primary: 35Q51, 35Q53; Secondary: 49S05, 81V25.
Key words and phrases. Solitons, Variational approximation, Quasi-particles, sine-Gordon equation, Nonlinear-wave quantization.

as, e.g., the zero-dispersion limit [7], the theory of non-integrable equations is on a case-by-case basis making approximate techniques, such as the CGD, indispensable.

In the present work, we extend the results of [6] to account for the shape change of solitons (and the corresponding, now deformable, quasi-particles) during their interaction. Specifically, we apply the CGD, for the purposes of which we now consider the trajectories *and* widths of the (sine-Gordon) solitons unknown in advance, to obtain the equations of motion of the deformable point particles. Then, we discuss the solution of the latter equations and show how these can be used to shed light on the particle-like dynamics of the solitons.

2. **Coarse-Grain Description of a One-Quasi-Particle Field.** First, let us briefly review the CGD by considering the evolution of a single quasi-particle (QP) with two *collective variables* (CVs): the spatial position of its center, X, and the inverse of its width, a. This section essentially summarizes the early work of Rice [8] and prepares us for the discussion that follows. The "shape" of the QP, for the sine-Gordon equation, is given by the well-known one-soliton profile [2]:

$$\Phi(\xi; a) = 4\arctan[\exp(a\xi)], \tag{2}$$

where ξ is the local coordinate in the moving frame (e.g., $\xi = x - vt$ for the one-soliton solution with velocity v). So, if we consider a wave profile consisting of a single soliton, then the CGD will consist of one QP with trajectory $x = X(t)$, and we cannot, in general, neglect the time dependence of the second CV a. Thus, we suppose that

$$u(x,t) \approx \Phi[x - X(t); a(t)] \tag{3}$$

gives a reasonable approximation to the one-soliton solution of (1).

On the basis of (3), it follows that the *discrete* (or coarse-grain) Lagrangian [i.e., the result of evaluating the Lagrangian of the field based on the ansatz (3)] is just

$$\mathbb{L} \equiv \frac{1}{2}\int_{-\infty}^{+\infty} u_t^2 - u_x^2 + 2(\cos u - 1)\,\mathrm{d}x = \frac{1}{2}\mathbb{M}\dot{X}^2 - 4a + \frac{\pi^2}{3}\frac{\dot{a}^2}{a^3} - \frac{4}{a}, \tag{4}$$

where

$$\mathbb{M} := \int_{-\infty}^{+\infty} [\Phi'(\xi; a)]^2\,\mathrm{d}\xi = 8a. \tag{5}$$

Note that we use block capital letters to denote the discrete versions of the corresponding continuous quantities. Consequently, the Euler–Lagrange equations for the two (unknown) CVs are

$$\frac{\delta\mathbb{L}}{\delta X} = -\frac{\mathrm{d}}{\mathrm{d}t}(\mathbb{M}\dot{X}) = 0, \tag{6a}$$

$$\frac{\delta\mathbb{L}}{\delta a} = 4(\dot{X}^2 - 1) - \pi^2\frac{\dot{a}^2}{a^4} + \frac{4}{a^2} - \frac{2\pi^2}{3}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{a}}{a^3}\right) = 0. \tag{6b}$$

This system lends itself to far-reaching interpretation, e.g., (6a) is exactly Newton's 2nd law for a particle with mass $\mathbb{M}$ and trajectory $X(t)$. Thus, we have the natural concept of "pseudomass" of a QP, which is given by the expression (5).

In addition, notice that the system (6) has a stationary solution for a and $\dot{X}$, i.e., a solution such that $\dot{a} = \ddot{X} = 0$. Then, letting $\dot{X} = \mathit{const} =: v$, it follows from the second equation that $a = 1/\sqrt{1 - v^2}$, which is nothing else but the formula for the Lorentz (or relativistic) contraction the QP experiences. So, a is related to the width of the QP, which is $1/a$, loosely speaking; though, for what follows, we leave this semantical detail understood and refer to a simply as the "width." In this light, (6a) is now the *relativistic* version of Newton's 2nd law for a point particle with rest mass 8 or, equivalently, relativistic mass $8/\sqrt{1-v^2}$, which is consistent with the definition of pseudomass of the QP given in (5).

3. **Coarse-Grain Description of a Two-Quasi-Particle Field.** Now consider the case of two interacting QPs of the type given by (2). The CGD rests on the assumption that any possible interaction terms (i.e., terms that depend on the parameters of both QPs simultaneously) can be neglected without incurring a large error in the approximation because the interaction is accounted for through the solution of the coarse-grain variational problem (see, e.g., [6, 9]). Thus, the two-soliton wave profile can be assumed to be the mere superposition of two QPs:

$$u(x,t) \approx \Phi[x - X_1(t); a_1(t)] \pm \Phi[x - X_2(t); a_2(t)], \tag{7}$$

