On frame indifferent formulation of the Maxwell–Cattaneo model of finite-speed heat conduction

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A B S T R A C T

A material-invariant (frame indifferent) version of the Maxwell–Cattaneo law is proposed in which the relaxation rate of the heat flux is given by Oldroyd's upper-convected derivative. It is shown that the new formulation allows for the elimination of the heat flux, thus yielding a single equation for the temperature field. This feature is to be expected from a truly frame indifferent description.

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1. Introduction

One of the most successful models in continuum physics is Fourier's law of heat conduction

$$\boldsymbol{q} = -\kappa \nabla T, \tag{1}$$

where $\boldsymbol{q}$ is the thermal flux vector, $T(\mathbf{x}, t)$ is temperature, and $\kappa(> 0)$ stands for the thermal conductivity. Many other constitutive relations in continuum mechanics are modeled on the same concept of a linear connection between different characteristics, e.g., the relation between the stress and strain tensors in elasticity and the stress and rate of strain tensors in a viscous liquid. One of the main shortcomings of Fourier's law is that it leads to a parabolic equation for the temperature field. This means that any initial disturbance is felt instantly throughout the entire medium. This behavior is said to contradict the principle of causality.

To correct this unrealistic feature, which is known as the 'paradox of heat conduction' (PHC), various modifications of Fourier's law have been proposed over the years, not all of which have been successful (see the illuminating discussion in Jordan et al. (2008)). Of these, the best known is the Maxwell–Cattaneo (MC) law (Joseph and Preziosi, 1989, 1990; Chandrasekharaiah, 1998; Jou et al., 1998)

$$(1 + \tau_0 \partial_t)\boldsymbol{q} = -\kappa \nabla T, \tag{2}$$

where ∂_t stands for the partial time derivative. Here, the thermal relaxation characteristic time $\tau_0(> 0)$ represents the time lag required to establish steady heat conduction in a volume element once a temperature gradient has been imposed across it (Chandrasekharaiah, 1998). The new term can be properly termed "thermal inertia". It should be mentioned that the value of τ_0 has been experimentally determined for a number of materials (Joseph and Preziosi, 1989, 1990; Caviglia et al., 1992). And although τ_0 turns out to be very small in many instances, e.g., τ_0 is of the order of picoseconds for most metals

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(Chandrasekharaiah, 1998; Antaki, 1995), there are several materials where this is not the case, most notably sand (21 s), H acid (25 s), NaHCO₃ (29 s), and biological tissue (1–100 s) (Chandrasekharaiah, 1998; Antaki, 1995).

The partial time derivative added by Cattaneo in the constitutive relationship between the heat flux and the temperature succeeded in resolving the main shortcoming of the Fourier model, rendering the heat-conduction equation to a damped hyperbolic equation. As a result, the propagation of a heat disturbance has a finite speed in such a model.

The success of the Maxwell–Cattaneo (MC) model showed the direction for generalization of Fourier's model to correct the temporal behavior of the solution. There are, however, more problems with Fourier law which have not yet gained the notoriety of the above discussed PHC; e.g., higher spatial gradients may be required for the constitutive relation between the heat flux and the temperature as discussed in Christov (2007), but this goes beyond the framework of the present work. Here we are concerned with properties of the time-dependent terms in the constitutive relationship for the propagation of heat, namely with the correct form of the relaxation of the heat flux introduced by Cattaneo.

The MC law involves the mere partial time derivative and should be considered only as a 'place holder' for a more complete formulation that acknowledges the mandatory invariance for any type of constitutive relation. In mechanics of continua, fields and transformations are called 'objective' when they transform according to specific rules when changing the coordinate system (see, e.g., Chadwick (1999); Marsden and Hughes (1994)). There are different modifications of the time derivative that are objective, and it is still not a settled issue which one has to be preferred when trying to make 'objective' (or material indifferent) a constitutive law involving relaxation.