where Φ is given by (2). Here, the signs "+" and "−" refer to the so-called soliton–soliton (S-S) and soliton–antisoliton (S-A) solutions, respectively. Notice that we now have four CVs, namely the two widths $a_1(t)$, $a_2(t)$, and the two trajectories $X_1(t)$, $X_2(t)$. On the basis of (7), we can easily evaluate the discrete Lagrangian of the field of two QPs each described by two CVs:

$$\begin{aligned}
\mathbb{L} = 4a_1(\dot{X}_1^2 - 1) + \frac{\pi^2}{3}\frac{\dot{a}_1^2}{a_1^3} - \frac{4}{a_1} + 4a_2(\dot{X}_2^2 - 1) + \frac{\pi^2}{3}\frac{\dot{a}_2^2}{a_2^3} - \frac{4}{a_2} \\
\mp 4a_1 a_2(1 - \dot{X}_1\dot{X}_2)\int_{-\infty}^{+\infty} \operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,\mathrm{d}x \\
\pm 4\dot{a}_1\dot{a}_2\int_{-\infty}^{+\infty}(x - X_1)(x - X_2)\operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,\mathrm{d}x \\
\mp 4\dot{a}_1 a_2\dot{X}_2\int_{-\infty}^{+\infty}(x - X_1)\operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,\mathrm{d}x \\
\mp 4a_1\dot{a}_2\dot{X}_1\int_{-\infty}^{+\infty}(x - X_2)\operatorname{sech}[a_1(x - X_1)]\operatorname{sech}[a_2(x - X_2)]\,\mathrm{d}x \\
\mp 4\int_{-\infty}^{+\infty}\frac{\sinh[a_1(x - X_1)]\sinh[a_2(x - X_2)] - 1}{\cosh^2[a_1(x - X_1)]\cosh^2[a_2(x - X_2)]}\,\mathrm{d}x,
\end{aligned} \tag{8}$$

where the integrals involving any one of the CVs have already been evaluated just as in (4), and (as before) the upper sign in any $\pm$ and $\mp$ expression corresponds to the S-S case and the bottom sign to the S-A case.

4. **An Asymptotic Approach.** Unlike the simplified two-CV case considered in [10], the exact evaluation of the integrals in (8) representing the interaction is very difficult. Although some may be evaluated by the residue method, the result is an infinite series involving trigonometric functions. Therefore, in this section, we obtain closed-form expressions for them by considering a distinguished limit. This is one of the main results of the present work.

To this end, we will operate within the approximation of small phase speeds. The zeroth-order model is what we called the "classical" approximation in [6]. In the latter case, the CVs related to the widths (scale) of the QPs are trivially equal to a constant, namely $a_1(t) = a_2(t) \equiv 1\,\forall t \geq 0$. In the present work, we include some of the effects of the variability of the a_is within the fourth order of approximation with respect to the small parameter

$$\varepsilon := \max\{\dot{X}_1(0), \dot{X}_2(0)\}. \tag{9}$$

Note that, in the S-S case, $\dot{X}_i(t) \leq \dot{X}_i(0)\ \forall t \geq 0$. However, in the S-A case, an initial phase speed of 0.1 can increase up to 0.6 in the moment when the two quasi-particles annihilate each other (i.e., their trajectories meet) [6]. Thus, the Lorentz contraction of the QPs is now significant, so we must incorporate the effects of the width CVs (even if we must still assume somewhat small phase speeds) in order to unravel the physics of the interaction of the QPs.

From Section 2, we know that for two *non-interacting* QPs $\dot{X}_i(t) \equiv \dot{X}_i(0)\ \forall t \geq 0$ and $a_i(t) = (1-\dot{X}_i^2)^{-1/2}$, thus $1 - a_i = \mathcal{O}(\dot{X}_i^2) = \mathcal{O}(\varepsilon^2)$. This leads us to assume the following forms of the asymptotic expansions of the widths CVs:

$$a_i(t) = 1 - \varepsilon^2\gamma_i(t) + \mathcal{O}(\varepsilon^4) \quad \Rightarrow \quad \dot{a}_i(t) = -\varepsilon^2\dot{\gamma}_i(t) + \mathcal{O}(\varepsilon^4). \tag{10}$$

Notice that $\gamma_i \neq \dot{X}_i$; the γ_is are to be determined. Based upon the latter, we have the following expansions, all within $\mathcal{O}(\varepsilon^5)$ [1], of the different terms that are needed in computing the asymptotic expressions of the unevaluated integrals in (8):

$$a_1a_2 = 1 - \varepsilon^2(\gamma_1+\gamma_2) + \varepsilon^4\gamma_1\gamma_2, \qquad a_i^2 = 1 - 2\varepsilon^2\gamma_i + \varepsilon^4\gamma_i^2, \tag{11a}$$

$$\dot{a}_ia_k = -\varepsilon^2\dot{\gamma}_i + \varepsilon^4\dot{\gamma}_i\gamma_k, \qquad \dot{a}_1\dot{a}_2 = \varepsilon^4\dot{\gamma}_1\dot{\gamma}_2, \qquad \dot{a}_i^2 = \varepsilon^4\dot{\gamma}_i^2, \tag{11b}$$

$$a_1a_2(1-\dot{X}_1\dot{X}_2) = 1 - \varepsilon^2(\gamma_1+\gamma_2+\dot{Z}_1\dot{Z}_2) + \varepsilon^4[\gamma_1\gamma_2 + \dot{Z}_1\dot{Z}_2(\gamma_1+\gamma_2)], \tag{11c}$$

where, for convenience, we have let $Z_i(t)$ be such that

$$X_i(t) = X_i(0) + \varepsilon Z_i(t) + \mathcal{O}(\varepsilon^2), \quad Z_i(0) = 0. \tag{12}$$

In what follows, we use Z_i only when it appears through its derivative because, unlike $\dot{X}_i$, the quantity $\dot{Z}_i$ is of unit order with respect to ε.

4.1. **Asymptotic Lagrangian of a Two-Quasi-Particle Field.** To complete the two-CV coarse-grain description of the two-QP problem we need to evaluate the remaining integrals in the expression of the discrete Lagrangian given in (8). However, as discussed above, the only way to do that is via an asymptotic expansion in terms of the small parameter ε. We begin this task by observing that

$$\begin{aligned}
&\cosh[a_1(x-X_1)]\\
&= \cosh(x-X_1)\cosh[\varepsilon^2\gamma_1(x-X_1)] - \sinh(x-X_1)\sinh[\varepsilon^2\gamma_1(x-X_1)]\\
&= \cosh(x-X_1)\{2\sinh^2[\tfrac{1}{2}\varepsilon^2\gamma_1(x-X_1)]+1\} - \sinh(x-X_1)\sinh[\varepsilon^2\gamma_1(x-X_1)]\\
&= \cosh(x-X_1)\Big\{1 - \varepsilon^2\gamma_1(x-X_1)\tanh(x-X_1) + \tfrac{1}{2}\varepsilon^4\gamma_1^2(x-X_1)^2\Big\} + \mathcal{O}(\varepsilon^6).
\end{aligned} \tag{13}$$

Note that the last line of the above equation is obtained using the one-term Taylor series expansion of sinh, which is valid within (and, in fact, is also valid beyond) the necessary $\mathcal{O}(\varepsilon^5)$ asymptotic order. Then,

$$\begin{aligned}
\operatorname{sech}[a_1(x-X_1)] = \operatorname{sech}(x-X_1)\big\{&1 + \varepsilon^2\gamma_1(x-X_1)\tanh(x-X_1)\\
&+ \varepsilon^4\gamma_1^2(x-X_1)^2[\tfrac{1}{2} - \operatorname{sech}^2(x-X_1)]\big\} + \mathcal{O}(\varepsilon^6).
\end{aligned} \tag{14}$$

And, a similar expression can be derived for the term $\operatorname{sech}[a_2(x-X_2)]$.

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[1]Note that we must keep terms of order $\mathcal{O}(\varepsilon^4)$ throughout the derivation so that the final equations for the X_is and a_is are asymptotically accurate within $\mathcal{O}(\varepsilon^4)$.

Making use of the above formulas, we obtain the following approximation of the integrand of the first unevaluated integral in (8):

$$\begin{aligned}\operatorname{sech}[a_1(x-X_1)]\operatorname{sech}[a_2(x-X_2)] &= \operatorname{sech}(x-X_1)\operatorname{sech}(x-X_2)\\ &\times\big\{1+\varepsilon^2\gamma_1(x-X_1)\tanh(x-X_1)+\varepsilon^2\gamma_2(x-X_2)\tanh(x-X_2)\\ &+\varepsilon^4\gamma_1^2(x-X_1)^2[\tfrac{1}{2}-\operatorname{sech}^2(x-X_1)]+\varepsilon^4\gamma_2^2(x-X_2)^2[\tfrac{1}{2}-\operatorname{sech}^2(x-X_2)]\\ &+\varepsilon^4\gamma_1\gamma_2(x-X_1)(x-X_2)\tanh(x-X_1)\tanh(x-X_2)\big\}+\mathcal{O}(\varepsilon^6). \quad (15)\end{aligned}$$

Then, for the integral itself, which we denote by I_1, we have

$$I_1 = G_{10}(z)+\varepsilon^2(\gamma_1+\gamma_2)G_{11}(z)+\varepsilon^4(\gamma_1^2+\gamma_2^2)G_{12}(z)+\varepsilon^4\gamma_1\gamma_2 G_{13}(z)+\mathcal{O}(\varepsilon^6), \quad (16)$$

where $z(t) := X_2(t) - X_1(t)$, and $G_{10}(z)$, $G_{11}(z)$, $G_{12}(z)$, $G_{13}(z)$ are given by equations (23) in Appendix A.