With respect to the MC law, the application of principle of objectivity appears to have been initiated by Fox (1969). However, Straughan and Franchi (1984) showed that the Fox's choice of Jaumannian derivative (called alternatively, Jaumann rate) led to physically questionalble results in the problem of Bénard convection. Subsequently, Lebon and Cloot (1984) treated the same problem, but with a modified version of the objective time derivative, and uncovered different qualitative behavior of Bénard convection. It should be noted, that the objective rate used by Lebon and Cloot (1984) was once again a version of the Jaumann rate. These two works showed that it was not enough to take just any objective derivative, but rather, further considerations have to be taken into account. For example, Haupt (2002, p. 321) pointed out that if the Jaumann rate is taken as the objective derivative of the stress tensor in Maxwell's model of viscoelastic liquids, then the solution is not unique. Haupt (2002) goes on to argue that the Oldroyd objective rate normally leads to well posed problems.[1] This gives additional support for the main idea of the present paper: to use an Oldroyd-type objective rate in the MC model.

In all objective rates, the convective derivative is present, which makes the models Galilean invariant. The importance of this fact was argued by Christov and Jordan (2005) who showed that without the convective terms, the Maxwell–Cattaneo law leads to paradoxical evolution of thermal waves in a moving frame. This means that the correct direction of improving the model is to use one of the invariant derivatives (objective rates). The use of the material derivative allowed Christov and Jordan (2005) to remove the above mentioned paradoxical feature. However, there remained an undesirable feature of their formulation: the impossibility to derive a single equation for the temperature. At the same time, if one formulates the model in the so-called 'referential description' (Chadwick, 1999; Marsden and Hughes, 1994; Truesdell, 1977), the objective rate is simply the partial time derivative, and the latter commutes with the spatial gradients, thus allowing one to derive a single equation for the temperature. Our intuitive understanding is that this property should somehow be preserved in the spatial description, provided that the pertinent objective rate is employed.

In this paper, we present the case of using another Lie derivative, namely Oldroyds' upper-convected derivative. It was introduced by Oldroyd (1949) for tensors of second rank, and has been widely used in fluid mechanics for modeling Maxwell viscoelastic liquids. The model, which involves the upper-convected derivative, is reffered to as the "Oldroyd-B viscoelastic liquid" (Bird et al., 1987). To adapt Oldroyd's idea to heat conduction, we re-derive the objective rate of a vector density and use the result to modify the MC law. We show that the proposed upper-convected form of the MC law allows one to eliminate **q**, thus yielding a single equation for the temperature field.

Here is to be mentioned that a radically different approach to the finite-speed heat conduction was originated by Green in Naghdi in 1991 (see Green and Naghdi (1995a,b)). In which the thermal displacement is used as the main variable. This is a very promising model and has been put to extensive testing (see Bargmann and Steinmann (2007) and the literature cited therein, and Bargmann et al., 2008 for an interesting specific example). The formulation developed here, based on Oldroyd's upper-convected derivative, should also be applicable in Green and Naghdi's model.

2. Invariant time derivative of a vector density

Directional and other invariant derivatives of tensors are investigated in numerous mathematical and physical works but in order to make the paper selfcontained and to clarify the physical meaning, we present here a brief account of the pertinent derivations.

Consider the 3D space and a fixed system of coordinates, $\{x^i\}$, in it. The fixed coordinate system is called the 'current configuration' in mechanics of continua (Chadwick, 1999; Truesdell, 1977). It can be assumed to be Cartesian without loosing the generality. Together with the fixed coordinate system, consider a generally curvilinear moving coordinate system, $\{\bar{x}^i\}$, that is embedded in the material continuum occupying the geometrical space in the sense that coordinate lines of

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[1] As pointed out by Marsden and Hughes (1994), the subject of different 'objective rates' is still a controversial one in continuum mechanics.

the moving system consists always of the same material particles. Then the transformation $x^j = f^j(\bar{x}^i; t)$ presents the law of motion of a material particle, parametrized by the coordinate $\bar{x}^j$. Assume that at time t, the two coordinate systems coinside. Then at time $t + \Delta t$, the law of motion gives the infinitesimal transformation $x^j = \bar{x}^j + v^j \Delta t$, which can be resolved for the material coordinates:

$$\bar{x}^i = x^i - v^i(x^j)\Delta t, \tag{3}$$

where v^i is the contravariant velocity vector.

$$\frac{\partial x^j}{\partial \bar{x}^i} = \delta_i^j + \Delta t \frac{\partial v^j}{\partial x^i} + o(\Delta t), \quad \frac{\partial \bar{x}^i}{\partial x^j} = \delta_j^i - \Delta t \frac{\partial v^i}{\partial x^j} + o(\Delta t). \tag{4}$$