Now, we turn to the next three integrals in (8) (denoted by I_2, I_3 and I_4, respectively), which can be approximated in the same fashion. However, it is important to realize that they are all multiplied by some power of the small parameter ε, consequently their expansion need not be carried out beyond the leading order. Thus, we have the following expressions:

$$I_2 = G_{20}(z)+\mathcal{O}(\varepsilon^2), \quad I_3 = G_{21}(z)+\mathcal{O}(\varepsilon^2), \quad I_4 = -G_{21}(z)+\mathcal{O}(\varepsilon^2), \quad (17)$$

where $G_{20}(z)$ and $G_{21}(z)$ are given by equations (24) in Appendix A.

The most intricate is the last integral in (8). To tackle it, we use the same reasoning as in (13) to expand the product of sinh functions and then use (14) for the sech^2 terms, arriving at the following expansion of the integrand:

$$\begin{aligned}\frac{\sinh[a_1(x-X_1)]\sinh[a_2(x-X_2)]-1}{\cosh^2[a_1(x-X_1)]\cosh^2[a_2(x-X_2)]} &= \operatorname{sech}^2(x-X_1)\operatorname{sech}^2(x-X_2)\\ &\times\Big\{\sinh(x-X_1)\sinh(x-X_2)-1+\varepsilon^2[\sinh(x-X_1)\sinh(x-X_2)-2]\\ &\times[\gamma_1(x-X_1)\tanh(x-X_1)+\gamma_2(x-X_2)\tanh(x-X_2)]\\ &+\varepsilon^4\Big(\gamma_1^2(x-X_1)^2\operatorname{sech}^2(x-X_1)\big[(1-\tfrac{1}{2}\cosh[2(x-X_1)])\\ &\times[1-\sinh(x-X_1)\sinh(x-X_2)]-2\sinh^3(x-X_1)\sinh(x-X_2)\big]\\ &+\gamma_2^2(x-X_2)^2\operatorname{sech}^2(x-X_2)\big[(1-\tfrac{1}{2}\cosh[2(x-X_2)])\\ &\times[1-\sinh(x-X_1)\sinh(x-X_2)]-2\sinh(x-X_1)\sinh^3(x-X_2)\big]\\ &-2\gamma_1\gamma_2(x-X_1)(x-X_2)\tanh(x-X_1)\tanh(x-X_2)\\ &\times[1-\sinh(x-X_1)\sinh(x-X_2)]\Big)\Big\}+\mathcal{O}(\varepsilon^6). \quad (18)\end{aligned}$$

Then, we obtain the following asymptotic expansion of the integral:

$$I_5 = G_{50}(z)+\varepsilon^2(\gamma_1+\gamma_2)G_{51}(z)+\varepsilon^4(\gamma_1^2+\gamma_2^2)G_{52}(z)-2\varepsilon^4\gamma_1\gamma_2 G_{53}(z)+\mathcal{O}(\varepsilon^6), \quad (19)$$

where $G_{50}(z)$, $G_{51}(z)$, $G_{52}(z)$, $G_{53}(z)$ are given by equations (25) in Appendix A.

At this point, we could construct a consistent approximation of the coarse-grain Lagrangian by expanding the remaining terms in (8) in ε, but then the Euler–Lagrange equations for the variables a_i may turn out to be computationally unstable because we would be altering the nonlinearity of the highest-order derivatives. Moreover, since the whole system is highly nonlinear to begin with, we do not gain much by finding asymptotic expansions of those terms. Therefore, we only use the asymptotic approximations (as constructed above) of the interaction integrals, which cannot be left as-is because they cannot be evaluated in closed form.

Then, one can easily construct the coarse-grain Lagrangian using (16), (17) and (19), which in terms of the original variables is

$$\begin{aligned}\mathbb{L} \approx{}& 4a_1(\dot{X}_1^2-1)+\frac{\pi^2}{3}\frac{\dot{a}_1^2}{a_1^3}-\frac{4}{a_1}+4a_2(\dot{X}_2^2-1)+\frac{\pi^2}{3}\frac{\dot{a}_2^2}{a_2^3}-\frac{4}{a_2}\\ &\mp 4(1-a_1-a_2)(\dot{X}_1\dot{X}_2-1)G_{10}\mp 4(1-a_1)(1-a_2)(G_{10}+G_{13}-2G_{53})\\ &\pm 4[\dot{a}_1\dot{a}_2G_{20}+(\dot{a}_2\dot{X}_1-\dot{a}_1\dot{X}_2)G_{21}]\mp 4[(1-a_1)^2+(1-a_2)^2](G_{12}+G_{52})\\ &\mp 4G_{50}\mp 4(2-a_1-a_2)[G_{51}-(2-a_1-a_2+\dot{X}_1\dot{X}_2)G_{11}]. \qquad (20)\end{aligned}$$

This result is a generalization of the Lagrangians derived in [10, 6]. For instance, we can recover the result in [6] by simply taking $a_1 = a_2 \equiv 1$, and should be able to recover the result in [10] by letting $X_2(t) = -X_1(t) = r(t)/2$. The most important feature of the Lagrangian (20) is that it gives us the opportunity to consider, for the first time, QPs whose widths vary *independently* during their interaction.

4.2. **Asymptotic Equations of Motion of a Two-Quasi-Particle Field.** The Euler–Lagrange equations for the extremization of the functional (20) are

$$\begin{aligned}\frac{\delta\mathbb{L}}{\delta X_1} ={}& \pm 4(1-a_1-a_2)\left[(\dot{X}_1\dot{X}_2-1)G'_{10}+\ddot{X}_2G_{10}+\dot{X}_2\dot{z}G'_{10}\right]\mp 4\dot{X}_2(\dot{a}_2-\dot{a}_1)G'_{21}\\ &-8(a_1\ddot{X}_1+\dot{a}_1\dot{X}_1)\pm 4(2-a_1-a_2)[G'_{51}-(2-a_1-a_2+\dot{X}_1\dot{X}_2)G'_{11}-\ddot{X}_2G_{11}-\dot{X}_2\dot{z}G'_{11}]\\ &\pm 4[(1-a_1)^2+(1-a_2)^2](G'_{12}+G'_{52})\pm 4G'_{50}\mp 4\dot{a}_1\dot{a}_2G'_{20}\mp 4(\dot{a}_2\dot{X}_1-\dot{a}_1\dot{X}_2)G'_{21}\mp 4\ddot{a}_2G_{21}\\ &\pm 4(1-a_1)(1-a_2)(G'_{10}+G'_{13}-2G'_{53})\mp 4(\dot{a}_1+\dot{a}_2)\dot{X}_2(G_{10}-G_{11})=0, \qquad (21a)\end{aligned}$$

$$\begin{aligned}\frac{\delta\mathbb{L}}{\delta a_1} ={}& 4(\dot{X}_1^2-1)+\frac{4}{a_1^2}+\pi^2\frac{\dot{a}_1^2}{a_1^4}-\frac{2\pi^2}{3}\frac{\ddot{a}_1}{a_1^3}\pm 4(1-a_2)(G_{10}+G_{13}-2G_{53})\\ &\pm 8(1-a_1)(G_{12}+G_{52})\pm 4(G_{51}-G_{10})\mp 8(2-a_1-a_2)G_{11}\\ &\pm 4\dot{X}_1\dot{X}_2(G_{10}-G_{11})\mp 4(\ddot{a}_2G_{20}+\dot{a}_2\dot{z}G'_{20}-\ddot{X}_2G_{21}-\dot{X}_2\dot{z}G'_{21})=0, \qquad (21b)\end{aligned}$$

$$\begin{aligned}\frac{\delta\mathbb{L}}{\delta X_2} ={}& \mp 4(1-a_1-a_2)\left[(\dot{X}_1\dot{X}_2-1)G'_{10}-\ddot{X}_1G_{10}-\dot{X}_1\dot{z}G'_{10}\right]\pm 4\dot{X}_1(\dot{a}_2-\dot{a}_1)G'_{21}\\ &-8(a_2\ddot{X}_2+\dot{a}_2\dot{X}_2)\mp 4(2-a_1-a_2)[G'_{51}-(2-a_1-a_2+\dot{X}_1\dot{X}_2)G'_{11}+\ddot{X}_1G_{11}+\dot{X}_1\dot{z}G'_{11}]\\ &\mp 4[(1-a_1)^2+(1-a_2)^2](G'_{12}+G'_{52})\mp 4G'_{50}\pm 4\dot{a}_1\dot{a}_2G'_{20}\pm 4(\dot{a}_2\dot{X}_1-\dot{a}_1\dot{X}_2)G'_{21}\pm 4\ddot{a}_1G_{21}\\ &\mp 4(1-a_1)(1-a_2)(G'_{10}+G'_{13}-2G'_{53})\mp 4(\dot{a}_1+\dot{a}_2)\dot{X}_1(G_{10}-G_{11})=0, \qquad (21c)\end{aligned}$$

$$\begin{aligned}\frac{\delta\mathbb{L}}{\delta a_2} ={}& 4(\dot{X}_2^2-1)+\frac{4}{a_2^2}+\pi^2\frac{\dot{a}_2^2}{a_2^4}-\frac{2\pi^2}{3}\frac{\ddot{a}_2}{a_2^3}\pm 4(1-a_1)(G_{10}+G_{13}-2G_{53})\\ &\pm 8(1-a_2)(G_{12}+G_{52})\pm 4(G_{51}-G_{10})\mp 8(2-a_1-a_2)G_{11}\\ &\pm 4\dot{X}_1\dot{X}_2(G_{10}-G_{11})\mp 4(\ddot{a}_1G_{20}+\dot{a}_1\dot{z}G'_{20}+\ddot{X}_1G_{21}+\dot{X}_1\dot{z}G'_{21})=0. \qquad (21d)\end{aligned}$$

Unfortunately, unlike the system we obtained for the classical approximation [6], this one is *highly* nonlinear and stiff. What is even worse is that it is not resolved with respect to the highest order derivatives, i.e., the equation for $\ddot{X}_1$ features $\ddot{X}_2$, etc. Thus, feeding the system, as it stands, directly into Mathematica's `NDSolve` (as done in [6]) does not provide good results because of the latter difficulties. In the absence of specialized solvers for systems such as (21), we propose a semi-empirical approach to overcome this difficulty until we can devise a better numerical method.