Let $\boldsymbol{A}$ represent some mechanical quantity, e.g., stress vector, electric field, temperature flux, etc. For all these mechanical characteristics, the actual observable is the following integral (see, e.g., Schrödinger (1950))

$$\int_D \boldsymbol{A} \mathrm{d}^3 x = \int_{\bar{D}} \boldsymbol{A} \mathrm{d}^3 \bar{x}. \tag{5}$$

The principle of material invariance requires that this integral be invariant under coordinate transformation, which means that the vector $\boldsymbol{A}$ is a tensor density (or what is called 'relative tensor'). We use the old gothic fonts to distinguish the components of a vector density from the components of a regular vector, hence $\mathfrak{A}^k$ are the contravariant components of $\boldsymbol{A}$. In component form, the integral in the left-hand side can be rewritten as

$$\int_D \mathfrak{A}^k \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \equiv \int_{\bar{D}} \frac{\partial \bar{x}^k}{\partial x^j} J \mathfrak{A}^j \mathrm{d}\bar{x}^1 \mathrm{d}\bar{x}^2 \mathrm{d}\bar{x}^3,$$

where a summation is understood if an index appears once as a superscript and once as a subscript. Respectively, J is the Jacobian of the coordinate transformation,

$$J = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right| = 1 + \Delta t \frac{\partial \bar{v}^i}{\partial x^i} + o(\Delta t). \tag{6}$$

Being reminded that D is an arbitrary region, one finds the transformation rule for a vector density in contravariant components

$$\bar{\mathfrak{A}}^k = J \frac{\partial \bar{x}^k}{\partial x^l} \mathfrak{A}^l. \tag{7}$$

Material invariance (see Oldroyd (1949)) requires that in constitutive laws that are non-local (i.e., containing relaxation and/or retardation), the total time variance of a tensor density is used, namely

$$\frac{\eth \mathfrak{A}^j}{\eth t} = \lim_{\Delta t \to 0} \frac{\bar{\mathfrak{A}}^j(\bar{x}^k; t + \Delta t) - \mathfrak{A}^j(x^k; t)}{\Delta t}. \tag{8}$$

Using a Taylor series for the first term in the numerateor on the r.h.s. of Eq. (8) with Eq. (3) acknowledged, yields within the order $o(\Delta t)$ the following

$$\bar{\mathfrak{A}}^j(\bar{x}^k; t + \Delta t) = \bar{\mathfrak{A}}^j(x^k; t) + \Delta t \left[\frac{\partial \bar{\mathfrak{A}}^j}{\partial t} + v^l \frac{\partial \bar{\mathfrak{A}}^j}{\partial \bar{x}^l} \right] = \bar{\mathfrak{A}}^j(x^k; t) + \Delta t \left[\frac{\partial \mathfrak{A}^j}{\partial t} + v^l \frac{\partial \mathfrak{A}^j}{\partial x^l} \right], \tag{9}$$

where the fact is also acknowledged that at any moment of time t, vectors $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ and their gradients coinside.

Now, the contravariant components $\mathfrak{A}^k$ transform according to the rule given in Eq. (7), which gives within the asymptotic order $o(\Delta t)$ the following

$$\bar{\mathfrak{A}}^j(x^k; t) = \left(1 + \frac{\partial v^i}{\partial x^i} \Delta t \right) \left[\mathfrak{A}^j(x^k) - \frac{\partial v^k}{\partial x^m} \mathfrak{A}^m(x^k) \Delta t \right] = \mathfrak{A}^j(x^k) + \frac{\partial v^i}{\partial x^i} \mathfrak{A}^j(x^k) \Delta t - \frac{\partial v^k}{\partial x^m} \mathfrak{A}^m(x^k) \Delta t. \tag{10}$$