First, we note that, when a_1 and a_2 are constants, the equations are still quite stiff and unresolved but `NDSolve` (using an implicit integrator) can compute a solution. However, if we take $a_i = const\ \forall t$, we are back to the classical approximation, so instead we will compute the time dependence of the widths $a_i(t)$ using a functional iteration. In other words, we begin with an initial guess: $a_1^0(t) = a_2^0(t) = 1$, then solve $\delta\mathbb{L}/\delta X_1 = 0$ and $\delta\mathbb{L}/\delta X_2 = 0$ for $X_1^0(t)$ and $X_2^0(t)$, where the a_is are taken as known functions: e.g., on the first iteration we simply set $a_i(t) = a_i^0(t)$. Subsequently, we compute new approximations for the width CVs (motivated by the functional forms for the case of non-interacting QPs), namely $a_i^{n+1}(t) = \big\{1 - [\dot{X}_i^n(t)]^2\big\}^{-1/2}$ and repeat the process, beginning with $n = 0$. Our experiments suggest that the iteration converges very quickly; about five iterations are sufficient to reach a fixed point in the S-S case. However, we must note that, in the S-A case, difficulties may arise leading to non-physical solutions such as ones that have discontinuous trajectories. This means that the S-A case requires a different approach, which will be discussed elsewhere. For the remainder of this paper, we focus on the S-S case, which is sufficient for demonstrating the physical importance of introducing the second set of CVs, i.e., the widths a_i.

To this end, a good representative example of the semi-empirical approach to the system (21) must involve QPs with different initial speeds and, therefore, widths. In addition, the phase speeds should not be close to unity (the speed of light in this non-dimensional model), in order for our asymptotic approximations of the interaction integrals in the discrete Lagrangian to remain valid. Very small phase speeds are not of interest because then the equations of motion derived in [6] are sufficient to obtain solutions in very good quantitative agreement with the exact ones. Thus, we chose the following initial conditions for our representative example:

$$X_1(0) = -10, \quad \dot{X}_1(0) = 0.3, \qquad X_2(0) = 10, \quad \dot{X}_2(0) = -0.5. \tag{22}$$

The results for this case are shown in Figure 1 after five steps of the semi-empirical functional iteration using 5th-order implicit Runge–Kutta for the time-integration.

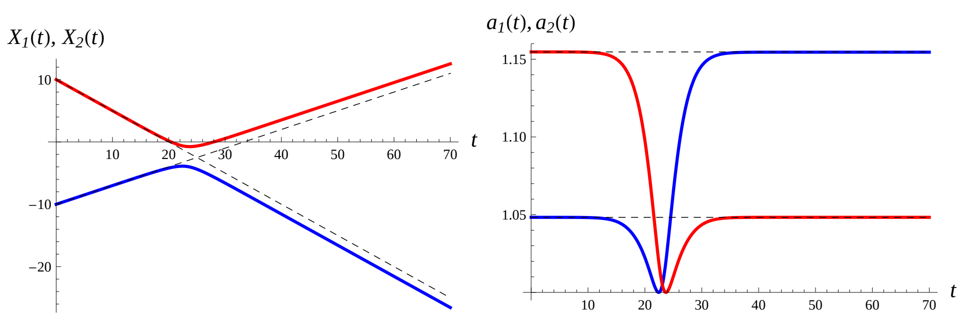

FIGURE 1. (Color online) Evolution of the trajectories and widths of the QPs in the case of S-S interaction, subject to the initial data given in (22)

5. **Results and Discussion.** Now that we have determined (at least numerically) the *a priori* unknown collective variables (CVs) $X_1(t)$, $X_2(t)$, $a_1(t)$ and $a_2(t)$, we can construct the coarse-grain description (CGD) of the two-soliton wave profile. The result is shown in the left panel of Figure 2. Note that we have plotted the spatial derivative of the profile, u_x, because it provides a more intuitive understanding of

why the solitons are quasi-particles. In other words, since the sine-Gordon solitons are *topological* solitons, it is not immediately clear from looking at u how they are localized, but it becomes obvious once one looks at u_x.