After making use of Eq. (6) and neglecting the higher order terms in (Δt), Eq. (10) yields

$$\frac{\eth \mathfrak{A}^j}{\eth t} \overset{\text{def}}{=} \lim_{\Delta t \to 0} \frac{\bar{\mathfrak{A}}^j(x^k) - \mathfrak{A}^j(x^k)}{\Delta t} = \frac{\partial \mathfrak{A}^j}{\partial t} + \mathscr{L}_v \mathfrak{A}^j = \frac{\partial \mathfrak{A}^j}{\partial t} + v^k \frac{\partial \mathfrak{A}^j}{\partial x^k} - \frac{\partial v^j}{\partial x^m} \mathfrak{A}^m + \frac{\partial v^i}{\partial x^i} \mathfrak{A}^j, \tag{11}$$

where $\mathscr{L}_v$ is the Lie derivative along the vector field v^i (see Lovelock and Rund (1989) for a mixed tensor density of arbitrary rank). The first term in the invariant derivative is the partial time derivative which accounts for the changes of the components as functions of time. The second term represents the changes due to the fact that the coordinate system and the associated basis are also changing with time (being 'convected' with the velocity field of the material continuum). In abstract vector notations, valid in any coordinate system, the above derived objective time derivative of a vector density has the form

$$\frac{\eth \boldsymbol{A}}{\eth t} = \frac{\partial \boldsymbol{A}}{\partial t} + v \cdot \nabla \boldsymbol{A} - \boldsymbol{A} \cdot \nabla v + (\nabla \cdot v) \boldsymbol{A}. \tag{12}$$

For a pointed exposition of different issues connected with invariant derivatives, we refer the reader to Triffeault (2001) and the literature cited therein. Actually, what Oldroyd did amounted to taking the directional derivative of a contravariant tensor density along the contravariant velocity vector of the material point at which the constitutive relation was written, which is a generalization of the advective (i.e., the nonlinear) part of the usual material derivative.

Following established terminology (Bird et al., 1987), we can call Eq. (12) 'the upper convected' material derivative (objective rate) of vector $\boldsymbol{A}$. Note that if $\boldsymbol{A}$ was not a tensor density, but an absolute tensor, then the last term in Eq. (12) would be absent (see, also Triffeault (2001)). As shown by Oldroyd (1949), there is a difference in the invariant derivatives of a contravariant and a covariant tensor, and the choice was left open to additional mechanical considerations. In fact, this is a much deeper question, one which goes beyond the scope of the present letter. The issues connected with the choice between the upper convected and lower convected derivative are still debated in the literature, and the verdict seems to be that the choice is dictated by the particular application. The upper convected Oldroyd derivative appears to be relevant in most of the cases found in the literature. We refer the reader to Bird et al. (1987) for details. Accordingly, we limit our attention in the present work to the upper-convected formulation of the invariant time derivative.

3. Frame indifferent Maxwell–Cattaneo law

The main idea of the present work is to replace the partial time derivative in the MC law with the upper-convected Oldroyd derivative, which is a frame indifferent objective rate. For the purpose of present work, it is important to keep in mind that the heat flux is actually a vector density, rather than an absolute vector.

Since for a scalar T, the invariant time derivative is simply the total (material, susbtantial, convective, etc.) derivative, then the material invariant form of the balance law for the internal energy reads

$$\rho c_p \left(\frac{\partial T}{\partial t} + \boldsymbol{v} \cdot \nabla T \right) = -\nabla \cdot \boldsymbol{q}. \tag{13a}$$

v Here, ρ is the mass density and c_p is the specific heat, where both of these parameters are taken here to be constant.

Now in addition to Eq. (13a), we have the frame-indifferent generalization of Fourier's law with relaxation of the heat flux, namely

$$\tau_0 \left[\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{q} - \boldsymbol{q} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v}) \boldsymbol{q} \right] + \boldsymbol{q} = -\kappa \nabla T, \tag{13b}$$

where we recall that τ_0 is the relaxation time of heat flux and $\kappa(> 0)$ stands for the thermal conductivity.

We call Eq. (13a) the frame-indifferent MC model of heat conduction. This model is a straightforward generalization of the model from Christov and Jordan (2005), which is based on the material derivative. That model was shown to be irreducible, in the sense that the flux cannot be eliminated (i.e., a single equation for the temperature field cannot be derived); again, see (Christov and Jordan, 2005). Hence, it is important to interrogate the new model for reducibility.

3.1. Reduction to a single temperature equation

Now we will show the consequences of taking the upper-convected derivative *in lieu* of the mere material derivative. Following the gist of a similar derivation for the displacement current given in Christov (2006), we show that the flux vector $\boldsymbol{q}$ can be eliminated between the two Eqs. (13a) and (13b). To this end we take the operation div of the first term in the latter to obtain

$$\nabla \cdot [\boldsymbol{q}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{q} - \boldsymbol{q} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v}) \boldsymbol{q}] = (\nabla \cdot \boldsymbol{q})_t + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{q}) + \nabla \boldsymbol{v} : \nabla \boldsymbol{q} - \nabla \boldsymbol{q} : \nabla \boldsymbol{v} - \boldsymbol{q} \cdot \nabla(\nabla \cdot \boldsymbol{v})$$
$$+ \boldsymbol{q} \cdot \nabla(\nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{q}) = (\nabla \cdot \boldsymbol{q})_t + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{q}) + (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{q}) = (\nabla \cdot \boldsymbol{q})_t + \nabla \cdot [(\nabla \cdot \boldsymbol{q}) \boldsymbol{v}]. \tag{14}$$

This allows us to get from Eq. (13b) the following

$$\tau_0 \{ (\nabla \cdot \boldsymbol{q})_t + \nabla \cdot [(\nabla \cdot \boldsymbol{q}) \boldsymbol{v}] + \nabla \cdot \boldsymbol{q} \} = -\nabla \cdot (\kappa \nabla T), \tag{15}$$

and to substitute Eq. (13a) in Eq. (15), arriving thus at a single equation for the temperature field, namely

$$\tau_0 [T_{tt} + 2\boldsymbol{v} \cdot \nabla T_t + \boldsymbol{v}_t \cdot \nabla T + (T_t + \boldsymbol{v} \cdot \nabla T)(\nabla \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla(\boldsymbol{v} \cdot \nabla T)] + T_t + \boldsymbol{v} \cdot \nabla T = \nabla \cdot (\lambda \nabla T), \quad \lambda = \frac{\kappa}{\rho c_p}, \tag{16}$$

where λ is called the 'coefficient of heat diffusion'.

The result in Eq. (16) is very important. It shows that if the proper invariant time derivative is used for the relaxation term in the MC law, then a single equation for the temperature can be derived, just as is the case of the linearized MC law, which involves a mere partial time derivative.

3.2. Galilean invariance

The main point raised by Christov and Jordan (2005) was that the 'local' MC law with the partial time derivative of the heat flux has the following undesirable property: it is not Galiean invariant. It is, of course, natural to expect that a material

invariant (frame indifferent) formulation like the one presented above retains Galilean invariance. Indeed, if we introduce the following change of variables, corresponding to a frame, moving with a constant velocity $\boldsymbol{V}$,

$$\xi = \boldsymbol{x} - \boldsymbol{V}\tau, \quad t = \tau, \quad T(\boldsymbol{x},t) = \Theta(\xi,\tau), \quad \boldsymbol{v} = \boldsymbol{V} + \boldsymbol{u},$$

then we find that

$$
\begin{aligned}
T_t &= \Theta_\tau - \nabla \boldsymbol{V} \cdot \nabla_\xi \Theta, \\
T_{tt} &= \Theta_{\tau\tau} - 2\boldsymbol{V} \cdot \nabla_\xi \Theta_\tau + \boldsymbol{VV} : \nabla_\xi \nabla_\xi \Theta \\
\nabla_x T_t &= \nabla_\xi \Theta_\tau - \boldsymbol{V} \cdot \nabla_\xi \nabla_\xi \Theta \\
\boldsymbol{v}_t &= \boldsymbol{u}_\tau - \boldsymbol{V} \cdot \nabla_\xi u.
\end{aligned}
\tag{17}
$$

In the above formulas, the products of the type $\boldsymbol{VV}$ have the ubiquitous meaning of dyads. Introducing these expressions in Eq. (16) we get

$$
\begin{aligned}
\tau_0[\Theta_{\tau\tau} &- 2\boldsymbol{V} \cdot \nabla_\xi \Theta_\tau + \boldsymbol{VV} : \nabla_\xi \nabla_\xi \Theta + (\boldsymbol{u}_\tau - \boldsymbol{V} \cdot \nabla_\xi \boldsymbol{u}) \cdot \nabla_\xi \Theta + 2(\boldsymbol{V} + \boldsymbol{u}) \cdot (\nabla_\xi \Theta_\tau - \boldsymbol{V} \cdot \nabla_\xi \nabla_\xi \Theta) + \Theta_\tau(\nabla_\xi \cdot \boldsymbol{u}) \\
&+ \boldsymbol{VV} : \nabla_\xi \nabla_\xi \Theta + \boldsymbol{V} \cdot \nabla_\xi(\boldsymbol{u} \cdot \nabla_\xi \Theta) + \boldsymbol{Vu} : \nabla_\xi \nabla_\xi \Theta + \boldsymbol{v} \cdot \nabla_\xi(\boldsymbol{v} \cdot \nabla_\xi \Theta)] + \Theta_\tau - \boldsymbol{V} \cdot \nabla_\xi \Theta + (\boldsymbol{V} + \boldsymbol{u}) \cdot \nabla_\xi \Theta = \nabla_\xi \cdot (\lambda \nabla_\xi \Theta),
\end{aligned}
\tag{18}
$$