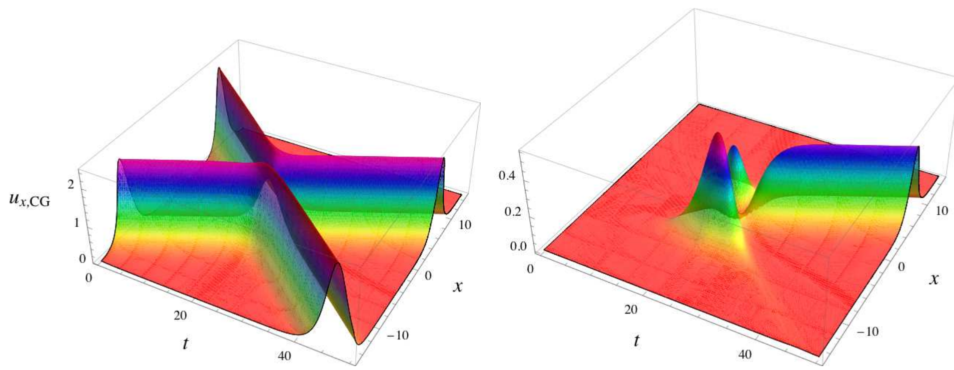

FIGURE 2. (Color online) Space-time plots of $u_x(x,t)$ (left) and the absolute error in $u(x,t)$ (right) in the case of the S-S interaction subject to the same initial condition as in Figure 1

In the right panel of Figure 2, we give a space-time plot of the absolute error in u due to the approximation in the CGD. Naturally, the approximation is not perfect because we chose an example with significant phase speeds. Since, the present approach to the two-CV discrete Lagrangian is asymptotically $\mathcal{O}(\varepsilon^4)$, where ε is the speed of the faster QP (recall equation (9)), the expected error in the CGD wave profile should be on the order of $(0.5)^4 = 0.0625 \approx 6.3\%$. Indeed, the maximal relative error in $u(x,t)$ (i.e., the maximum of the profile in the right panel of Figure 2 divided by 4π) is 4.6%, validating our asymptotic approach.

Notice that the maximal error is located near the instant of time at which the QPs stop. The reason for this is that the analytical solution exhibits a slight elevation of the profile near the latter instant of time. This is not observed in the wave profile constructed via the CGD. Most likely, the absence of this elevation is the result of our semi-empirical functional-iteration approach to solving the system (21). The exact solution suggests that the a_is increase as the QPs approach each other, while in the semi-empirical approximation they do not—as Figure 1 shows, the values of the a_is decrease to unity when the corresponding QPs stops moving. This is due to the particular functional form of the a_is we assumed in Section 4.2.

However, the most important thing to take away from the CDG of the two-QP field is its account of the soliton interaction. Clearly, the trajectories of the QPs, which correspond to the approximate trajectories of the centers of the solitons, do *not* cross. Therefore, the solitons (QPs) *scatter* in the S-S case. Yet, because of the shape of the two-soliton profile, one might hastily conclude that, during the S-S interaction, the solitons pass through each other, and there is no information to the contrary provided by the analytical solution. The present result extends those of [6, 10] to show that in the process there is an exchange of phase speeds such that momentum is conserved. Therefore, the QPs must exchange their pseudomasses (represented by the shape parameters a_i), which is precisely what we observe in the computations (see the right panel of Figure 1). Since this is a 1D problem, the QPs cannot scatter in an arbitrary direction, as they would be able to in 2D or 3D. So, the first QP must absorb the energy in the second one's "internal mode" (as a_i is sometimes referred to [2]) and vice versa, which means that the only way to conserve momentum is indeed through an exchange of the pseudomasses. Therefore, the returning QP assumes the identity of the incoming QP, so it may appear that the returning one is the incoming one passing through.

6. **Summary.** We used the coarse-grain description developed in [6] to determine the dynamics of a pair of quasi-particles (QPs), described by two collective variables (CVs) each, corresponding to interacting sine-Gordon solitons. By introducing a "width" CV for each QP, we were able to: (i) improve the approximation properties of the coarse-grain description, (ii) understand the physics of QP interactions better, showing that QPs actually scatter and exchange their "identity" during a collision, which gives the illusion that they pass through each other.

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Appendix A. **Exact Expressions for Various Integrals.** Recall, for convenience, we set $z := X_2 - X_1$. Then, the functions used in defining the asymptotic series of the first integral in the discrete Lagrangian for two QPs given by (8) are

$$G_{10}(z) = \int_{-\infty}^{+\infty} \operatorname{sech}(x - X_1) \operatorname{sech}(x - X_2)\, \mathrm{d}x = 2z \operatorname{csch} z, \tag{23a}$$

$$G_{11}(z) = \int_{-\infty}^{+\infty} \frac{(x - X_1)\sinh(x - X_1)}{\cosh^2(x - X_1)\cosh(x - X_2)}\, \mathrm{d}x = z^2 \coth z \operatorname{csch} z, \tag{23b}$$

$$\begin{aligned} G_{12}(z) &= \int_{-\infty}^{+\infty} (x - X_1)^2 \left[\tfrac{1}{2} - \operatorname{sech}^2(x - X_1)\right] \operatorname{sech}(x - X_1) \operatorname{sech}(x - X_2)\, \mathrm{d}x \\ &= \tfrac{1}{3}\left[\tfrac{1}{4}\pi^2 z + z^3 - \tfrac{1}{2}\pi^2 \coth z + \left(\tfrac{1}{2}\pi^2 z + 2z^3\right)\operatorname{csch}^2 z\right] \operatorname{csch} z, \end{aligned} \tag{23c}$$

$$\begin{aligned} G_{13}(z) &= \int_{-\infty}^{+\infty} \frac{(x - X_1)(x - X_2)\sinh(x - X_1)\sinh^2(x - X_2)}{\cosh^2(x - X_1)\cosh(x - X_2)}\, \mathrm{d}x \\ &= \tfrac{1}{12}\left\{z\left(2z^2 - \pi^2\right)[3 + \cosh(2z)] + 2\pi^2 \sinh(2z)\right\} \operatorname{csch}^3 z. \end{aligned} \tag{23d}$$

Those needed for the second, third and fourth integrals are

$$G_{20}(z) = \int_{-\infty}^{+\infty} \frac{(x - X_1)(x - X_2)}{\cosh(x - X_1)\cosh(x - X_2)}\, \mathrm{d}x = \tfrac{1}{6}(\pi^2 - 2z^2) z \operatorname{csch} z, \tag{24a}$$

$$G_{21}(z) = \int_{-\infty}^{+\infty} (x - X_1) \operatorname{sech}(x - X_1) \operatorname{sech}(x - X_2) \, \mathrm{d}x = z^2 \operatorname{csch} z. \tag{24b}$$

Finally, those needed for the fifth (last) integral are

$$\begin{aligned} G_{50}(z) &= \int_{-\infty}^{+\infty} \frac{\sinh(x - X_1) \sinh(x - X_2) - 1}{\cosh^2(x - X_1) \cosh^2(x - X_2)} \, \mathrm{d}x \\ &= \left[2 - z \coth\left(\tfrac{1}{2} z\right)\right] \operatorname{csch}^2\left(\tfrac{1}{2} z\right), \end{aligned} \tag{25a}$$

$$\begin{aligned} G_{51}(z) &= \int_{-\infty}^{+\infty} \frac{[\sinh(x - X_1) \sinh(x - X_2) - 2](x - X_1) \tanh(x - X_1)}{\cosh^2(x - X_1) \cosh^2(x - X_2)} \, \mathrm{d}x \\ &= (2 - 4z^2 + 4z \coth z) \operatorname{csch}^2 z - 6z^2 \operatorname{csch}^4 z \\ &\quad - \tfrac{1}{2} \coth z \operatorname{csch}^3 z [1 + 5z^2 + (z^2 - 1) \cosh(2z) - 2z \sinh(2z)], \end{aligned} \tag{25b}$$

$$\begin{aligned} G_{52}(z) &= \int_{-\infty}^{+\infty} \Bigg\{ \frac{(x - X_1)^2 \{1 - \frac{1}{2} \cosh[2(x - X_1)]\}[1 - \sinh(x - X_1) \sinh(x - X_2)]}{\cosh^4(x - X_1) \cosh^2(x - X_2)} \\ &\qquad - \frac{2(x - X_1)^2 \sinh^3(x - X_1) \sinh(x - X_2)}{\cosh^4(x - X_1) \cosh^2(x - X_2)} \Bigg\} \mathrm{d}x \\ &= \tfrac{1}{288} \operatorname{csch}^4 z \Big\{ 72(-3 + 2\pi^2 + 6z^2) - 264z(\pi^2 + 4z^2) \coth z + \operatorname{csch} z \\ &\quad \times \big[30z(\pi^2 + 4z^2) + 60z(\pi^2 + 4z^2) \cosh(2z) - 24z(\pi^2 + 4z^2) \cosh(3z) \\ &\quad + 6z(\pi^2 + 4z^2) \cosh(4z) + 2(12 - 11\pi^2) \sinh(2z) \\ &\quad + 24(3 + 2\pi^2 + 6z^2) \sinh(3z) - (12 + 13\pi^2 + 72z^2) \sinh(4z)\big] \Big\} \end{aligned} \tag{25c}$$

$$\begin{aligned} G_{53}(z) &= \int_{-\infty}^{+\infty} \Bigg\{ (x - X_1)(x - X_2) \tanh(x - X_1) \tanh(x - X_2) \\ &\qquad \times \frac{1 - \sinh(x - X_1) \sinh(x - X_2)}{\cosh^2(x - X_1) \cosh^2(x - X_2)} \Bigg\} \mathrm{d}x \\ &= \tfrac{1}{384} \operatorname{csch}^5\left(\tfrac{1}{2} z\right) \operatorname{sech}\left(\tfrac{1}{2} z\right) \big(3z(4 - 5\pi^2 + 10z^2) \cosh z \\ &\quad - 2z(\pi^2 - 2z^2) \cosh(2z) + z\left[2(z^2 - 6) - \pi^2\right] \cosh(3z) \\ &\quad + 2\{6z^3 - 3\pi^2 z + \pi^2 [5 \sinh z + 2 \sinh(2z) + \sinh(3z)]\}\big) \end{aligned} \tag{25d}$$

Received July 2008; revised June 2009.

E-mail address: `christov@u.northwestern.edu`
E-mail address: `christov@louisiana.edu`