and we can rewrite the above expression as follows

$$
\begin{aligned}
\tau_0 \Big[\Theta_{\tau\tau} &- 2\boldsymbol{V} \cdot \widetilde{\nabla_\xi \Theta_\tau} + \boldsymbol{VV} : \widehat{\nabla_\xi \nabla_\xi \Theta} + \boldsymbol{u}_\tau \cdot \nabla_\xi \Theta - \overline{\overline{(\boldsymbol{V} \cdot \nabla_\xi \boldsymbol{u}) \cdot \nabla_\xi \Theta}} + 2\boldsymbol{V} \cdot \widetilde{\nabla_\xi \Theta_\tau} + 2\boldsymbol{u} \cdot \nabla_\xi \Theta_\tau - 2\overline{\boldsymbol{uV} : \nabla_\xi \nabla_\xi \Theta} \\
&- 2\boldsymbol{VV} : \widehat{\nabla_\xi \nabla_\xi \Theta} + \Theta_\tau(\nabla_\xi \cdot \boldsymbol{u}) + \boldsymbol{VV} : \widehat{\nabla_\xi \nabla_\xi \Theta} + \overline{\overline{(\boldsymbol{V} \cdot \nabla_\xi \boldsymbol{u}) \cdot \nabla_\xi \Theta}} + \overline{\boldsymbol{Vu} : \nabla_\xi \nabla_\xi \Theta} \\
&+ \overline{\boldsymbol{Vu} : \nabla_\xi \nabla_\xi \Theta} + \boldsymbol{v} \cdot \nabla_\xi(\boldsymbol{v} \cdot \nabla_\xi \Theta)] + \Theta_\tau - \underline{\boldsymbol{V} \cdot \nabla_\xi \Theta} + \underline{\boldsymbol{V} \cdot \nabla_\xi \Theta} + \boldsymbol{u} \cdot \nabla_\xi \Theta = \nabla_\xi \cdot (\lambda \nabla_\xi \Theta).
\end{aligned}
\tag{19}
$$

Canceling the like terms, we get

$$
\tau_0[\Theta_{\tau\tau} + \boldsymbol{u}_\tau \cdot \nabla_\xi \Theta + 2\boldsymbol{u} \cdot \nabla_\xi \Theta_\tau + \Theta_\tau(\nabla_\xi \cdot \boldsymbol{u}) + \boldsymbol{u} \cdot \nabla_\xi(\boldsymbol{u} \cdot \nabla_\xi \Theta)] + \Theta_\tau + \boldsymbol{u} \cdot \nabla_\xi \Theta = \nabla_\xi \cdot (\lambda \nabla_\xi \Theta),
\tag{20}
$$

which has *exactly* the same form as Eq. (16), but for the variables defined in the moving frame. Thus, we have proved the Galilean invariance of the proposed model.

Actually, the new model is not just Galilean invariant: it is also invariant in *any* accelerating and generally deforming coordinate system connected with the motion of the material continuum in which the thermal wave propagates.

The fact that we were able to derive a single equation for the temperature speaks strongly in favor of the choice of Oldroyd's upper-convective derivative as the objective rate for the MC law.

4. Summary

In this paper, the generalization of Fourier's law known as the Maxwell–Cattaneo (MC) law, has been re-examined from the point of view of material invariance. It has been argued that the time derivative of the heat flux should, in fact, be Oldroyd's upper-convected derivative, in order to preserve the material invariance of the model. A detailed derivation of Oldroyd's type of Lie derivative is presented for the case of a contravariant vector density. The reformulated MC law has been shown to yield a single equation for the temperature field, which distinguishes it favorably from the other invariant formulations, including the formulation that makes use of the standard material derivative.

Acknowledgement

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